Math 7360 **Homework Problems** Fall 2004

Assignment (1) Due: Sept 8, 2004

1. Compute the mean and the variance directly from their definitions for the binomial, the Poisson, and the geometric random variables.
2. Compute the mean and the variance directly from their definitions for the Gaussian and the exponential random variables.
3. The gamma function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \quad \mathrm{Re}(z) > 0.$$

Show that $\Gamma(z) = (z-1)\Gamma(z-1)$ for $\mathrm{Re}(z) > 1$ and $\Gamma(n) = (n-1)!$ for any integer $n \geq 1$. Sketch the graph of the function $y = \Gamma(x)$, $x > 0$.

4. For $\alpha, \lambda > 0$, define a function

$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0. \end{cases}$$

Show that $f(x)$ is a probability density function. Compute the mean and the variance of a random variable whose density function is given by the function $f(x)$.

5. (optional) Let X be a random variable whose distribution function is given by the Cantor function. Find the variance of X.

Assignment (2) Due: Sept 14, 2004

6. Show that half-open intervals, closed intervals, and the Cantor set are all Borel sets.
7. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Suppose $A_n \in \mathcal{F}$, $n = 1, 2, \ldots$, is a decreasing sequence and let $A = \cap_{n\geq 1} A_n$. Show that $P(A) = \lim_{n\to\infty} P(A_n)$. Is the conclusion still true if the sequence is not assumed to be decreasing?
8. Let X be a real-valued function defined on Ω with a σ-field $\mathcal{F}$. Show that the following are equivalent:
(a) $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{B}(\mathbb{R})$.
(b) $X^{-1}((-\infty, x]) \in \mathcal{F}$ for all $x \in \mathbb{R}$.
(c) $X^{-1}((-\infty, r]) \in \mathcal{F}$ for all $r \in \mathbb{Q}$
9. (optional) Let $\Omega_n = \{0, 1\}$ and $P_n(\{0\}) = P_n(\{1\}) = 1/2$ for $n = 1, 2, \ldots$ Apply the Kolmogorov extension theorem to get a probability measure $P = P_1 \times P_2 \times \cdots \times P_n \times \cdots$ on $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n \times \cdots$ Let

$$A = \big\{(\omega_1, \omega_2, \ldots, \omega_n, \ldots) \,|\, \omega_n = 0 \text{ for infinitely many } n'\text{s}\big\}.$$

Find the probability $P(A)$.

10. (optional) In #9, change P_n's to $P_n(\{0\}) = 1/n^2$, $P_n(\{1\}) = 1 - 1/n^2$. Find $P(A)$.

Assignment (3) Due: Sept 24, 2004

11. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of extended real-valued measurable functions on Ω. Show that the function $\inf_n f_n$ is measurable. (Note: This fact implies that the functions $\sup_n f_n$, $\liminf_n f_n$, and $\limsup_n f_n$ are all measurable.)

12. Let $F(x)$ be the distribution function of a random variable X. Prove the following equalities for any $a < b$:

$$P(a < X < b) = F(b-) - F(a),$$
$$P(a \le X < b) = F(b-) - F(a-),$$
$$P(a \le X \le b) = F(b) - F(a-).$$

13. Let X be a random variable. Prove that $\lim_{a\to\infty} P(|X| > a) = 0$.

14. (optional) For any $0 < t_1 < t_2 < \cdots < t_n$, define

$$\mu_{t_1,t_2,\ldots,t_n}(C) = \frac{1}{\sqrt{(2\pi)^n t_1(t_2-t_1)\cdots(t_n-t_{n-1})}} \times \int_C \exp\left[-\frac{1}{2}\Big(\frac{x_1^2}{t_1} + \frac{(x_2-x_1)^2}{t_2-t_1} + \cdots + \frac{(x_n-x_{n-1})^2}{t_n-t_{n-1}}\right] dx_1 dx_2 \cdots dx_n.$$

Show that the family $\{\mu_{t_1,t_2,\ldots,t_n};\ 0 < t_1 < t_2 < \cdots < t_n,\ n = 1, 2, \ldots\}$ satisfies the Kolmogorov consistency conditions.

Assignment (4) Due: Oct 1, 2004

15. Give an example to show that the strict inequality in Fatou's lemma can occur.

16. Suppose X is a random variable with finite expectation. Show that

$$\lim_{x\to\infty} xP(X > x) = \lim_{x\to-\infty} xP(X < x) = 0. \qquad (*)$$

17. Let X be a random variable satisfying the condition in Equation $(*)$. Does it follow that X has finite expectation?

18. (optional) Let X be a random variable with standard normal distribution. For $c > 0$, define

$$X_c = \begin{cases} X, & \text{if } |X| \le c; \\ 0, & \text{otherwise.} \end{cases}$$

Find the distribution function of X_c.

Assignment (5) Due: Oct 11, 2004

19. Prove or disprove the statement: If $\mathcal{A}$ and $\mathcal{B}$ are two independent families of subsets of Ω, then $\sigma(\mathcal{A})$ and $\sigma(\mathcal{B})$ are also independent.
20. Suppose X and Y are independent random variables. Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable. Show that $f(X)$ and $f(Y)$ are also independent.
21. (optional) Show that the following three assertions are equivalent:
(a) $X_n \to 0$ in probability, (b) $E\dfrac{|X_n|}{1+|X_n|} \to 0$, (c) $E(|X_n| \wedge 1) \to 0$.
22. Let $\{E_n\}$ be a sequence of events and let $p_n = P(E_n)$. Find the necessary and sufficient condition on the sequence $\{p_n\}$ such that $1_{E_n} \to 0$ in probability.
23. Let $\{E_n\}$ be a sequence of independent events and let $p_n = P(E_n)$. Find the necessary and sufficient condition on the sequence $\{p_n\}$ such that $1_{E_n} \to 0$ almost surely.

