Solutions to some homework problems

Homework (1) Due: 9/3/03

1.2.3, 1.2.4, 1.3.1, 1.3.3, 1.3.4, 1.4.1, 1.4.5, 1.5.1, 1.5.2, 1.5.6

1.3.1

The sample space is $\Omega = \{LF, LW, BF, BW\}$. We have the given probabilities $P(LF) = 0.5, P(BF) = 0.2, P(BW) = 0.2$. Hence $P(LW) = 0.1$ and so,

$$P(W) = P(LW) + P(BW) = 0.1 + 0.2 = 0.3,$$
$$P(B) = P(BF) + P(BW) = 0.2 + 0.2 = 0.4.$$

Therefore, we get

$$P(W \cup B) = P(W) + P(B) - P(W \cap B) = 0.3 + 0.4 - 0.2 = 0.5.$$

1.4.1

Using the given probabilities, we can easily find

$$P(H_0) = P(LH_0) + P(BH_0) = 0.1 + 0.4 = 0.5,$$
$$P(B) = P(BH_0) + P(BH_1) + P(BH_2) = 0.4 + 0.1 + 0.1 = 0.6,$$
$$P(L \cup H_2) = P(L) + P(H_2) - P(L \cap H_2) = 0.4 + 0.3 - 0.2 = 0.5.$$

1.4.5

Consider the following probability table:

	H_0	H_1	H_2
F	a	b	c
V	d	e	f

By assumption the numbers satisfy the equations:

$$a + d = \frac{1}{3} \tag{1}$$
$$b + e = \frac{1}{3} \tag{2}$$
$$c + f = \frac{1}{3} \tag{3}$$
$$a + b + c = \frac{5}{12}. \tag{4}$$

(a) To find three solutions to these equations, try simple numbers, e.g., let $a = 0$ and $b = c = 5/24$ so that Equation (4) is satisfied. Then we get $d = 1/3$, $e = f = 1/8$. Then permute a, b, c and d, e, f accordingly to get other two solutions. Thus we have the following tables:

	H_0	H_1	H_2
F	0	5/24	5/24
V	1/3	1/8	1/8

	H_0	H_1	H_2
F	5/24	0	5/24
V	1/8	1/3	1/8

	H_0	H_1	H_2
F	5/24	5/24	0
V	1/8	1/8	1/3

(b) Being given $a = 1/4$, $e = 1/6$, we can easily get $b = 1/6$, $c = 0$, $d = 1/12$, $f = 1/3$ and so we have the table:

	H_0	H_1	H_2
F	1/4	1/6	0
V	1/12	1/6	1/3

Comment: The equations (1) (2) (3) (4) have general solutions given by

$$\begin{aligned} a &= s + t - \frac{1}{4}, \\ b &= \frac{1}{3} - s, \\ c &= \frac{1}{3} - t, \\ d &= \frac{7}{12} - s - t, \\ e &= s, \\ f &= t, \end{aligned}$$

where s and t are parameters satisfying the inequalities:

$$0 \leq s \leq \frac{1}{3}, \quad 0 \leq t \leq \frac{1}{3}, \quad \frac{1}{4} \leq s + t \leq \frac{7}{12}.$$

In particular, when $s = t$, the above inequalities are reduced to $\frac{1}{8} \leq t \leq \frac{7}{24}$ and we have the following table:

	H_0	H_1	H_2
F	$2t - \frac{1}{4}$	$\frac{1}{3} - t$	$\frac{1}{3} - t$
V	$\frac{7}{12} - 2t$	t	t

Thus there is such a table for each value of t in the interval $[1/8, 7/24]$.

