Math 7360-1 Homework

Homework 1: Due September 3, 2010

1. State the probability mass functions of the distributions: (a) binomial, (b) Poisson, and (c) geometric.
2. Derive the means and variances of the distributions in Problem 1.
3. State the density functions of the distributions: (a) normal, (b) exponential, and (c) gamma.
4. Derive the means and variances of the distributions in Problem 3.
5. State the density function of a Cauchy random variable X and show that X has no expectation.
6. Let X be a standard normal random variable. Define a random variable Y by

$$Y(\omega) = \begin{cases} a, & \text{if } X(\omega) \geq 0, \\ b, & \text{if } X(\omega) < 0, \end{cases}$$

where a, b are two constants. Let $Z = X + Y$.
(a) Find the density function of Z.
(b) Use part (a) to find $E(Z)$.

Homework 2: Due September 17, 2010

7. Let $F(x)$ be the distribution function of a random variable X. Express

$$P(a < X \leq b), \quad P(a < X < b), \quad P(a \leq X < b), \quad P(a \leq X \leq b)$$

in terms of the distribution function $F(x)$.

8. Let X be a random variable with distribution function given by the Cantor function. Find the mean and variance of X.
9. Let $X_i, i = 1, 2,$ be a random variable with distribution, mean, and variance given by μ_i, m_i, and σ_i^2, respectively. Let X be a random variable with distribution

$$\mu = a\mu_1 + b\mu_2,$$

where $a, b \geq 0$ and $a + b = 1$. Find the mean m and variance σ^2 of X.

10. Let $f(t)$ be a continuous function on $[a, b]$. Prove that the function

$$F(x) = \int_a^x f(t)\, dt, \quad a \leq x \leq b,$$

is absolutely continuous.
(Note: The conclusion is true for an integrable function $f(t)$. But the proof is harder.)

Homework 3: Due October 1, 2010

11. Let X be a standard Gaussian random variable with postive part X^+ and negative part X^-. Find (a) the distribution functions of X^+ and X^-, (b) means and variances of X^+ and X^-, (c) the covariance of X^+ and X^-.
12. Prove that $L^p([0,1]) \subset L^q([0,1])$ for any $1 \le q \le p \le \infty$. Check whether the following equality holds:
$$L^\infty([0,1]) = \bigcap_{1 \le p < \infty} L^p([0,1]).$$
13. Prove that $\ell^p \subset \ell^q$ for any $1 \le p \le q \le \infty$. Check whether the following equality holds:
$$\ell^\infty = \bigcup_{1 \le p < \infty} \ell^p.$$
14. Let $0 < p < 1$. Show that $|f| = \big(\int_{[0,1]} |f(t)|^p \, dt \big)^{1/p}$ does not define a norm on the space $L^p([0,1])$.
15. Let $0 < p < 1$. Show that $d(f,g) = \int_{[0,1]} |f(t) - g(t)|^p \, dt$ is a metric on the space $L^p([0,1])$.
16. Suppose the joint density function of X and Y is given by
$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \left(\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho \frac{x-\mu_1}{\sigma_1} \frac{y-\mu_2}{\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2} \right) \right],$$
where μ_1, μ_2 are real numbers and σ_1, σ_2 are positive numbers. Show that X and Y have the normal distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively, and that ρ is the correlation coefficient of X and Y.

Homework 4: Due October 15, 2010

17. Let $\mathcal{M}$ be the space of random variables on a probability space. Show that
$$d_1(X,Y) = E\frac{|X-Y|}{1+|X-Y|} \quad \text{and} \quad d_2(X,Y) = E(|X-Y| \wedge 1)$$
define two metrics on $\mathcal{M}$.
18. Let d_1 and d_2 be the metrics in Problem 17. Check whether there exist constants $\alpha, \beta > 0$ such that $\alpha d_2(X,Y) \le d_1(X,Y) \le \beta d_2(X,Y)$ for all $X, Y \in \mathcal{M}$.
19. Prove the equivalence: (1) $X_n \to X$ in prob., (2) $d_1(X_n X) \to 0$, (3) $d_2(X_n X) \to 0$.
20. Let S_n be a binomial random variable with parameters n and p. Prove that
$$E\left[\left(\frac{S_n}{n} - p \right)^4 \right] = \frac{1}{n^3} p(1-p)(1-6p+6p^2) + \frac{3}{n^2} p^2 (1-p)^2.$$
21. Suppose X_n, $n \ge 1$, is a sequence of random variables on a probability space. Prove or disprove the statement: "$X_n \to 0$ in distribution implies $X_n \to 0$ in probability."
22. Suppose X and Y are uncorrelated random variables, each taking two values. Does it follow that they are independent?

