Math 7360-1 Homework

Homework (1): Due 9/18/2015 (Friday)

1. Derive the equality $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\, dx = \sqrt{2\pi}$.
2. Find the volume of $\{x \in \mathbb{R}^n;\ |x| \le 1\}$.
3. Find the surface measure of $\{x \in \mathbb{R}^n;\ |x| = 1\}$.
4. Let Z be a standard normal random variable. For a given number $a > 0$, define

$$X(\omega) = \begin{cases} Z(\omega), & \text{if } |Z(\omega)| \le a, \\ a, & \text{if } |Z(\omega)| > a. \end{cases}$$

   Find the distribution function of X and use it to evaluate EX.
5. Let (X, Y) be uniformly distributed on the triangle with vertices at $(0, 0)$, $(1, 0)$, and $(0, 1)$. Find $E[X|Y = y]$ and $E[X|Y]$.
6. Show that the strong law of large numbers implies the weak law of large numbers.
7. Let $\mathcal{F}$ be a σ-field and $A_n \in \mathcal{F}$, $n = 1, 2, \ldots$. Show that the set

$$\{\omega \in \Omega \,|\, \omega \in A_k \text{ for infinitely many } k'\text{s}\}$$

   belongs to $\mathcal{F}$.
8. Let $\Omega = [0, 1]$. Consider the following collection $\mathcal{A}$ of subsets of Ω:

$$\{x\}, \quad x \in [0, 1] \cap \mathbb{Q}.$$

   Describe the σ-field $\sigma[\mathcal{A}]$ generated by $\mathcal{A}$.

Homework (2): Due 10/9/2015 (Friday)

9. Let $I\!N$ be the set of natural numbers and $\mathcal{F}$ the collection of subsets A of $I\!N$ such that either A or A^c is a finite set. Define a function P on $\mathcal{F}$ by

$$P(A) = \begin{cases} 0, & \text{if } A \text{ is finite,} \\ 1, & \text{if } A^c \text{ is finite.} \end{cases}$$

Show that $\mathcal{F}$ is a field and P does not satisfy the continuity condition.

10. Let $\{a_n\}$ be a sequence of postive numbers with sum a. For a sequence $\{x_n\}$ of real numbers, define a function

$$G(x) = \sum_{n=1}^{\infty} a_n 1_{[x_n,\infty)}(x), \quad -\infty < x < \infty.$$

Prove that $\lim_{x\to-\infty} G(x) = 0$ and $\lim_{x\to\infty} G(x) = a$.

11. Find a singularly continuous distribution function F on $(-\infty,\infty)$ such that $F(x) > 0$ for all $x \in (-\infty,\infty)$.

12. Let F be the Cantor function on $[0,1]$. Evaluate the Riemann-Stieltjes integrals $\int_0^1 x\,dF(x)$ and $\int_0^1 x^2\,dF(x)$.

13. Let μ be the probability measure on $[0,1]$ given by the Cantor function. Find a sequence $\{s_n\}_{n=1}^{\infty}$ of simple functions such that $s_n(x) \to x$ as $n \to \infty$ for all $x \in [0,1]$. Then use this sequence to evaluate the Lebesgue integral $\int_{[0,1]} x\,d\mu(x)$.

14. Prove the following equality for any $0 < s < u < t$,

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{1}{\sqrt{(2\pi)^3 s(u-s)(t-u)}} \exp\left[-\frac{1}{2}\Big(\frac{x^2}{s} + \frac{(y-x)^2}{u-s} + \frac{(z-y)^2}{t-u}\Big)\right] dy \\ &= \frac{1}{\sqrt{(2\pi)^2 s(t-s)}} \exp\left[-\frac{1}{2}\Big(\frac{x^2}{s} + \frac{(z-x)^2}{t-s}\Big)\right]. \end{aligned}$$

15. Prove that $X_n \to 0$ in probability as $n \to \infty$ if and only if $E\big(\frac{|X_n|}{1+|X_n|}\big) \to 0$ as $n \to \infty$.

16. Let S_n be the number of heads in the first n independent trials of a coin with $P(H) = p$. Prove that $E\big[(S_n - np)^4\big] = np(1-p)(1-6p+6p^2) + 3n^2p^2(1-p)^2$.