1.5.1

The probabilities as stated in the questions can be misunderstood as probabilities of intersection events. But they are actually conditional probabilities:

$$(a) \qquad P(H_0|B) = \frac{P(H_0 \cap B)}{P(B)} = \frac{0.4}{0.6} = \frac{2}{3},$$

$$(b) \qquad P(L|H_1) = \frac{P(L \cap H_1)}{P(H_1)} = \frac{0.1}{0.2} = \frac{1}{2},$$

$$(c) \qquad P(H_0^c|L) = 1 - P(H_0|L) = 1 - \frac{P(H_0 \cap L)}{P(L)} = 1 - \frac{0.1}{0.4} = \frac{3}{4}.$$

1.5.2

We assume that the die is unloaded so that each side has the same probability $\frac{1}{6}$ of facing upward when landed. Use the definition of conditional to get

$$(a) \qquad P(R_3|G_1) = \frac{P(R_3 \cap G_1)}{P(G_1)} = \frac{1/6}{5/6} = \frac{1}{5},$$

$$(b) \qquad P(R_6|G_3) = \frac{P(R_6 \cap G_3)}{P(G_3)} = \frac{1/6}{3/6} = \frac{1}{3},$$

$$(c) \qquad P(G_3|E) = \frac{P(G_3 \cap E)}{P(E)} = \frac{2/6}{3/6} = \frac{2}{3},$$

$$(d) \qquad P(E|G_3) = \frac{P(E \cap G_3)}{P(G_3)} = \frac{2/6}{3/6} = \frac{2}{3}.$$

1.5.6

(a) It is given that $P(L) = 0.16$, $P(H) = 0.10$, and $P(L \cap H)/P(L \cup H) = 0.10$. Thus $P(L \cup H) = 10P(L \cap H)$. Apply the equality

$$P(L \cup H) = P(L) + P(H) - P(L \cap H)$$

to obtain

$$10P(L \cap H) = 0.16 + 0.10 - P(L \cap H),$$

which can be easily solved to get $P(L \cap H) = \frac{0.26}{11} = \frac{13}{550}$.

(b) Use the probability $P(H \cap L) = \frac{13}{550}$ from (a) to get

$$P(H|L) = \frac{P(H \cap L)}{P(L)} = \frac{13/550}{0.16} = \frac{13}{88}.$$

Homework (2) Due: 9/10/03

1.6.3(g), 1.6.4(g), 1.8.1, 1.8.2, 1.8.3(c), 1.8.4, 1.8.7.

Extra: A system consists of five components as follows:

The system works if components 1, 2, and 3 all work or if components 4 and 5 both work. Assume that the components are mutually independent and each component has a probability of p, $0 < p < 1$, of working. Find the probability that the system works.

1.6.4(g)

By assumption, $P(C) = 1/2$, $P(C \cap D) = 1/3$, and C and D are independent. Hence $1/3 = P(C \cap D) = P(C)P(D) = (1/2)P(D)$ and so $P(D) = 2/3$. Therefore,

$$P(C \cup D) = P(C) + P(S) - P(C \cap D) = 1/2 + 2/3 - 1/3 = 5/6.$$

1.8.2

(a) Note that the letters can be repeated. Hence there are $4 \times 4 \times 4 = 64$ three-letter words in this language.

(b) If each letter only appears once in each word, then there are $4! = 24$ four-letter words.

1.8.3(c)

The order is taken into account (although the answer does not depend on the ordering). Hence the total number of possible outcomes is 52×51. To count the number of favorable outcomes, first choose two suits, in which there are $\binom{4}{2}$ possibilities. Then with the chosen two suits, there are 13 possibilities that the two cards are of the same type. Finally we need to multiply by 2 since the order is taken into account. Therefore, the number of favorable outcomes is

$$\binom{4}{2} \times 13 \times 2.$$

Thus the probability that two cards are the same type but different suits is

$$\frac{\binom{4}{2} \times 13 \times 2.}{52 \times 51} = \frac{1}{17}.$$

Extra

Let E_i denote the event that component i works, $i = 1, 2, 3, 4, 5$. Then the event E that the system works is given by

$$E = (E_1 \cap E_2 \cap E_3) \cup (E_4 \cap E_5).$$

Since the five components work independent with probability p, we get

$$\begin{aligned} P(E) &= P\big[(E_1 \cap E_2 \cap E_3) \cup (E_4 \cap E_5)\big] \\ &= P(E_1 \cap E_2 \cap E_3) + P(E_4 \cap E_5) - P(E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5) \\ &= p^3 + p^2 - p^5. \end{aligned}$$

