Math 7311 Homework (1) Due Sept 7, 2001

1. Let X be a nonempty set. Define $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$.
   (a) Show that any subset of X is both open and closed.
   (b) Describe all compact subsets of X, i.e., give a necessary and sufficient condition for a subset of X to be compact.

2. Let $0 < p < 1$. For $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, define

$$d(x, y) = \sum_{k=1}^{n} |x_k - y_k|^p.$$

   Show that d is a metric on $\mathbb{R}^n$.

3. On $\mathbb{R}$, define two metrics d_1 and d_2 by

$$d_1(x, y) = |x - y|, \qquad d_2(x, y) = |\arctan x - \arctan y|.$$

   Show that d_1 and d_2 induce the same topology on $\mathbb{R}$.

4. Suppose R is a complete metric space.
   (a) Let $S_n = S[x_n, r_n]$ be a decreasing sequence of closed spheres in R such that $\lim_{n\to\infty} r_n = 0$. Show that $\cap_{n=1}^{\infty} S_n$ consists of a single point.
   (b) Let A_n be a decreasing sequence of closed subsets of R. Is it true that $\cap_{n=1}^{\infty} A_n \neq \emptyset$?

5. Consider the metrics d_1 and d_2 on $\mathbb{R}$ defined in Problem 3. Show that $\mathbb{R}$ is incomplete with respect to the metric d_2. (Thus two metric spaces may be homeomorphic, while one is complete and the other is incomplete.)

Math 7311 **Homework (2)** Due Sept 14, 2001

1. What is the completion of $\mathbb{R}$ with respect to the metric

$$\rho(x, y) = |\tan^{-1} x - \tan^{-1} y|?$$

2. Let c be the set of all sequences of real numbers. For $x = (x_1, x_2, \ldots, x_n, \ldots)$ and $y = (y_1, y_2, \ldots, y_n, \ldots)$ in c, define

$$\rho(x, y) = \sup_{n \geq 1} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

Show that ρ is a metric on c. Is the metric space (c, ρ) complete?

3. A nonempty subset A of a metric space is called *bounded* if its diameter $d(A)$

$$d(A) = \sup_{x,y \in A} \rho(x, y)$$

is finite. Prove that A is bounded if and only if there exist $x_0 \in A$ and $0 < r < \infty$ such that $A \subset S(x_0, r)$.

4. Show that a totally bounded subset of a metric space is bounded.

5. Let $\{a_n\}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n^2 < \infty$. The associated Hilbert cube is defined by

$$Q = \{(x_1, x_2, \ldots, x_n, \ldots);\ |x_n| \leq a_n\, \forall n \geq 1\}.$$

Show that Q is totally bounded with respect to the metric on ℓ^2.

Math 7311 Homework (3) Due Sept 24, 2001

1. Let M be a bounded subset of $C[a,b]$. Prove that the set of all functions
$$F(x) = \int_a^x f(t)\,dt, \quad a \le x \le b,$$
with $f \in M$ is precompact.

2. Check whether the set $\{\sin 2\pi nx;\ n = 1, 2, 3, \ldots\} \subset C[0,1]$ is equi-continuous.

3. Check whether the set $\{x^n;\ n = 1, 2, 3, \ldots\} \subset C[0,1]$ is equi-continuous.

4. Let X and Y be topological spaces and let f be a continuous function from X into Y. Prove that if A is compact, then $f(A)$ is also compact.

5. Let Λ be an index set. For each $\lambda \in \Lambda$, let f_λ be a real-valued function defined on a metric space.
(a) Show that if f_λ's are lower semi-continuous, then $\sup_{\lambda\in\Lambda} f_\lambda$ is also lower semi-continuous. Is the conclusion true for $\inf_{\lambda\in\Lambda} f_\lambda$?
(b) Show that if f_λ's are upper semi-continuous, then $\inf_{\lambda\in\Lambda} f_\lambda$ is also upper semi-continuous. Is the conclusion true for $\sup_{\lambda\in\Lambda} f_\lambda$?

Math 7311 Homework (4) Due Oct 1, 2001

1. A collection F of functions on $[a,b]$ is called *pointwise bounded* if for any $x \in [a,b]$ there exists a constant $0 < K < \infty$ (depending on x) such that

$$|f(x)| \le K, \qquad \forall f \in F.$$

(a) Prove that if $F \subset C[a,b]$ is uniformly bounded, then it is pointwise bounded.

(b) Prove that if $F \subset C[a,b]$ is pointwise bounded and there exist constants $\varepsilon, \delta > 0$ such that

$$|x-y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon, \quad \forall f \in F, \tag{$\star$}$$

then F is uniformly bounded.