Assignment (6) Due: Oct 19, 2004

24. Let X be a standard normal random variable. Find the variance of the truncation X_c of X at level $c > 0$.
25. Let X and Y be independent random variables with the same distribution. Show that the truncations X_c and Y_c of X and Y, respectively, at level $c > 0$ are also independent and identically distributed.
26. Let $X \geq 0$ and $E(X^k) < \infty$. Prove the equalities

$$\lim_{x\to\infty} x^k P(X > x) = 0, \quad E(X^k) = \int_0^\infty k x^{k-1} P(X > x)\, dx.$$

27. (optional) Let $X \geq 0$ and $EX < \infty$. Is it possible that $EX = 1 + \sum_{n=1}^\infty P(X \geq n)$?

Assignment (7) Due: Nov 9, 2004

28. Prove that $\displaystyle\sum_{n>y} \frac{1}{n^2} \leq \frac{2}{y}$ for any $y > 0$.
29. Let $\alpha > 1$ and $k(n) = [\![\alpha^n]\!]$, the greatest integer $\leq \alpha^n$. Show that $\displaystyle\lim_{n\to\infty} \frac{k(n+1)}{k(n)} = \alpha$.
30. Choose a sequence $X_n, n \geq 1$, of points independently and randomly from the unit interval $[0, 1]$. Let $S_n = X_1 + X_2 + \cdots + X_n$. Find the limits of S_n/n with respect to (a) L^1-convergence, (b) convergence in probability, and (c) convergence almost surely.
31. Do the same thing as in Problem 30 for a sequence $\{X_n\}_{n=1}^\infty$ of independent random variables with the same standard normal distribution.
32. Let $\{X_n\}$ be a sequence of independent random variables and for each n, X_n is uniformly distributed on an interval $[-a_n, a_n]$. Find conditions on a_n such that the random series $\sum_{n=1}^\infty X_n$ converges almost surely.

Assignment (8) Due: Nov 16, 2004

33. Let $\{X_n\}$ be a sequence of independent random variables and for each n, X_n has the distribution $P(X_n = n) = P(X_n = -n) = a_n$, $P(X_n = 0) = 1 - 2a_n$. Find conditions on a_n such that the random series $\sum_{n=1}^{\infty} X_n$ converges almost surely.
34. Let X be uniformly distributed on the interval $[-1, 1]$. Show that the characteristic function of X is given by $\varphi(t) = \dfrac{\sin t}{t}$.
35. Suppose X is a random variable with finite expectation. Show that its characteristic function $\varphi(t)$ is a Lipschitz function, i.e., there exists a constant $L \geq 0$ such that $|\varphi(s) - \varphi(t)| \leq L|s - t|$ for all s and t.
36. Prove that for any $f \in L^1(\mu * \nu)$ the following equality holds

$$\int_{\mathbb{R}} f(x)\, d(\mu * \nu)(x) = \int_{\mathbb{R}^2} f(y + z)\, d\mu(y) d\nu(z).$$

37. (optional) Let X be a random variable with distribution function $F(x)$ and let Z be a standard normal random variable. Assume that X and Z are independent. Show that $X + Z$ is a continuous random variable with density function given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}(x-y)^2}\, dF(y).$$

38. (optional) Let Φ_δ be the normal distribution function with mean 0 and variance δ^2. For a distribution function F, define $F_\delta = F * \Phi_\delta$. Prove that if a is a continuity point of F, then $\lim_{\delta \to 0} F_\delta(a) = F(a)$.

Assignment (9) Due: Nov 23, 2004

39. Let μ_n be the Gaussian measure with mean a_n and variance σ_n^2. Find conditions on a_n and σ_n such that the family $\{\mu_n\}$ is tight.
40. Prove or disprove the claim: If $X_n \to X$ in distribution and $Y_n \to Y$ in distribution, then $X_n + Y_n \to X + Y$ in distribution.
41. Let $X_n \to X$ in distribution and $Y_n \to Y$ in distribution. Assume that X_n and Y_n are independent for each n, and X and Y are also independent. Prove that $X_n + Y_n \to X + Y$ in distribution.
42. Prove that $\displaystyle\lim_{n \to \infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}$.
(Note: See the hint on page 198 of the textbook by K. Itô.)
43. (optional) Prove that if X_n converges to X in distribution and Y_n converges to a constant c in distribution, then $X_n + Y_n$ converges to $X + c$ in distribution.

Assignment (10) Due: Nov 30, 2004

44. Let X be a random variable with the density function $f(x) = \max\{1 - |x|, 0\}$. Find the characteristic function of X.

45. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent random variables with the distributions $X_1 \sim N(0,1)$ and $X_n \sim N(0, 2^{n-2})$, $n \geq 2$. Let

$$X_{nk} = \frac{X_k}{\sqrt{\sum_{i=1}^{n} \mathrm{Var}(X_i)}}, \quad 1 \leq k \leq n.$$

   (a) Show that the triangular array $\{X_{nk}\}$ does not satisfy the Lindeberg condition.

   (b) Show that the triangular array $\{X_{nk}\}$ satisfies the central limit theorem.

46. Check whether the binomial distribution $b(1, p)$ is stable.

47. Check whether the Poisson distribution is stable.