1.8.7

Suppose there are n boxes and 5 of them have hidden special mark. A player is to scratch five boxes in a ticket. If all five boxes show the special mark, then the ticket is a winner. There are $\binom{n}{5}$ possible choices and only one choice is a winner. Therefore the probability that a ticket is a winner is $p_n = 1/\binom{n}{5}$. The question is to find n so that $p_n \approx 0.01$. The first few values of p_n are given in the following table:

n	p_n
5	1
6	0.1666666666
7	0.047619047
8	0.017857142
9	0.007936507937
10	0.003968253968

Obviously, p_n decreases as n increases. Thus the best n to get $p_n \approx 0.01$ is $n = 9$. To design such an instant lottery game, we can put a statement in the ticket saying "In the nine boxes, five of them have special mark. Scratch five boxes, if they all have the mark, you are an instant winner."

Comment. Let us try another design which has n boxes and k of them have hidden special mark. A player scratches five boxes. If all five boxes show the special mark, then the ticket is a winner. Then the probability that a ticket is a winner is $p_{n,k} = \binom{k}{5}/\binom{n}{5}$. The question is whether we can improve the above answer by allowing different value of k, other than $k = 5$. Below are some tables for the first few values of n. There is no need to consider the cases $n = 5, 6, 7, 8, 9$.

Case 1: $n = 10$

k	$p_{10,k}$
5	0.003968253968
6	0.023809523

Case 2: $n = 11$

k	$p_{11,k}$
5	0.002164502165
6	0.012987012

Case 3: $n = 12$

k	$p_{12,k}$
5	0.0012626263
6	0.007575757576
7	0.026515151

There is no point to go beyond $n = 12$ since there will be too many boxes. From the above tables we see that the values of n and k that give $p_{n,k} \approx 0.01$ are given by $n = 9, k = 5$ and $n = 11, k = 6$. The case with $n = 9, k = 5$ is a better choice.

Homework (3) Due: 9/17/03

Problems 1, 4, 7, 10, 13, 16, 19, 22, and 25 from the handout list of supplementary problems on counting.

1

A six-digit number cannot begin with 0 and so there are 9 possibilities for the first digit. The next five digits can be any number from $\{0, 1, \ldots, 9\}$. Therefore there are

$$9 \times 10 \times 10 \times 10 \times 10 \times 10 = 900,000$$

six-digit numbers. To count the number of those six-digit numbers which contain the digit 5, consider the complement, namely, those six-digit numbers which contain no 5. Since 5 is excluded, the number of those six-digit numbers which contain no 5 is

$$8 \times 9 \times 9 \times 9 \times 9 \times 9 = 472,392.$$

It follows that there are $900,000 - 472,392 = 427,608$ six-digit numbers which contain the digit 5.

7

Note the catch word "with replacement." To find the probability of at least king, consider the complement, i.e., no king. The total number of outcomes in drawing 4 cards with replacement is $52 \times 52 \times 52 \times 52 = 52^4$. Since there are 4 kings, the number of favorable outcomes with no king is $48 \times 48 \times 48 \times 48 = 48^4$. Therefore, the probability of at least one king is

$$1 - \frac{48^4}{52^4} = 0.273975.$$

10

Think of a row of 8 positions and you want to put 5 dashes and 3 dots in these positions with each position receiving exactly one dash or one dot. First choose 5 positions for the dashes and then put dots to the remaining positions. Thus there are

$$\binom{8}{5} \times 1 = 56$$

different ways, i.e., 56 different messages. If you cannot see the solution this way, try another way. Imaging that those 5 dashes have different colors and 3 dots also have different colors. Then there are 8! permutations for different messages. Now, erase the colors. Note that 5! permutations of dashes followed by 3! permutations of dots all give the same message. Thus the number of different messages is

$$\frac{8!}{5!3!} = 56.$$

16

The total number of outcomes is 10!. To find the number of favorable outcomes, think of the 5 boys holding hands together in a row and the same for the 5 girls. Then there are 2 ways they can sit in a row. Next, let the boys separate hands and change seats among themselves. This results in 5! different ways. Similarly, there are 5! different ways for the girls. Hence the number of favorable outcomes is $2 \times 5! \times 5!$. Therefore, the answer to this problem is

