1* (10 pts) Let $X$ be a random variable with $P(X = -1) = P(X = 1) = \frac{1}{2}$ and let $Y$ be a standard normal random variable. Assume that $X$ and $Y$ are independent. Find the distribution function $F$ of $X + Y$ and, if it exists, find the density function $f$ of $X + Y$. (Note: Do not use the convolution.)

2. Let $X$ be a Poisson random variable with parameter $\lambda$ and let $Y$ be a standard normal random variable. Assume that $X$ and $Y$ are independent. Find the distribution function $F$ of $X + Y$ and, if it exists, find the density function $f$ of $X + Y$. (Note: Do not use the convolution. Moreover, whenever you encounter an infinite series, you need to check its convergence.)

3. (§1.3, #10) For each $x \in [0, 1]$, we have

$$2F\left(\frac{x}{3}\right) = F(x), \quad 2F\left(\frac{2}{3} + \frac{x}{3}\right) - 1 = F(x).$$

4* (15 pts) (§1.3, #11) Calculate

$$\int_0^1 x \, dF(x), \quad \int_0^1 x^2 \, dF(x), \quad \int_0^1 e^{itx} \, dF(x).$$

5. (§2.1, #4) If $\Omega$ is countable, then $\mathcal{S}$ is generated by the singletons, and conversely.

6* (20 pts) (§2.2, #2) Let $\Omega$ be the space of natural numbers. For each $E \subset \Omega$, let $N_n(E)$ be the cardinality of the set $E \cap [0, n]$ and let $C$ be the collection of $E$’s for which the following limit exists:

$$\mathcal{P}(E) = \lim_{n \to \infty} \frac{N_n(E)}{n}.$$  

The limit $\mathcal{P}(E)$ is called the “asymptotic density” of $E$. Show that $\mathcal{P}$ is finitely additive on $C$, but $C$ is not a field.

7. (§2.2, #11) An atom of any measure $\mu$ on $\mathcal{B}^1$ is a singleton $\{x\}$ such that $\mu(\{x\}) > 0$. The number of atoms of any $\sigma$-finite measure is countable. For each $x$, we have $\mu(\{x\}) = F(x) - F(x^-)$

8* (10 pts) (§2.2, #25) Let $f$ be measurable with respect to $\mathcal{F}$, and $Z$ be contained in a null set. Define

$$\tilde{f} = \begin{cases} f, & \text{on } Z^c, \\ K, & \text{on } Z, \end{cases}$$

where $K$ is a constant. Then $\tilde{f}$ is measurable with respect to $\mathcal{F}$ provided that $(\Omega, \mathcal{F}, \mathcal{P})$ is complete. Show that the conclusion may be false otherwise.

H(1) for 55 pts: The asterisked problems in #1 to #8 are due Sept 25, 2007.
9. (§3.1, #4) Let \( \theta \) be uniformly distributed on \([0, 1]\). For each distribution function \( F \), define \( G(y) = \sup\{x : F(x) \leq y\} \). Then \( G(\theta) \) has the distribution function \( F \).

10. (§3.1, #6) Is the range of a random variable necessarily Borel or Lebesgue measurable?

11* (10 pts) (§3.2, #1) If \( X \geq 0 \) a.e. on \( \Lambda \) and \( \int_{\Lambda} X \, dP = 0 \), then \( X = 0 \) a.e. on \( \Lambda \).

12. (§3.2, #2) If \( E(|X|) < \infty \) and \( \lim_{n \to \infty} P(\Lambda_n) = 0 \), then \( \lim_{n \to \infty} \int_{\Lambda_n} X \, dP = 0 \).

13* (20 pts) (§3.2, #5) For any \( r > 0 \), \( E(|X|^r) < \infty \) if and only if
\[
\sum_{n=1}^{\infty} n^{r-1} P(|X| \geq n) < \infty.
\]

14. (§3.2, #11) If \( E(X^2) = 1 \) and \( E(|X|) \geq a > 0 \), then \( P\{X \geq \lambda a\} \geq (1 - \lambda)^2 a^2 \) for \( 0 \leq \lambda \leq 1 \).

15* (15 pts) (§3.2, #16) For any distribution function \( F \) and any \( a \geq 0 \), we have
\[
\int_{-\infty}^{\infty} [F(x+a) - F(x)] \, dx = a.
\]

16. (§3.3, #8) Let \( \{X_j, 1 \leq j \leq n\} \) be independent with distribution functions \( \{F_j, 1 \leq j \leq n\} \). Find the distribution functions of \( \max_{1 \leq j \leq n} X_j \) and \( \min_{1 \leq j \leq n} X_j \).