Remark: Note that Equation ($\star$) is obviously satisfied when F is equi-continuous. Moreover, by (a) and (b) the necessary and sufficient conditions in the Arzelà-Ascoli theorem can be replaced by "pointwise bounded and equi-continuous."

- Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in a vector space V are said to be *equivalent* if there exist constants $a, b > 0$ such that

$$a\|x\|_1 \le \|x\|_2 \le b\|x\|_1, \qquad \forall x \in V.$$

2. Show that two norms in a vector space are equivalent if and only if they induce the same topology.

3. For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, define

$$\|x\|_p = \Big(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\Big)^{1/p}, \quad 1 \le p < \infty,$$
$$\|x\|_\infty = \max_{1 \le k \le n} |x_k|.$$

Show that all norms $\|\cdot\|_p, 1 \le p \le \infty$, in $\mathbb{R}^n$ are equivalent.

4. Prove that any two norms in $\mathbb{R}^2$ are equivalent. (In fact, this statement is true for any finite dimensional vector space.)

5. Let $\ell^2 = \{(x_1, x_2, \ldots, x_n, \ldots)\,;\, \sum_{n=1}^\infty |x_n|^2 < \infty\}$. Check whether ℓ^2 is complete with respect to the norm defined by

$$\|(x_1, x_2, \ldots, x_n, \ldots)\| = \left(\sum_{n=1}^\infty \frac{1}{n}|x_n|^2\right)^{1/2}.$$

Math 7311 Homework (5) Due Oct 8, 2001

1. Let $(\cdot,\cdot)$ be an inner product in a vector space X and let $\|\cdot\|$ be the norm induced by $(\cdot,\cdot)$, i.e., $\|x\| = \sqrt{(x,x)}$. Prove the *parallelogram law*:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2), \quad \forall x, y \in X.$$

2. For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, define

$$\|x\|_p = \Big(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\Big)^{1/p}, \quad 1 \le p < \infty,$$
$$\|x\|_\infty = \max_{1\le k\le n} |x_k|.$$

When $p = 2$, we know that $\|\cdot\|_2$ is induced by an inner product. How about when $p \neq 2$? Namely, does there exist an inner product $\langle\cdot,\cdot\rangle$ such that $\|x\|_p = \sqrt{\langle x,x\rangle}$ for all $x \in \mathbb{R}^n$?

3. Does the parallelogram law imply that the norm is induced by an inner product?

4. (a) Given a Banach space X, let $\{S^n\}$ be a decreasing sequence of closed spheres in X. Prove that $\bigcap_n S_n$ is nonempty. (Note: It is not assumed that the radius of S_n tends to 0 as $n \to \infty$.)
(b) Is the conclusion in (a) still valid when X is an incomplete normed space?

5. (a) Find the Fourier series of the function

$$f(x) = \begin{cases} -1, & -\pi < x \le 0; \\ 1, & 0 < x \le \pi. \end{cases}$$

(b) Use the result of (a) and the Parseval identity to derive the equality

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n-1)^2} + \cdots = \frac{\pi^2}{8}.$$

(c) Use the result of (b) to derive the equality

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6}.$$

Math 7311 Homework (6) Due Oct 17, 2001

1. We have a Theorem "*If M is a closed subspace of a Hilbert space H, then H is the orthogonal direct sum of M and its orthogonal complement $M^{\perp}$.*" Is the conclusion still true when M is not assumed to be closed?

2. Let $M = \{(x_1, x_2, \ldots, x_n, \ldots) \in \ell^2;\ x_1 = x_2\}$. Show that M is a subspace of ℓ^2. Is it closed? What is its orthogonal complement?

3. Let $M = \{(x_1, x_2, \ldots, x_n, \ldots) \in \ell^2;\ x_n = 0 \text{ for all even } n\}$. Show that M is a subspace of ℓ^2. Is it closed? What is its orthogonal complement?

4. Let X be the set of all rational numbers in the interval $[0, 1]$. An *elementary subset* of X is defined to be a finite union of disjoint intervals of the form $[a, b), [a, 1] \subset X$. Let $\mathcal{F}$ be the collection of all elementary subsets of X. Define

$$\mu([a, b)) = b - a, \quad \mu([a, 1]) = 1 - a$$

and extend μ to $\mathcal{F}$ by finite additivity. Prove that μ is not σ-additive on $\mathcal{F}$.

5. (§27) Suppose A is measurable and $\mu^*(A) = 0$. Show that every subset of A is also measurable.