$$\frac{2 \times 5! \times 5!}{10!} = 0.0079365.$$

25

This problem is equivalent to the following: Draw 26 cards from an ordinary deck of 52 cards, what is the probability that they contain exactly 13 red cards (and so also exactly 13 black cards!) The total number of outcomes is $\binom{52}{26}$. The number of favorable outcomes is $\binom{26}{13} \times \binom{26}{13}$, for first choosing 13 red cards and then choosing 13 black cards. Therefore, the answer to this problem is

$$\frac{\binom{26}{13} \times \binom{26}{13}}{\binom{52}{26}} = 0.218125504.$$

Homework (4) Due: 9/22/03

2.2.1, 2.2.3, 2.3.2, 2.3.4, 2.3.6, 2.3.8, 2.4.2, 2.5.2, 2.5.4, 2.6.2, 2.6.6, 2.7.5

2.2.1

(a) The condition $c(1+\frac{1}{2}+\frac{1}{4})=1$ gives the value $c=\frac{4}{7}$.

(b) With this value of c, we get $P(N\le 1)=\frac{4}{7}(1+\frac{1}{2})=\frac{6}{7}$.

2.3.2

Note that the random variable K has binomial distribution with parameters n and p.

(a) The probability mass function of K is given by

$$P(K=x)=\binom{n}{x}p^x(1-p)^{n-x},\quad x=0,1,2,\ldots,n.$$

(b) With $p=0.8$ we have $P(K\ge 1)=1-P(K=0)=1-(0.2)^n$. We need to find n such that $1-(0.2)^n\ge 0.95$, which can be easily solved to be $n\ge 1.86$. Hence the smallest n is given by $n=2$.

2.3.4

The random variable X has geometric distribution.

(a) $P_X(x)=p(1-p)^{x-1},\quad x=1,2,3,\ldots.$

(b) $P(X>4)=\sum_{x=5}^{\infty}p(1-p)^{x-1}$. This is a geometric series with the first term $a=p(1-p)^4$ and the ratio $r=1-p$. Hence the sum is $p(1-p)^4/(1-(1-p))=(1-p)^4$. An easier way to find the answer is to observe that the event $\{X>4\}$ occurs if and only if the dog fails to catch the frisbee in the first four throws, which happens with the probability $(1-p)^4$.

2.3.6

Note that the random variable B has Poisson distribution with parameter $\lambda=T/5$.

(a) The probability mass function of B is given by

$$P(B=k)=e^{-T/5}\frac{(T/5)^k}{k!},\quad k=0,1,2,\ldots,n,\ldots.$$

(b) With $T=2$ we get

$$P(B=3)=e^{-2/5}\frac{(2/5)^3}{3!}=0.00715.$$

(c) With $T = 10$ we get

$$P(B = 0) = e^{-10/5} = e^{-2} = 0.13534.$$

(d) Since $P(B \geq 1) = 1 - P(B = 0) = 1 - e^{-T/5}$, we need to find T such that

$$1 - e^{-T/5} \approx 0.99.$$

Hence $e^{-T/5} \approx 0.01$ and so $-T/5 \approx \ln 0.01$, which gives $T = 23$.

2.3.8

The random variable L has Pascal distribution with parameter $r = 6$ and $p = 0.75$.

(a) $P_L(x) = \binom{x-1}{5}(0.75)^6(0.25)^{x-6}, \quad x = 6, 7, 8, \ldots.$

(b) $P(L = 10) = \binom{9}{5}(0.75)^6(0.25)^4 = 0.0876.$

(c) $P(L \geq 9) = 1 - \sum_{x=6}^{8} \binom{x-1}{5}(0.75)^6(0.25)^{x-6} = 0.3215.$

2.4.2

(a) The graph is a step function with a jump of 0.2 at $x = -1$, a jump of 0.5 at $x = 0$, and a jump of 0.3 at $x = 1$.

