Supplementary problems on counting

The following problems are taken from the book "Fundamentals of Probability" by S. Ghahramani, Prentice Hall, 1996.

1. How many six-digit numbers are there? How many of them contain the digit 5?
2. In how many different ways can 15 offices be painted with four different colors?
3. In how many ways can we draw five cards from an ordinary deck of 52 cards (a) with replacement; (b) without replacement?
4. Two fair dice are thrown. What is the probability that the outcome is a 6 and an odd number?
5. A multiple-choice test has 15 questions each having four possible answers, out of which only one is correct. If the questions are answered at random, what is the probability of getting all of them right?
6. How many divisors does $55,125$ have? (Hint: $55,125 = 3^2 5^3 7^2$.)
7. Suppose that four cards are drawn from an ordinary deck of 52 cards successively, with replacement and at random. What is the probability of at least one king?
8. A campus telephone extension has four digits. How many different extensions with no repeated digits exist? Of these, (a) how many do not start with a 0; (b) how many do not have 01 as the first two digits?
9. How many permutations of the set $\{a, b, c, d\}$ begin with a and end with c?
10. How many different messages can be sent by five dashes and three dots?
11. Let A be the set of all sequences of $0'$s, $1'$s, and $2'$s of length 12.
    (a) How many elements are there in A?
    (b) How many elements of A have exactly six $0'$s and six $1'$s?
    (c) How many elements of A have exactly three $0'$s, four $1'$s, and five $2'$s?
12. Six fair dice are tossed. What is the probability that at least two of them show the same face?
13. Find the number of distinguishable permutations of the letters MISSISSIPPI.
14. In drawing nine cards with replacement from an ordinary deck of 52 cards, what is the probability of three aces of spaces, three queens of hearts, and three kings of clubs?
15. If we put five math, six biology, eight history, and three literature books on a bookshelf at random, what is the probability that all the math books are together?
16. Five boys and five girls sit in a row at random. What is the probability that the boys are together and the girls are together?
17. If n balls are randomly placed into n cells, what is the probability that each cell will be occupied?

18. Jim has 20 friends. If he decides to invite six of them to his birthday party, how many choices does he have?
19. A panel consists of 20 men and 25 women. How many choices do we have for a jury of six men and six women from this panel?
20. From an ordinary deck of 52 cards, five are drawn randomly. What is the probability of exactly three face cards?
21. Find the coefficient of x^3y^4 in the expansion of $(2x - 4y)^7$.
22. If five numbers are selected at random from the set $\{1, 2, 3, \ldots, 20\}$, what is the probability that their minimum is larger than 5?
23. Find the values of $\displaystyle\sum_{i=0}^{n} 2^i \binom{n}{i}$ and $\displaystyle\sum_{i=0}^{n} x^i \binom{n}{i}$.
24. What is the coefficient of $x^2y^3z^2$ in the expansion of $(2x - y + 3z)^7$?
25. An ordinary deck of 52 cards is divided into two equal sets randomly. What is the probability that each set contains exactly 13 red cards?

Special discrete random variables

X	$P_X(x)$	EX	$\text{Var}(X)$
uniform on $\{x_1, \ldots, x_n\}$	$\frac{1}{n}, \ x = x_1, \ldots, x_n$	$\frac{x_1 + \cdots + x_n}{n}$	$\frac{x_1^2 + \cdots + x_n^2}{n} - \frac{(x_1 + \cdots + x_n)^2}{n^2}$
Bernoulli with parameter p	$p^x(1-p)^{1-x}, \ x = 0, 1$	p	$p(1-p)$
binomial with parameters n and p	$\binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, 2, \ldots, n$	np	$np(1-p)$
Poisson with parameter λ	$e^{-\lambda} \frac{\lambda^x}{x!}$, $x = 0, 1, 2, \ldots, n, \ldots$	λ	λ
geometric with parameter p	$p(1-p)^{x-1}$, $x = 1, 2, \ldots, n, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Pascal with parameters r and p	$\binom{x-1}{r-1} p^r (1-p)^{x-r}$, $x = r, r+1, \ldots, r+n, \ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$

1. Binomial expansion: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

2. Exponential function: $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \quad -\infty < x < \infty$

3. Geometric series: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$

4. Negative binomial series: $(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \quad |x| < 1$

Special continuous random variables

X	$f_X(x)$	EX	$\text{Var}(X)$
Uniform on (a,b)	$\begin{cases} \frac{1}{b-a}, & a<x<b; \\ 0, & \text{otherwise.} \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal with parameters μ and σ^2	$\frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-(x-\mu)^2/2\sigma^2}$, $-\infty < x < \infty$	μ	σ^2
Exponential with parameter $\lambda > 0$	$\begin{cases} \lambda e^{-\lambda x}, & x \ge 0; \\ 0, & x \le 0. \end{cases}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma with parameters $\alpha > 0$ and $\lambda > 0$	$\begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, & x > 0; \\ 0, & x \le 0. \end{cases}$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$

1. Theorem: $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

2. Gamma function: $\displaystyle\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx,\ \alpha > 0$

3. Formulas:

   (a) $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$.

   (b) $\Gamma(n) = (n-1)!,\ \ n \ge 1$ an integer.

   (c) $\Gamma(1/2) = \sqrt{\pi}$.