Math 7311 Homework (7) Due Oct 26, 2001

1. Let $\mathcal{F}$ be a field and let μ be finitely additive on $\mathcal{F}$. Prove that μ is σ-additive if and only if for any $A, A_1, A_2, \ldots, A_n, \ldots \in \mathcal{F}$ with $A \subset \cup_{n=1}^{\infty} A_n$, the following holds
$$\mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

2. Let $X = [0,1)$. A set $A \subset X$ is called *elementary* if it is a disjoint union of intervals of the form
$$A = \bigcup_{i=1}^{n} [a_i, b_i). \qquad (\star)$$
Show that the collection $\mathcal{E}$ of elementary sets is a field. For A given by Equation $(\star)$, define
$$m(A) = \sum_{i=1}^{n} (b_i - a_i).$$
Show that m is well-defined for each $A \in \mathcal{E}$ and is σ-additive on $\mathcal{E}$.

- A real-valued function F on $\mathbb{R}$ is called a *distribution function* if it is non-decreasing, right continuous, $\lim_{x\to-\infty} F(x) = 0$, and $\lim_{x\to\infty} F(x) = 1$.

3. Let $\mathcal{E}$ be the collection of elementary subsets of $\mathbb{R}$. Let F be a distribution function. For an elementary set A given by Equation $(\star)$ (with an obvious modification), define
$$\mu(A) = \sum_{i=1}^{n} (F(b_i-) - F(a_i-)),$$
where $F(c-)$ denotes the left-hand limit at c. Show that μ is σ-additive on $\mathcal{E}$.

- A measure m on a σ-field $\mathcal{G}$ is called *complete* if $m(A) = 0$ and $B \subset A$ imply that $B \in \mathcal{G}$.

4. Let μ be σ-additive on a field $\mathcal{F}$. Let μ^* be the outer measure of μ and $\mathcal{M}$ the colloection of all measurable sets. Show that μ^* is complete on $\mathcal{M}$.

5. Show that the Lebesgue measure on $\mathbb{R}$ is translation invariant.

Math 7311 Homework (8) Due Nov 5, 2001

1. Is Egorov's theorem still true when $\mu(X) < \infty$ is not assumed?

- A sequence $\{f_n\}$ of measurable functions is said to *converge in measure* to a measurable function f if for any $\varepsilon > 0$, there exists an N such that

$$\mu\big(\{x : |f_n(x) - f(x)| > \varepsilon\}\big) < \varepsilon, \qquad \forall n \geq N.$$

2. Prove that the following statements are equivalent:
   (1) f_n converges to f in measure.
   (2) For any $\varepsilon, \delta > 0$, there exists an N such that

$$\mu\big(\{x : |f_n(x) - f(x)| > \varepsilon\}\big) < \delta, \qquad \forall n \geq N.$$

   (3) For any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu\big(\{x : |f_n(x) - f(x)| > \varepsilon\}\big) = 0.$$

3. (a) Assume $\mu(X) < \infty$. Show that convergence almost everywhere implies convergence in measure, i.e., if f_n converges almost everywhere to f, then f_n converges in measure to f.
   (b) Still assume that $\mu(X) < \infty$. Is the converse in part (a) true?
   (c) Now, suppose $\mu(X) < \infty$ is not assumed. Is the implication in part (a) still true?

- For a sequence $\{A_n\}$ of sets, define

$$\overline{\lim}_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

It is quite easy to convince yourself that $x \in \overline{\lim}_{n\to\infty} A_n$ if and only if $x \in A_n$ for infinitely many n's. Thus the set $\overline{\lim}_{n\to\infty} A_n$ can be interpreted intuitively as

$$\overline{\lim}_{n\to\infty} A_n = \{x : x \in A_n \text{ for infinitely many } n\text{'s}\}.$$

4. Let $\{A_n\}$ be a sequence of measurable sets such that $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. Show that $\mu\big(\overline{\lim}_{n\to\infty} A_n\big) = 0$.

5. Prove that if f is Riemann integrable on $[a, b]$, then it is Lebesgue integrable on $[a, b]$ and

$$(L) \int_{[a,b]} f(x)\, dm(x) = (R) \int_a^b f(x)\, dx,$$

where (L) and (R) denote the Lebesgue and Riemann integrals, respectively, and m is the Lebesgue measure on $[a, b]$.

Math 7311 **Homework (9)** Due Nov 12, 2001

1. Can the inequality in Fatou's lemma be replaced by equality?

2. Does the Lebesgue dominated convergence theorem remain true when the condition "$|f_n| \leq g$, *μ-a.e. for an integrable function* g" is not assumed?

3. Suppose f is an integrable function. Let $\{A_n\}$ be a decreasing sequence of measurable sets and let $A = \bigcap_{n=1}^{\infty} A_n$. Does it follow that $\displaystyle\int_A f\, d\mu = \lim_{n\to\infty} \int_{A_n} f\, d\mu$?