17* (10 pts) (§3.3, #10) If \( X \) and \( Y \) are independent and some \( p > 0 \) : \( E(|X + Y|^p) < \infty \), then \( E(|X|^p) < \infty \) and \( E(|Y|^p) < \infty \).

18* (10 pts) (§4.1, #1) \( X_n \to +\infty \) a.e. if and only if \( \forall M > 0 : P\{X_n < M \} \to 0 \).

19. (§4.1, #4) Let \( f \) be a bounded uniformly continuous function in \( \mathcal{R} \). Then \( X_n \to 0 \) in pr. implies \( E\{f(X_n)\} \to f(0) \).

20* (10 pts) (§4.1, #7) If \( X_n \to X \) in pr. and \( X_n \to Y \) in pr., then \( X = Y \) a.e.

21. (§4.1, #9) Give an example in which \( E(X_n) \to 0 \) but there does not exist any subsequence \( \{n_k\} \to \infty \) such that \( X_{n_k} \to 0 \) in pr.

22. (§4.1, #13) If \( \sup_n X_n = +\infty \) a.e., there need exist no subsequence \( \{X_{n_k}\} \) that diverges to \( +\infty \) in pr.

23* (20 pts) (§4.2, #4) For any sequence of r.v.’s \( \{X_n\} \) there exists a sequence of constants \( \{A_n\} \) such that \( X_n/A_n \to 0 \) a.e.

24. (§4.2, #6) Cauchy convergence of \( \{X_n\} \) in pr. (or in \( L^p \)) implies the existence of an \( X \) (finite a.e.), such that \( X_n \) converges to \( X \) in pr. (or in \( L^p \)).
25* (15 pts) (§4.2, #12) Prove that the probability of convergence of a sequence of independent r.v.'s is equal to zero or one.

26. (§4.2, #13) If \( \{X_n\} \) is a sequence of independent and identically distributed r.v.'s not constant a.e., then \( P\{X_n \text{ converges}\} = 0 \).

27. (§5.1, #4) If \( \{X_n\} \) are independent r.v.'s such that the fourth moments \( E(X_n^4) \) have a common bound, then

\[
\frac{S_n - E(S_n)}{n} \to 0 \quad \text{a.e.}
\]

28. (§5.2, #1) For any sequence of r.v.'s \( \{X_n\} \), and any \( p \geq 1 \):

\[
X_n \to 0 \text{ a.e.} \Rightarrow \frac{S_n}{n} \to 0 \text{ a.e.},
\]

\[
X_n \to 0 \text{ in } L^p \Rightarrow \frac{S_n}{n} \to 0 \text{ in } L^p.
\]

29* (10 pts) (§5.2, #2) Even for a sequence of independent r.v.'s \( \{X_n\} \),

\[
X_n \to 0 \text{ in pr.} \not\Rightarrow \frac{S_n}{n} \to 0 \text{ in pr.}
\]

30* (10 pts) (§5.2, #4) For any \( \delta > 0 \), we have

\[
\lim_{n \to \infty} \sum_{|k-np| > n \delta} \binom{n}{k} p^k (1-p)^{n-k} = 0
\]

uniformly in \( p : 0 < p < 1 \).

31. (§5.2, #9) A median of the r.v. \( X \) is any number \( \alpha \) such that

\[
P\{X \leq \alpha\} \geq \frac{1}{2}, \quad P\{X \geq \alpha\} \geq \frac{1}{2}.
\]

Show that such a number always exists but need not be unique.

32. (§5.2, #10) Let \( \{X_n, 1 \leq n \leq \infty\} \) be arbitrary r.v.'s and for each \( n \) let \( m_n \) be a median of \( X_n \). Prove that if \( X_n \to X_\infty \) in pr. and \( m_\infty \) is unique, then \( m_n \to m_\infty \). Furthermore, if there exists any sequence of real numbers \( \{c_n\} \) such that \( X_n - c_n \to 0 \) in pr., then \( X_n - m_n \to 0 \) in pr.

33* (10 pts) (§5.3, #7) For arbitrary \( \{X_n\} \), if \( \sum_n E(|X_n|) < \infty \), then \( \sum_n X_n \) converges absolutely a.e.
34* (10 pts) (§5.3, #9) Let \( \{X_n\} \) be independent and identically distributed, taking the values 0 and 2 with probability \( \frac{1}{2} \); then

\[
\sum_{n=1}^{\infty} \frac{X_n}{3^n}
\]

converges a.e. Prove that the limit has the Cantor d.f. discussed in Sec. 1.3.

35. (§5.4, #1) If \( E(X_1^+) = +\infty \), \( E(X_1^-) < \infty \), then \( S_n/n \to +\infty \) a.e.

\[\text{H(4) for 40 pts: The asterisked problems in #27 to #35 are due Nov 8, 2007.}\]

36* (10 pts) Suppose \( X_n \to X \) in distribution and \( Y_n \) converges to a constant \( a \) in distribution. Then \( X_n + Y_n \to X + a \) in distribution.