(b) The probability mass function of X is given by

$$P_X(x) = \begin{cases} 0.2, & \text{if } x = -1; \\ 0.5, & \text{if } x = 0; \\ 0.3, & \text{if } x = 1; \\ 0, & \text{elsewhere.} \end{cases}$$

2.5.2

(a) The probability mass function of C is given by

c	20	30
$P_C(c)$	0.6	0.4

(b) $E(C) = 20 \times 0.6 + 30 \times 0.4 = 24$ cents.

2.5.4

$E(X) = (-1) \times 0.2 + 0 \times 0.5 + 1 \times 0.3 = 0.1.$

2.6.6

Note that C takes the values 20 and $20 + 0.5x$ with $x = 1, 2, 3, \ldots$ First we find

$$\begin{aligned} P(C = 20) &= P(M \leq 30) \\ &= \sum_{x=1}^{30} \frac{1}{30}\left(\frac{29}{30}\right)^{x-1} \\ &= 1 - \left(\frac{29}{30}\right)^{30}. \end{aligned}$$

On the other hand, for $x = 1, 2, 3, \ldots$, we have

$$P(C = 20 + 0.5x) = P(M = 30 + x) = \frac{1}{30}\left(\frac{29}{30}\right)^{29+x}.$$

Therefore, the probability mass function of C is given by

$$P_C(c) = \begin{cases} 1 - \left(\frac{29}{30}\right)^{30}, & \text{if } c = 20; \\ \frac{1}{30}\left(\frac{29}{30}\right)^{29+x}, & \text{if } c = 20 + 0.5x, \\ & x = 1, 2, 3, \ldots \end{cases}$$

Homework (5) Due: 10/8/03

2.8.1, 2.8.3, 2.8.5, 2.8.6, 2.8.9, 2.9.2, 2.9.4, 2.9.5

2.8.1

(a) $E[N] = 0 \times 0.2 + 1 \times 0.7 + 2 \times 0.1 = 0.9$.

(b) $E[N^2] = 0^2 \times 0.2 + 1^2 \times 0.7 + 2^2 \times 0.1 = 1.1$.

(c) $\mathrm{Var}[N] = E[N^2] - (E[N])^2 = 1.1 - 0.9^2 = 0.29$.

(d) $\sigma_N = \sqrt{\mathrm{Var}[N]} = \sqrt{0.29} = 0.5385$.

2.8.5

Observe that from the form of the given PMF, the random variable X is binomial with parameters $n = 4$ and $p = \frac{1}{2}$. Therefore,

(a) $\sigma_X = \sqrt{npq} = \sqrt{4 \times \frac{1}{2} \times \frac{1}{2}} = 1$.

(b) $\mu_X = np = 4 \times \frac{1}{2} = 2$. Hence

$$
\begin{aligned}
P[\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X] &= P[1 \leq X \leq 3] \\
&= P[X = 1] + P[X = 2] + P[X = 3] \\
&= \binom{4}{1}\left(\frac{1}{2}\right)^4 + \binom{4}{2}\left(\frac{1}{2}\right)^4 + \binom{4}{3}\left(\frac{1}{2}\right)^4 \\
&= \frac{7}{8}.
\end{aligned}
$$

2.8.9

From Problem 2.6.5, let X be the number of times that a packet is transmitted by the source and let T be the time required until the packet is successfully received. The random variables X and T are related by the equality $T = 2X - 1$. Moreover, X is a geometric random variable with parameter $p = 1 - q$, where q is the probability of errors. Note that for a random variable Y with finite variance, the following equality holds for any constants a and b,

$$\mathrm{Var}[aY + b] = a^2 \mathrm{Var}[Y].$$

Therefore, we have

$$\mathrm{Var}[T] = 4\mathrm{Var}[X] = 4 \times \frac{q}{p^2} = \frac{4q}{p^2}.$$

Hence $\sigma_T = \frac{2\sqrt{q}}{p} = \frac{2\sqrt{q}}{1-q}$. Now, we need to find the largest value of q so that

$$\frac{2\sqrt{q}}{1-q} < 2.$$

This inequality is equivalent to $\sqrt{q} < 1 - q$. Square both sides to get

$$q^2 - 3q + 1 > 0.$$

The equation $q^2 - 3q + 1 = 0$ has two roots $\frac{3 \pm \sqrt{5}}{2}$. Thus the solution of the last inequality is given by $q > \frac{3+\sqrt{5}}{2}$ and $q < \frac{3-\sqrt{5}}{2}$. Since $0 < q < 1$, q is at most $\frac{3-\sqrt{5}}{2} = 0.38197$.

