Math 7360-1 Homework Fall 2007

1* (10 pts) Let X be a random variable with $P(X = -1) = P(X = 1) = \frac{1}{2}$ and let Y be a standard normal random variable. Assume that X and Y are independent. Find the distribution function F of $X + Y$ and, if it exists, find the density function f of $X + Y$. (Note: Do not use the convolution.)

2. Let X be a Poisson random variable with parameter λ and let Y be a standard normal random variable. Assume that X and Y are independent. Find the distribution function F of $X + Y$ and, if it exists, find the density function f of $X + Y$. (Note: Do not use the convolution. Moreover, whenever you encounter an infinite series, you need to check its convergence.)

3. (§1.3, #10) For each $x \in [0, 1]$, we have

$$2F\Big(\frac{x}{3}\Big) = F(x), \quad 2F\Big(\frac{2}{3} + \frac{x}{3}\Big) - 1 = F(x).$$

4* (15 pts) (§1.3, #11) Calculate

$$\int_0^1 x\, dF(x), \quad \int_0^1 x^2\, dF(x), \quad \int_0^1 e^{itx}\, dF(x).$$

5. (§2.1, #4) If Ω is countable, then $\mathcal{S}$ is generated by the singletons, and conversely.

6* (20 pts) (§2.2, #2) Let Ω be the space of natural numbers. For each $E \subset \Omega$, let $N_n(E)$ be the cardinality of the set $E \cap [0, n]$ and let $\mathcal{C}$ be the collection of E's for which the following limit exists:

$$\mathcal{P}(E) = \lim_{n\to\infty} \frac{N_n(E)}{n}.$$

The limit $\mathcal{P}(E)$ is called the "asymptotic density" of E. Show that $\mathcal{P}$ is finitely additive on $\mathcal{C}$, but $\mathcal{C}$ is not a field.

7. (§2.2, #11) An *atom* of any measure μ on $\mathcal{B}^1$ is a singleton $\{x\}$ such that $\mu(\{x\}) > 0$. The number of atoms of any σ-finite measure is countable. For each x, we have $\mu(\{x\}) = F(x) - F(x-)$

8* (10 pts) (§2.2, #25) Let f be measurable with respect to $\mathcal{F}$, and Z be contained in a null set. Define

$$\widetilde{f} = \begin{cases} f, & \text{on } Z^c, \\ K, & \text{on } Z, \end{cases}$$

where K is a constant. Then $\widetilde{f}$ is measurable with respect to $\mathcal{F}$ provided that $(\Omega, \mathcal{F}, \mathcal{P})$ is complete. Show that the conclusion may be false otherwise.

H(1) for 55 pts: The asterisked problems in #1 to #8 are due Sept 25, 2007.

9. (§3.1, #4) Let θ be uniformly distributed on $[0,1]$. For each distribution function F, define $G(y) = \sup\{x : F(x) \le y\}$. Then $G(\theta)$ has the distribution function F.

10. (§3.1, #6) Is the range of a random variable necessarily Borel or Lebesgue measurable?

11* (10 pts) (§3.2, #1) If $X \ge 0$ a.e. on Λ and $\int_\Lambda X\,dP = 0$, then $X = 0$ a.e. on Λ.

12. (§3.2, #2) If $E(|X|) < \infty$ and $\lim_{n\to\infty} P(\Lambda_n) = 0$, then $\lim_{n\to\infty} \int_{\Lambda_n} X\,dP = 0$.

13* (20 pts) (§3.2, #5) For any $r > 0$, $E(|X|^r) < \infty$ if and only if

$$\sum_{n=1}^{\infty} n^{r-1} P(|X| \ge n) < \infty.$$

14. (§3.2, #11) If $E(X^2) = 1$ and $E(|X|) \ge a > 0$, then $P\{|X| \ge \lambda a\} \ge (1-\lambda)^2 a^2$ for $0 \le \lambda \le 1$.

15* (15 pts) (§3.2, #16) For any distribution function F and any $a \ge 0$, we have

$$\int_{-\infty}^{\infty} [F(x+a) - F(x)]\,dx = a.$$

16. (§3.3, #8) Let $\{X_j, 1 \le j \le n\}$ be independent with distribution functions $\{F_j, 1 \le j \le n\}$. Find the distribution functions of $\max_{1\le j\le n} X_j$ and $\min_{1\le j\le n} X_j$.