37. (§6.1, #1) If \( f \) is a ch.f., and \( G \) a d.f. with \( G(0-) = 0 \), then the following functions are all ch.f.'s:

\[
\int_0^1 f(ut) \, du, \quad \int_0^\infty f(ut) e^{-u} \, du, \quad \int_0^\infty e^{-|t|u} \, dG(u),
\]

\[
\int_0^\infty e^{-t^2u} \, dG(u), \quad \int_0^\infty f(ut) \, dG(u).
\]

38* (10 pts) (§6.1, #11) Let \( X \) have the normal distribution \( \Phi \). Find the d.f., p.d., and ch.f. of \( X^2 \).

39. (§6.1, #12) Let \( \{X_j, 1 \leq j \leq n\} \) be independent r.v.'s each having the d.f. \( \Phi \). Find the ch.f. of \( \sum_{j=1}^n X_j^2 \) and show that the corresponding p.d. is

\[
2^{-n/2}\Gamma(n/2)^{-1}x^{(n/2)-1}e^{-x^2/2}, \quad x > 0.
\]

This is called in statistics the "\( \chi^2 \) distribution with \( n \) degrees of freedom".

40. (§6.2, #1) Show that

\[
\int_0^\infty \left( \frac{\sin x}{x} \right)^2 \, dx = \frac{\pi}{2}.
\]

41* (10 pts) (§5.2, #4) If \( f(t)/t \in L^1(-\infty, \infty) \), then for each \( \alpha > 0 \) such that \( \pm\alpha \) are points of continuity of \( F \), we have

\[
F(\alpha) - F(-\alpha) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \alpha t}{t} f(t) \, dt.
\]
42. (§6.3, #8) Interpret the remarkable trigonometric identity
\[
\frac{\sin t}{t} = \prod_{n=1}^{\infty} \cos \frac{t}{2^n}
\]
in terms of ch.f.’s anbd hence by addition of independent r.v.’s.

43. (§6.4, #4) Let \(X_n\) have the binomial distribution with parameter \((n, p_n)\), and suppose \(np_n \to \lambda \geq 0\). Prove that \(X_n\) converges in dist. to the Poisson d.f. with parameter \(\lambda\). (In the old days this was called the law of small numbers.)

44* (10 pts) (§6.4, #5) Let \(X_\lambda\) have the Poisson distribution with parameter \(\lambda\). Prove that
\[
\frac{X_\lambda - \lambda}{\sqrt{\lambda}} \to \Phi \quad \text{as} \quad \lambda \to \infty.
\]

45. (§7.2, #8) For each \(j\) let \(X_j\) have the uniform distribution in \([-j, j]\). Show that Lindeberg’s condition is satisfied and state the resulting central limit theorem.

46* (10 pts) List and state ten theorems in this course in the order of importance according to your opinion.

47* (10 pts) State ten other theorems in this course and give one simple example for each theorem.

H(5) for 60 pts: The asterisked problems in #36 to #47 are due Dec 6, 2007.

48. (§7.6, #5) Show that \(f(t) = (1 - b) / (1 - be^{it})\), \(0 < b < 1\), is an infinitely divisible characteristic function.

49. (§7.6, #6) Show that the d.f. with density \(\beta^\alpha \Gamma(\alpha)^{-1} x^{\alpha-1} e^{-\beta x}, \alpha > 0, \beta > 0,\) in \((0, \infty)\), and 0 otherwise, is infinitely divisible.

50. Check whether the distribution is infinitely divisible for (1) binomial, (2) geometric, (3) uniform on \([-1, 1]\).

51. Find the Lévy components of a compound Poisson distribution. (Note: The ch. f. of such a distribution is given by \(\Phi(t) = e^{\lambda(\varphi(t))^{-1}}\), where \(\lambda > 0\) and \(\varphi(t) = Ee^{it\xi_1}\).)

52. Find the Lévy components of a symmetric stable distribution. (Note: The ch. f. of such a distribution is given by \(\varphi(t) = e^{-c|t|^p}, c > 0, 0 < p \leq 2\).)

53. Suppose \(X\) and \(Y\) are independent stable random variables. Does it follow that \(X + Y\) is also stable?

54. Suppose \(X\) and \(Y\) are independent infinitely divisible random variables. Does it follow that \(X + Y\) is also infinitely divisible?

55. Let \((X, Y)\) be uniformly distributed on the unit disk \:\{\(x, y\); \(x^2 + y^2 \leq 1\}\}. Find the conditional expectation \(E[X|Y]\).