4. (a) Show that the function $f(x) = \dfrac{\sin x}{x}$ is not Lebesgue integrable on $\mathbb{R}$.

   (b) Show that the improper Riemann integral $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$ exists and has the value

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi.$$

   (Hint: For part (b) use contour integral in Complex Analysis.)

5. (a) Let $\mu(X) < \infty$. Prove that $L^p(\mu) \subset L^q(\mu)$ for any $1 \leq q \leq p \leq \infty$. Is it true that

$$L^{\infty}(\mu) = \bigcap_{1 \leq p < \infty} L^p(\mu)?$$

   (b) Do we still have the inclusion in Part (a) when $\mu(X) = \infty$?

Math 7311 Homework (10) Due Nov 19, 2001

1. Let X be the interval $[a, b]$, and let μ be ordinary Lebesgue measure on the line.
   (a) Prove that the set $\mathcal{P}$ of all polynomials on $[a, b]$ with rational coefficients is dense in $L^p(X, \mu)$ for any $1 \leq p < \infty$. (Hence $L^p(X, \mu)$ is separable.)
   (b) Show that the space $L^\infty(X, \mu)$ is not separable.

2. Let X be an uncountable set and let μ be the counting measure on X. Determine the separability of the space $L^p(X, \mu)$ for $1 \leq p \leq \infty$.

3. Give an example of a sequence of functions $\{f_n\}$ which converges everywhere on $[0, 1]$, but does not converge in the mean.

4. Give an example of a sequence of functions $\{f_n\}$ which converges uniformly, but does not converge in the mean or in the mean square.

5. Show that convergence in the mean needs not imply convergence in the mean square, whether or not $\mu(X) < \infty$.

Math 7311 Homework (11) Due Nov 26, 2001

1. Let μ be an outer regular measure on a metric space. Let B be a Borel set with $\mu(B) < \infty$. Show that for any $\varepsilon > 0$, there exists a closed set $F \subset B$ such that $\mu(F) > \mu(B) - \varepsilon$.

2. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f \in L^1(\mu)$. Prove that for any $\varepsilon > 0$, there exists $\delta > 0$ such that
$$\left| \int_A f \, d\mu \right| < \varepsilon$$
for all $A \in \mathcal{F}$ with $\mu(A) < \delta$.
(Note: The special case when f is a simple function has been proved in class. You may assume this fact.)

3. Let $f \in L^1([a,b])$ and define
$$F(x) = \int_a^x f(t) \, dt, \qquad a \leq x \leq b.$$
Show that F is a continuous function.

4. Let $x_1, x_2, \ldots, x_n, \ldots$ be the set of all rational points in $\mathbb{R}$, enumerated in any way, and let $h_n = 1/2^n$. Define a function
$$F(x) = \sum_{x_n < x} h_n, \qquad x \in \mathbb{R}.$$
(a) Show that F is discontinuous at every rational point and continuous at every irrational point.
(b) Show that F is a distribution function. (cf. Homework (7))
(c) What is the measure μ arising from F? (cf. Problem 3 in Homework (7))
(d) Show that the function
$$f(x) = \begin{cases} n, & \text{if } x = x_n, \ n = 1, 2, \ldots, \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$
belongs to $L^p(\mu)$ for all $1 \leq p < \infty$, but not to $L^\infty(\mu)$. Find $\|f\|_1$ and $\|f\|_2$.
(e) In the spirit of (d), describe all functions in $L^p(\mu)$.

5. Find the one-sided upper and lower derivatives $\overline{D}_+ f$, $\underline{D}_+ f$, $\overline{D}_- f$, $\underline{D}_- f$ of the function
$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0 \end{cases}$$
at the point $x = 0$.

Math 7311 Homework (12) Due Dec 3, 2001

1. Compute the positive, negative, and total variation functions of the function

$$f(x) = x(x-1), \quad 0 \le x \le 1.$$

2. Prove that the function

$$f(x) = \begin{cases} x^\alpha \sin \dfrac{1}{x^\beta}, & \text{if } 0 < x \le 1, \\ 0, & \text{if } x = 0 \end{cases}$$

is of bounded variation on $[0,1]$ if $\alpha > \beta$ but not if $\alpha \le \beta$.

3. Check directly from the definition of absolute continuity that the Cantor function is not absolutely continuous.

4. Check directly from the definition of absolute continuity that the function

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & \text{if } x \ne 0, \\ 0, & \text{if } x = 0 \end{cases}$$

is not absolutely continuous on any interval containing the point $x = 0$.

5. Let f be absolutely continuous on the interval $[\varepsilon, 1]$ for each $\varepsilon > 0$. Does the continuity of f at 0 imply that f is absolutely continuous on $[0,1]$? What if f is also of bounded variation on $[0,1]$?