2.9.4

First note that $P(B) = 1 - P(X = 0) = \frac{15}{16}$. By definition,

$$P_{X|B}(x) = P(X = x|B) = \frac{P(X = x, X \neq 0)}{P(B)} = \frac{16}{15} P(X = x, X \neq 0).$$

Hence $P_{X|B}(x)$ is nonzero only for $x = 1, 2, 3, 4$. Moreover, for such x's, we have

$$P(X = x, X \neq 0) = P(X = x) = \binom{4}{x} (1/2)^4.$$

Thus $P_{X|B}(x)$ is given by

$$P_{X|B}(x) = \begin{cases} \frac{1}{15}\binom{4}{x}, & x = 1, 2, 3, 4; \\ 0, & \text{otherwise.} \end{cases}$$

With these values of $P_{X|B}(x)$, we can compute $E[X|B]$ and Var$[X|B[$ as follows:

$$E[X|B] = \frac{1}{15}(4 + 12 + 12 + 4) = \frac{32}{15},$$

$$E[X^2|B] = \frac{1}{15}(4 + 24 + 36 + 16) = \frac{80}{15},$$

$$\text{Var}[X|B] = \frac{80}{15} - \left(\frac{32}{15}\right)^2 = \frac{2224}{225}.$$

Homework (6) Due: 10/17/03

4.1.1(b)(d), 4.2.1(a)(d), 4.2.4, 4.3.4, 4.4.1(a)(c), 4.4.3(c), 4.4.4(d)(e)

4.1.1

(b) Note that for a continuous random variable X with CDF $F_X(x)$, we have

$$P[a < X \leq b] = F_X(b) - F_X(a).$$

Use this equality to get

$$P[-1/2 < X \leq 3/4] = F_X(3/4) - F_X(-1/2) = 7/8 - 1/4 = 5/8.$$

(d) First note that the value of a must be a number in the interval $(-1, 1)$. Hence $P[X \leq a] = (a+1)/2 = 0.8$ and so $a = 0.6$.

Note: Some of you mistook $F_X(x)$ as the PDF of X.

4.2.1

(a) $\int_0^2 cx\,dx = 2c$. Hence $c = 1/2$ in order for $f_X(x)$ to be a PDF.

(d) To find the CDF $F_X(x)$, first note that $F_X(x) = 0$ for $x < 0$ and $F_X(x) = 1$ for $x > 2$. As for $0 \leq x \leq 2$, we have

$$F_X(x) = \int_0^x f_X(t)\,dt = \int_0^x \frac{1}{2}t\,dt = \frac{1}{4}x^2.$$

Therefore, the CDF $F_X(x)$ is given by

$$F_X(x) = \begin{cases} 0, & x < 0; \\ \dfrac{1}{4}x^2, & 0 \leq x \leq 2; \\ 1, & x > 2. \end{cases}$$

Note: Many of you got the following wrong answer

$$F_X(x) = \begin{cases} \frac{1}{4}x^2, & 0 \leq x \leq 2; \\ 0, & \text{elsewhere.} \end{cases}$$

This function is neither a CDF nor a PDF. Remember, for a CDF $F(x)$, we must have

$$\lim_{x \to -\infty} F(x) = 0, \qquad \lim_{x \to \infty} F(x) = 1.$$

4.2.4

Recall that a function $f(x)$ is a PDF if it satisfies the following two conditions:

(1) $f(x) \geq 0$ for all $-\infty < x < \infty$.