Homework 5: Due November 1, 2010

23. Let $\{E_n\}$ be a sequence of events and let $p_n = P(E_n)$. Find the necessary and sufficient condition on the sequence $\{p_n\}$ such that $1_{E_n} \to 0$ in probability.
24. Let $\{E_n\}$ be a sequence of independent events and let $p_n = P(E_n)$. Find the necessary and sufficient condition on the sequence $\{p_n\}$ such that $1_{E_n} \to 0$ almost surely.
25. Let X be a standard normal random variable and let X_c be the truncation of X at level $c > 0$. Find the mean and the variance of X_c.
26. Let $\{X_n\}$ be a sequence of independent random variables with the same uniform distribution on the interval $[0, 1]$. Suppose $f(x)$ is a continuous function on $[0, 1]$. Investigate the limit

$$\lim_{n\to\infty} \frac{1}{n}\{f(X_1) + f(X_2) + \cdots + f(X_n)\}$$

in L^1-convergence, almost sure convergence, and convergence in probability.
27. Let $\alpha > 1$ and let $k(n) = [\![\alpha^n]\!]$. Prove that $\lim_{n\to\infty} \frac{k(n+1)}{k(n)} = \alpha$.
28. *Let*$\{X_n\}$ be a sequence of independent and identically distributed random variables taking values 0 and 2 with probability $\frac{1}{2}$. Show that the random series

$$\sum_{n=1}^{\infty} \frac{1}{3^n} X_n$$

converges almost surely. Moreover, prove that the distribution function of the limit is the Cantor function.
29. Let $\{X_n\}$ be a sequence of independent and identically distributed random variables. Assume that X_n are nonconstant. Prove that $P\{\omega;\ X_n(\omega) \text{ converges}\} = 0$.
30. Prove the following equality

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}.$$

31. Let $f(x) = \frac{\sin x}{x}$, $x > 0$. Prove that $f \notin L^1(0, \infty)$. However, prove that the following improper integral exists and has the value

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

Homework 6: Due November 12, 2010

32. Let $\{X_n\}$ be a sequence of independent random variables with the distributions $P(X_n = n) = P(X_n = -n) = a_n$, $P(X_n = 0) = 1 - 2a_n$, $0 < a_n < \frac{1}{2}$, $n \geq 1$. Find conditions on $\{a_n\}$ so that $\sum_n X_n$ converges almost surely.

33. Suppose $\{\xi_n\}$ is a sequence of independent random variables with the same distribution $P(\xi_n = 1) = P(\xi_n = -1) = 1/2$. Find the condition on the constant α so that $\sum_n \frac{1}{n^\alpha}\xi_n$ converges almost surely.
34. Let $\{\zeta_n\}$ be a sequence of independent random variables having the same exponential distribution with parameter $\lambda > 0$. Find conditions on constants a_n so that $\sum_n a_n\zeta_n$ converges almost surely.
35. Prove or disprove the statement: If $\sum_n E|X_n| < \infty$, then $\sum_n X_n$ converges absolutely almost surely.
36. Let X be uniformly distributed on the interval $[-1,1]$. Show that the characteristic function of X is given by $\varphi(t) = \dfrac{\sin t}{t}$.
37. Let Φ_ϵ be the normal distribution function with mean 0 and variance ϵ^2. For a distribution function F, define $F_\epsilon = F * \Phi_\epsilon$. Prove that if a is a continuity point of F, then $\lim_{\epsilon\to 0} F_\epsilon(a) = F(a)$. Find the assertion when a is not a continuity point.

Homework 7

38. Let μ_n be the Gaussian measure with mean a_n and variance σ_n^2. Find conditions on a_n and σ_n such that the family $\{\mu_n\}$ is tight.
39. Let $\{X_n\}$ be independent Poisson random variables, each with parameter 1. By applying the central limit theorem to this sequence, prove that
$$\lim_{n\to\infty} \frac{1}{e^n}\sum_{k=0}^{n}\frac{n^k}{k!} = \frac{1}{2}.$$
40. Let $\{X_n\}_{n=1}^\infty$ be a sequence of independent random variables with the distributions $X_1 \sim N(0,1)$ and $X_n \sim N(0, 2^{n-2})$, $n \geq 2$. Let
$$X_{nk} = \frac{X_k}{\sqrt{\sum_{i=1}^n \mathrm{Var}(X_i)}}, \quad 1 \leq k \leq n.$$
Show that the triangular array $\{X_{nk}\}$ does not satisfy the Lindeberg condition.
41. Check whether the binomial distribution $b(1,p)$ is stable.
42. Let X_n be binomial with parameter (n, p_n) and suppose $np_n \to \lambda > 0$. Prove that X_n converges in distribution to the Poisson distribution with parameter λ.
43. For each j, let X_j have the uniform distribution in $[-j, j]$. Show that Lindeberg's condition is satisfied and state the resulting central limit theorem.
44. Find the Lévy components of a compound Poisson distribution. (The characteristic function of such a distribution is given by $\Phi(t) = e^{\lambda(\varphi(t)-1)}$, $\lambda > 0$, $\varphi(t) = Ee^{it\xi_1}$.)
45. Find the Lévy components of a symmetric stable distribution. (The characteristic function of such a distribution is given by $\varphi(t) = e^{-c|t|^p}$, $c > 0$, $0 < p \leq 2$.)
46. Let (X,Y) be uniformly distributed on the unit disk $\{(x,y);\, x^2+y^2 \leq 1\}$. Find the conditional expectation $E[X|Y]$.
47. State fifteen important theorems in this course. For each theorem, give examples and counterexamples.