Homework (3): Due 11/2/2015 (Monday)

17. (1) Show that $L^p(\Omega) \subset L^q(\Omega)$ for all $p \geq q$.
    (2) Check whether the equality $\bigcap_{1 \leq p < \infty} L^p(\Omega) = L^\infty(\Omega)$ is true.
    (3) Check whether the equality $\bigcup_{1 < p \leq \infty} L^p(\Omega) = L^1(\Omega)$ is true.

18. Let $F(x)$ be the distribution function of X. Show that the distribution functions of X^+ and X^- are given respectively by

$$F_+(x) = \begin{cases} 0, & \text{if } x < 0, \\ F(x), & \text{if } x \geq 0; \end{cases}$$

$$F_-(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1 - F\big((-x)^-\big), & \text{if } x \geq 0, \end{cases}$$

where $F(t^-)$ denotes the left-hand limit at t.

19. Let $\alpha > 1$ and $k(n) = [\![\alpha^n]\!]$, the integral part of α^n. Show that

$$\lim_{n\to\infty} \frac{k(n+1)}{k(n)} = \alpha.$$

20. Give an example of a sequence $\{X_n\}$ of identically distributed random variables which are pairwise independent, but not jointly independent.

21. Let $\{X_n\}$ be a sequence of independent and identically distributed random variables. Assume that X_n are nonconstant. Prove that $P\{\omega;\, X_n(\omega) \text{ converges}\} = 0$.

22. Let $\{E_n\}$ be a sequence of independent events and let $p_n = P(E_n)$. Find a necessary and sufficient condition on the sequence $\{p_n\}$ such that $1_{E_n} \to 0$ almost surely.

23. Let X_n, $n \geq 1$, be independent random variables with the distributions

$$P(X_n = n) = P(X_n = -n) = \frac{1}{2}a_n, \quad P(X_n = 0) = 1 - a_n, \quad n \geq 1,$$

where $0 < a_n < 1$ for all $n \geq 1$. Find conditions on $\{a_n\}$ so that $\sum_{n=1}^\infty X_n$ converges almost surely and yet $\sum_{n=1}^\infty \mathrm{var}(X_n) = \infty$.

24. Let ε_n, $n \geq 1$, be the Radamacher functions. Find a necessary and sufficient condition on $\{a_n\}$ so that $\sum_{n=1}^\infty a_n\varepsilon_n$ converges almost surely.

Homework (4): Due 11/23/2015 (Monday)

25. Prove that $\int_{-\infty}^{\infty} \left|\frac{\sin t}{t}\right| dt = \infty$ and derive the equality $\int_{-\infty}^{\infty} \frac{\sin t}{t}\, dt = \pi$.

26. Derive the equality $\int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \pi$.

27. Let a be a real number. Justify the informal change of variables $u = y - ia$ for the equality $\int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-ia)^2}\, dy = \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\, du$.

28. Derive the equality $\int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2}\, dx = \pi e^{-|t|}, \quad -\infty < t < \infty$.

29. Let μ_n be the Gaussian measure with mean a_n and variance σ_n^2. Find conditions on a_n and σ_n such that the family $\{\mu_n\}$ is tight.

30. Let $\{X_n\}$ be independent Poisson random variables, each with parameter 1. By applying the central limit theorem to this sequence, prove that

$$\lim_{n\to\infty} \frac{1}{e^n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}.$$

Homework (5): Due 12/2/2015 (Wednesday)

31. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent random variables with the distributions $X_1 \sim N(0,1)$ and $X_n \sim N(0, 2^{n-2})$, $n \geq 2$. Let

$$X_{nk} = \frac{X_k}{\sqrt{\sum_{i=1}^{n} \mathrm{Var}(X_i)}}, \quad 1 \leq k \leq n.$$

Show that the triangular array $\{X_{nk}\}$ does not satisfy the Lindeberg condition.

32. Check whether the uniform distribution on the interval $[-1, 1]$ is infinitely divisible.

33. Prove the equality $\int_{-\infty}^{\infty} \frac{1}{1+(x-y)^2} \frac{1}{1+y^2}\, dy = \frac{2}{4+x^2}\pi$.

34. Show that the Poisson distribution with parameter $\lambda > 0$ is not stable.

35. Prove that stable distributions are infinitely divisible.

36. Show that gamma distributions are not stable, but are infinitely divisible.