17* (10 pts) (§3.3, #10) If X and Y are independent and some $p > 0:\ E(|X+Y|^p) < \infty$, then $E(|X|^p) < \infty$ and $E(|Y|^p) < \infty$.

H(2) for 55 pts: The asterisked problems in #9 to #17 are due Oct 9, 2007.

18* (10 pts) (§4.1, #1) $X_n \to +\infty$ a.e. if and only if $\forall\, M > 0:\ P\{X_n < M \text{ i.o.}\} = 0$.

19. (§4.1, #4) Let f be a bounded uniformly continuous function in $\mathcal{R}$. Then $X_n \to 0$ in pr. implies $E\{f(X_n)\} \to f(0)$.

20* (10 pts) (§4.1, #7) If $X_n \to X$ in pr. and $X_n \to Y$ in pr., then $X = Y$ a.e.

21. (§4.1, #9) Give an example in which $E(X_n) \to 0$ but there does not exist any subsequence $\{n_k\} \to \infty$ such that $X_{n_k} \to 0$ in pr.

22. (§4.1, #13) If $\sup_n X_n = +\infty$ a.e., there need exist no subsequence $\{X_{n_k}\}$ that diverges to $+\infty$ in pr.

23* (20 pts) (§4.2, #4) For any sequence of r.v.'s $\{X_n\}$ there exists a sequence of constants $\{A_n\}$ such that $X_n/A_n \to 0$ a.e.

24. (§4.2, #6) Cauchy convergence of $\{X_n\}$ in pr. (or in L^p) implies the existence of an X (finite a.e.), such that X_n converges to X in pr. (or in L^p).

25* (15 pts) (§4.2, #12) Prove that the probability of convergence of a sequence of independent r.v.'s is equal to zero or one.

26. (§4.2, #13) If $\{X_n\}$ is a sequence of independent and identically distributed r.v.'s not constant a.e., then $P\{X_n \text{ converges}\} = 0$.

H(3) for 55 pts: The asterisked problems in #18 to #26 are due Oct 23, 2007.

27. (§5.1, #4) If $\{X_n\}$ are independent r.v.'s such that the fourth moments $E(X_n^4)$ have a common bound, then

$$\frac{S_n - E(S_n)}{n} \to 0 \quad \text{a.e.}$$

28. (§5.2, #1) For any sequence of r.v.'s $\{X_n\}$, and any $p \geq 1$:

$$X_n \to 0 \text{ a.e. } \Rightarrow \frac{S_n}{n} \to 0 \text{ a.e. },$$
$$X_n \to 0 \text{ in } L^p \Rightarrow \frac{S_n}{n} \to 0 \text{ in } L^p.$$

29* (10 pts) (§5.2, #2) Even for a sequence of independent r.v.'s $\{X_n\}$,

$$X_n \to 0 \text{ in pr. } \not\Rightarrow \frac{S_n}{n} \to 0 \text{ in pr.}$$

30* (10 pts) (§5.2, #4) For any $\delta > 0$, we have

$$\lim_{n\to\infty} \sum_{|k-np|>n\delta} \binom{n}{k} p^k (1-p)^{n-k} = 0$$

uniformly in $p : 0 < p < 1$.

31. (§5.2, #9) A median of the r.v. X is any number α such that

$$P\{X \leq \alpha\} \geq \frac{1}{2}, \quad P\{X \geq \alpha\} \geq \frac{1}{2}.$$

Show that such a number always exists but need not be unique.

32. (§5.2, #10) Let $\{X_n, 1 \leq n \leq \infty\}$ be arbitrary r.v.'s and for each n let m_n be a median of X_n. Prove that if $X_n \to X_\infty$ in pr. and m_∞ is unique, then $m_n \to m_\infty$. Furthermore, if there exists any sequence of real numbers $\{c_n\}$ such that $X_n - c_n \to 0$ in pr., then $X_n - m_n \to 0$ in pr.

33* (10 pts) (§5.3, #7) For arbitrary $\{X_n\}$, if $\sum_n E(|X_n|) < \infty$,then $\sum_n X_n$ converges absolutely a.e.

34* (10 pts) (§5.3, #9) Let $\{X_n\}$ be independent and identically distributed, taking the values 0 and 2 with probability $\frac{1}{2}$; then

$$\sum_{n=1}^{\infty} \frac{X_n}{3^n}$$

converges a.e. Prove that the limit has the Cantor d.f. discussed in Sec. 1.3.