(2) $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

From the given function $f_X(x)$ we have

$$\int_{-\infty}^{\infty} f_X(x)\, dx = \int_0^1 (ax^2 + bx)\, dx = \frac{a}{3} + \frac{b}{2}.$$

Hence by condition (2), the constants a and b must satisfy the equality $\frac{a}{3} + \frac{b}{2} = 1$, or equivalently, $b = 2\big(1 - \frac{a}{3}\big)$. Thus condition (2) implies that

$$f_X(x) = ax^2 + 2\Big(1 - \frac{a}{3}\Big)x = x\Big[ax + 2\Big(1 - \frac{a}{3}\Big)\Big], \quad \forall\, 0 \leq x \leq 1. \qquad (\star)$$

Now, the first factor x in the right hand side of equation $(\star)$ is positive. Hence condition (1) is satisfied if and only if

$$g(x) \equiv ax + 2\Big(1 - \frac{a}{3}\Big) \geq 0, \quad \forall\, 0 \leq x \leq 1. \qquad (\star\star)$$

Note that the graph of $y = g(x)$ is a straight line. Hence equation $(\star\star)$ is satisfied if and only if $g(0) \geq 0$ and $g(1) \geq 0$. But $g(0) = 2(1 - \frac{a}{3})$ and $g(1) = \frac{a}{3} + 2$. It follows that condition (1) is satisfied if and only if $1 - \frac{a}{3} \geq 0$ and $\frac{a}{3} + 2 \geq 0$, or equivalently, $-6 \leq a \leq 3$. Therefore, $f_X(x)$ is a PDF if and only if the constants a and b are given by

$$b = 2\Big(1 - \frac{a}{3}\Big), \quad -6 \leq a \leq 3.$$

Observe that in the ab-plane, this set represents a line segment.

4.3.4

We have $E[Y] = \int_0^2 y \cdot y/2\, dy = 4/3$ and $E[Y^2] = \int_0^2 y^2 \cdot y/2\, dy = 2$. Hence $E[Y] = 4/3$ and $\mathrm{Var}[Y] = 2 - (4/3)^2 = 2/9$.

4.4.4

The PDF of X is $f_X(x) = 1/10$ for $-5 \leq x < 5$ and 0 elsewhere. Hence

$$E[X^5] = \int_{-5}^{5} x^5 \cdot \frac{1}{10}\, dx = 0,$$

$$E[e^X] = \int_{-5}^{5} e^x \cdot \frac{1}{10}\, dx = \frac{1}{10}(e^5 - e^{-5}).$$

Homework (7)

4.5.1, 4.7.3, 4.8.2, 3.1.1, 3.2.1, 3.3.1, 3.4.1, 3.6.3, 3.7.6

Note: The second exam (10/27/03) will cover the following sections:

2.8, 2.9, 4.1, 4.2, 4.3, 4.4, 4.5, 4.7, 4.8, 3.1, 3.2, 3.3, 3.4, 3.6, 3.7

4.5.1

Recall the fact that if X is $N(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma}$ is $N(0,1)$. Hence

$$\begin{aligned} P[T > 100] &= P\Big[\frac{100-85}{10} < \frac{T-85}{10}\Big] = P\Big[1.5 < \frac{T-85}{10}\Big] \\ &= 1 - \Phi(1.5) = 1 - 0.9332 \\ &= 0.0668, \end{aligned}$$

$$\begin{aligned} P[T < 60] &= P\Big[\frac{T-85}{10} < \frac{60-85}{10}\Big] = P\Big[\frac{T-85}{10} < -2.5\Big] \\ &= \Phi(-2.5) = 1 - \Phi(2.5) = 1 - 0.99379 \\ &= 0.00621, \end{aligned}$$

$$\begin{aligned} P[70 \le T \le 100] &= P\Big[\frac{70-85}{10} \le \frac{T-85}{10} \le \frac{100-85}{10}\Big] \\ &= P\Big[-1.5 \le \frac{T-85}{10} \le 1.5\Big] \\ &= \Phi(1.5) - \Phi(-1.5) = \Phi(1.5) - \big(1 - \Phi(1.5)\big) \\ &= 2\Phi(1.5) - 1 = 2 \times 0.9332 - 1 \\ &= 0.8664. \end{aligned}$$

