Math 7360-2 Homework

1. State the probability mass functions of the distributions: (a) binomial, (b) Poisson, and (c) geometric.

2. Derive the means and variances of the distributions in Problem 1.

3. State the density functions of the distributions: (a) normal, (b) exponential, and (c) gamma.

4. Derive the means and variances of the distributions in Problem 3.

5. Let X be a standard normal random variable. Define a random variable Y by

$$Y(\omega) = \begin{cases} a, & \text{if } X(\omega) \geq 0, \\ b, & \text{if } X(\omega) < 0, \end{cases}$$

   where a, b are two constants. Let $Z = X + Y$.
   (1) Find the density function of Z.
   (2) Use part (1) to find the mean and variance of Z.
   (3) Find the covariance of X and Y.

6. Let X be a random variable with distribution function given by the Cantor function. Find the mean and variance of X.

7. Let $f(t)$ be a continuous function on $[a, b]$. Prove that the function

$$F(x) = \int_a^x f(t)\, dt, \quad a \leq x \leq b,$$

   is absolutely continuous.
   (Note: The conclusion is true for an integrable function $f(t)$. But the proof is harder.)

8. Suppose the joint density function of X and Y is given by

$$f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{x-\mu_1}{\sigma_1}\frac{y-\mu_2}{\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right)\right],$$

   where μ_1, μ_2 are real numbers, σ_1, σ_2 are positive numbers, and $-1 < \rho < 1$.
   (1) Show that X and Y have distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively,
   (2) Show that ρ is the correlation coefficient of X and Y.
   (3) Find the density function of $X + Y$.

9. Prove the following equality for any $0 < s < t < u$,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{(2\pi)^3 s(t-s)(u-t)}} \exp\left[-\frac{1}{2}\left(\frac{x^2}{s} + \frac{(y-x)^2}{t-s} + \frac{(z-y)^2}{u-t}\right)\right] dy$$
$$= \frac{1}{\sqrt{(2\pi)^2 s(u-s)}} \exp\left[-\frac{1}{2}\left(\frac{x^2}{s} + \frac{(z-x)^2}{u-s}\right)\right].$$

10. Let $\mathcal{M}$ be the space of random variables on a probability space. Show that

$$d_1(X,Y) = E\frac{|X-Y|}{1+|X-Y|} \quad \text{and} \quad d_2(X,Y) = E(|X-Y| \wedge 1)$$

define two metrics on $\mathcal{M}$.

11. Let d_1 and d_2 be the metrics in Problem 17. Check whether there exist constants $\alpha, \beta > 0$ such that $\alpha d_2(X,Y) \leq d_1(X,Y) \leq \beta d_2(X,Y)$ for all $X, Y \in \mathcal{M}$.

12. Prove the equivalence: (1) $X_n \to X$ in prob., (2) $d_1(X_n X) \to 0$, (3) $d_2(X_n X) \to 0$.

13. Let $\{E_n\}$ be a sequence of events and let $p_n = P(E_n)$. Find the necessary and sufficient condition on the sequence $\{p_n\}$ such that $1_{E_n} \to 0$ in probability.

14. Let $\{E_n\}$ be a sequence of independent events and let $p_n = P(E_n)$. Find the necessary and sufficient condition on the sequence $\{p_n\}$ such that $1_{E_n} \to 0$ almost surely.

15. Let X be a standard normal random variable and let X_c be the truncation of X at level $c > 0$. Find the mean and the variance of X_c.

16. Let $\{X_n\}$ be a sequence of independent random variables with the same uniform distribution on the interval $[0,1]$. Suppose $f(x)$ is a continuous function on $[0,1]$. Investigate the limit

$$\lim_{n\to\infty} \frac{1}{n}\{f(X_1) + f(X_2) + \cdots + f(X_n)\}$$

in L^1-convergence, almost sure convergence, and convergence in probability.

17. Let $\{X_n\}$ be a sequence of independent and identically distributed random variables. Assume that X_n are nonconstant. Prove that $P\{\omega;\, X_n(\omega) \text{ converges}\} = 0$.

18. Prove the following equality

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}.$$

19. Let $f(x) = \frac{\sin x}{x}$, $x > 0$. Prove that $f \notin L^1(0,\infty)$. However, prove that the following improper integral exists and has the value

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

20. Let $\{X_n\}$ be a sequence of independent random variables with the distributions $P(X_n = n) = P(X_n = -n) = a_n$, $P(X_n = 0) = 1 - 2a_n$, $0 < a_n < \frac{1}{2}$, $n \geq 1$. Find conditions on $\{a_n\}$ so that $\sum_n X_n$ converges almost surely.

21. Suppose $\{\xi_n\}$ is a sequence of independent random variables with the same distribution $P(\xi_n = 1) = P(\xi_n = -1) = 1/2$. Find the condition on the constant α so that $\sum_n \frac{1}{n^\alpha}\xi_n$ converges almost surely.

22. Let $\{\zeta_n\}$ be a sequence of independent random variables having the same exponential distribution with parameter $\lambda > 0$. Find conditions on constants a_n so that $\sum_n a_n \zeta_n$ converges almost surely.

23. Prove or disprove the statement: If $\sum_n E|X_n| < \infty$, then $\sum_n X_n$ converges absolutely almost surely.

24. Let X be uniformly distributed on the interval $[-1, 1]$. Show that the characteristic function of X is given by $\varphi(t) = \dfrac{\sin t}{t}$.

25. Let μ_n be the Gaussian measure with mean a_n and variance σ_n^2. Find conditions on a_n and σ_n such that the family $\{\mu_n\}$ is tight.

26. Let $\{X_n\}$ be independent Poisson random variables, each with parameter 1. By applying the central limit theorem to this sequence, prove that

$$\lim_{n\to\infty} \frac{1}{e^n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}.$$

27. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent random variables with the distributions $X_1 \sim N(0,1)$ and $X_n \sim N(0, 2^{n-2})$, $n \geq 2$. Let

$$X_{nk} = \frac{X_k}{\sqrt{\sum_{i=1}^{n} \mathrm{Var}(X_i)}}, \quad 1 \leq k \leq n.$$

Show that the triangular array $\{X_{nk}\}$ does not satisfy the Lindeberg condition.

28. Check whether the binomial distribution $b(1, p)$ is stable.

29. Let X_n be binomial with parameter (n, p_n) and suppose $np_n \to \lambda > 0$. Prove that X_n converges in distribution to the Poisson distribution with parameter λ.

30. Find the Lévy components of a compound Poisson distribution. (The characteristic function of such a distribution is given by $\Phi(t) = e^{\lambda(\varphi(t)-1)}$, $\lambda > 0$, $\varphi(t) = Ee^{it\xi_1}$.)

31. Find the Lévy components of a symmetric stable distribution. (The characteristic function of such a distribution is given by $\varphi(t) = e^{-c|t|^p}$, $c > 0$, $0 < p \leq 2$.)

32. State ten important theorems in this course. For each theorem, give examples and counterexamples.