35. (§5.4, #1) If $E(X_1^+) = +\infty$, $E(X_1^-) < \infty$, then $S_n/n \to +\infty$ a.e.

H(4) for 40 pts: The asterisked problems in #27 to #35 are due Nov 8, 2007.

36* (10 pts) Suppose $X_n \to X$ in distribution and Y_n converges to a constant a in distribution. Then $X_n + Y_n \to X + a$ in distribution.

37. (§6.1, #1) If f is a ch.f., and G a d.f. with $G(0-) = 0$, then the following functions are all ch.f.'s:

$$\int_0^1 f(ut)\,du, \quad \int_0^\infty f(ut)e^{-u}\,du, \quad \int_0^\infty e^{-|t|u}\,dG(u),$$

$$\int_0^\infty e^{-t^2 u}\,dG(u), \quad \int_0^\infty f(ut)\,dG(u).$$

38* (10 pts) (§6.1, #11) Let X have the normal distribution Φ. Find the d.f., p.d., and ch.f. of X^2.

39. (§6.1, #12) Let $\{X_j,\ 1 \le j \le n\}$ be independent r.v.'s each having the d.f. Φ. Find the ch.f. of $\sum_{j=1}^n X_j^2$ and show that the corresponding p.d. is

$$2^{-n/2}\Gamma(n/2)^{-1}x^{(n/2)-1}e^{-x^2/2}, \quad x > 0.$$

This is called in statistics the "χ^2 distribution with n degrees of freedom".

40. (§6.2, #1) Show that

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}.$$

41* (10 pts) (§5.2, #4) If $f(t)/t \in L^1(-\infty, \infty)$, then for each $\alpha > 0$ such that $\pm\alpha$ are points of continuity of F, we have

$$F(\alpha) - F(-\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin \alpha t}{t} f(t)\,dt.$$

42. (§6.3, #8) Interpret the remarkable trigonometric identity

$$\frac{\sin t}{t} = \prod_{n=1}^{\infty} \cos \frac{t}{2^2}$$

in terms of ch.f.'s anbd hence by addition of independent r.v.'s.

43. (§6.4, #4) Let X_n have the binomial distribution with parameter (n, p_n), and suppose $np_n \to \lambda \geq 0$. Prove that X_n converges in dist. to the Poisson d.f. with parameter λ. (In the old days this was called *the law of small numbers*.)

44* (10 pts) (§6.4, #5) Let X_λ have the Poisson distribution with parameter λ. Prove that $[X_\lambda - \lambda]/\lambda^{1/2}$ converges to in dist. to Φ as $\lambda \to \infty$.

45. (§7.2, #8) For each j let X_j have the uniform distribution in $[-j, j]$. Show that Lindeberg's condition is satisfied and state the resulting central limit theorem.

46* (10 pts) List and state ten theorems in this course in the order of importance according to your opinion.

47* (10 pts) State ten other theorems in this course and give one simple example for each theorem.

H(5) for 60 pts: The asterisked problems in #36 to #47 are due Dec 6, 2007.

48. (§7.6, #5) Show that $f(t) = (1-b)/(1-be^{it})$, $0 < b < 1$, is an infinitely divisible characteristic function.

49. (§7.6, #6) Show that the d.f. with density $\beta^\alpha \Gamma(\alpha)^{-1} x^{\alpha-1} e^{-\beta x}$, $\alpha > 0$, $\beta > 0$, in $(0, \infty)$, and 0 otherwise, is infinitely divisible.

50. Check whether the distribution is infinitely divisible for (1) binomial, (2) geometric, (3) uniform on $[-1, 1]$.

51. Find the Lévy components of a compound Poisson distribution. (Note: The ch. f. of such a distribution is given by $\Phi(t) = e^{\lambda(\varphi(t)-1)}$, where $\lambda > 0$ and $\varphi(t) = Ee^{it\xi_1}$.)

52. Find the Lévy components of a symmetric stable distribution. (Note: The ch. f. of such a distribution is given by $\varphi(t) = e^{-c|t|^p}$, $c > 0$, $0 < p \leq 2$.)

53. Suppose X and Y are independent stable random variables. Does it follow that $X+Y$ is also stable?

54. Suppose X and Y are independent infinitely divisible random variables. Does it follow that $X+Y$ is also infinitely divisible?

55. Let (X, Y) be uniformly distributed on the unit disk $\{(x, y);\, x^2 + y^2 \leq 1\}$. Find the conditional expectation $E[X|Y]$.