4.7.3

(a) Obviously, $F_Y(y) = 0$ for $y \le 0$. When $y > 0$ we have

$$\begin{aligned} F_Y(y) &= P[Y \le y] = P[X \le \sqrt{y}\,] \\ &= \int_0^{\sqrt{y}} 9e^{-9x}\,dx = -e^{-9x}\Big|_0^{\sqrt{y}} \\ &= 1 - e^{-9\sqrt{y}}. \end{aligned}$$

Note that the PDF is given by $f_Y(y) = \frac{d}{dy} F_Y(y)$. Hence

$$f_Y(y) = \begin{cases} \dfrac{9}{2\sqrt{y}}\, e^{-9\sqrt{y}}, & y > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Before we do parts (b) and (c), we state the first four moments of an exponential random variable X with parameter $\lambda > 0$:

$$E[X] = \frac{1}{\lambda}, \quad E[X^2] = \frac{2}{\lambda^2}, \quad E[X^3] = \frac{6}{\lambda^3}, \quad E[X^4] = \frac{24}{\lambda^4}.$$

These inequalities can be checked by integration by part (if you are very patient) or by using moment generating function, which we will study later in Section 7.3.

(b) $E[Y] = E[X^2] = \dfrac{2}{81}$.

(c) $E[Y^2] = E[X^4] = \dfrac{8}{2187}$ and so $\mathrm{Var}[Y] = E[Y^2] - (E[Y])^2 = \dfrac{20}{6561}$.

4.8.2

(a) First compute

$$P[Y < 2] = \int_0^2 0.2e^{-0.2y}\, dy = -e^{-0.2y}\Big|_0^2 = 1 - e^{-0.4}.$$

Since by definition $f_{Y|\{Y<2\}}(y) = \dfrac{f_Y(y)}{P[Y<2]}$ for $0 \le y < 2$ and 0 elsewhere, we get

$$f_{Y|\{Y<2\}}(y) = \begin{cases} \dfrac{0.2e^{-0.2y}}{1 - e^{-0.4}}, & 0 \le y < 2; \\ 0, & \text{elsewhere.} \end{cases}$$

(b) The conditional expectation $E[Y|Y < 2]$ is given by

$$E[Y|Y<2] = \int_{-\infty}^{\infty} y f_{Y|\{Y<2\}}(y)\, dy = \int_0^2 y \cdot \frac{0.2e^{-0.2y}}{1 - e^{-0.4}}\, dy.$$

Apply integration by part to get

$$E[Y|Y<2] = \frac{5 - 7e^{-0.4}}{1 - e^{-0.4}}.$$

Homework (8)

3.7.3(a), 5.1.1, 5.1.6, 5.2.1, 5.2.2, 5.2.3, 5.3.1, 5.3.3, 5.3.4, 5.3.6

Homework (9)

5.5.2, 5.5.4, 5.5.5, 5.7.2, 5.7.3, 5.7.6, 5.8.1

$\star$-1 Choose a number x randomly from $[0, 1]$. Then choose a number y randomly from $[0, x]$. Let X and Y be the first and second random points chosen, respectively. Find

(a) $E[Y]$,

(b) $f_{X,Y}(x, y)$,

(c) $f_Y(y)$

(d) $f_{X|Y}(x \mid y)$.

$\star$-2 Let X and Y be independent standard normal random variables and let $Z = X + Y$. Find (a) $f_Z(z)$, (b) $\text{Var}[Z]$, (c) $\rho_{X,Z}$.

Homework (10) Due: 11/17/03

7.1.3, 7.1.4, 7.1.5, 7.2.1, 7.2.3, 7.2.4, 7.3.1, 7.3.2, 7.3.3, 7.4.1, 7.4.2, 7.4.4

$\star$-1 Let X and Y be independent randorm variables with the same uniform distribution on the interval $[0, 1]$. Find the PDF of $W = X + Y$.

Homework (11) Due: 11/24/03

7.5.1, 7.5.2, 7.7.1, 7.7.2, 7.8.1

Homework (12) Due: 12/1/03

8.2.1, 8.2.4, 8.2.6, 8.4.1, 8.4.3