Math 7365-1 Homework

Homework (1)

1. Let $f(t)$ be a continuous function on $[a, b]$. Prove that

$$\int_a^b f(t)\, dB(t) = \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^{n} f(t_i^*)(B(t_i) - B(t_{i-1})), \qquad \text{in } L^2(\Omega),$$

   where the left-hand side is the Wiener integral of $f(t)$, $t_i^* \in [t_{i-1}, t_i]$, and Δ_n's are partitions of $[a, b]$.

2. Let $f(t)$ be a C^1-function on $[a, b]$, i.e., a continuously differentiable function on $[a, b]$. Prove the following equality:

$$\int_a^b f(t)\, dB(t) = f(t)B(t)\Big|_a^b - \int_a^b B(t) f'(t)\, dt.$$

3. Let X_t be an exponential Brownian motion given by

$$X_t = e^{\sigma B(t) + (\alpha - \frac{1}{2}\sigma^2)t},$$

   where σ and α are real numbers. Find all moments $E(X_t^n)$ for $n \geq 1$.

4. Find the power series of the function $f(x) = e^{e^x - 1}$. In particular, evaluate the first six terms.

5. Let $f(t)$ be a C^2-function on the real line and define the convolution

$$(p_t * f)(x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f(y)\, e^{-\frac{(x-y)^2}{2t}}\, dy.$$

   Prove that

$$\lim_{t \to 0} \frac{(p_t * f)(x) - f(x)}{t} = \frac{1}{2} f''(x).$$

Homework (2)

6. Let $B(t)$ be a Brownian motion. Define a stochastic process X_t by

$$X_t = \begin{cases} B(t), & \text{if } 0 \le t \le 1, \\ B(1), & \text{if } t > 1. \end{cases}$$

Show that X_t is a Markov proves and find its transition probabilities $\{P_{s,x}(t,\cdot)\}$.

7. Suppose a collection $\{P_{s,x}(t,\cdot);\, 0 \le s < t,\, x \in I\!R\}$ of probability measures satisfies the Chapman-Kolmogorv equation. For any $0 < t_1 < t_2 < \cdots < t_n$, define

$$\begin{aligned} &\mu_{t_1,t_2,\ldots,t_n}((-\infty, c_1] \times (-\infty, c_2] \times \cdots \times (-\infty, c_n]) \\ &= \int_{-\infty}^{c_1} \int_{-\infty}^{c_2} \cdots \int_{-\infty}^{c_n} P_{t_{n-1},x_{n-1}}(t_n, dx_n) \\ &\qquad \times P_{t_{n-2},x_{n-2}}(t_{n-1}, dx_{n-1}) \cdots\cdots P_{t_1,x_1}(t_2, dx_2)\nu(dx_1), \end{aligned}$$

where ν is a probability measure on $I\!R$. Show that

(a) the collection $\{\mu_{t_1,t_2,\ldots,t_n};\, 0 < t_1 < t_2 < \cdots < t_n,\, n \ge 1\}$ satisfies the Kolmogorov consistency condition.

(b) The stochastic process X_t with marginal distributions given by the collection $\{\mu_{t_1,t_2,\ldots,t_n};\, 0 < t_1 < t_2 < \cdots < t_n,\, n \ge 1\}$ is a Markov process.

8. Let $P_t(x,\cdot)$ be the Gaussian measure with mean $e^{-t}x$ and variance $\frac{1}{2}(1 - e^{-2t})$ and define

$$(T_t f)(x) = \int_{-\infty}^{\infty} f(y)\, P_t(x, dy).$$

Show that

$$\lim_{t \to 0} \frac{(T_t f)(x) - f(e^{-t}x)}{t} = \frac{1}{2} f''(x).$$

9. Let A be the matrix $A = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$. Evaluate the matrix e^{tA}.

10. Consider the stochastic differential equation $dX_t = X_t\, dB(t) + \alpha X_t\, dt,\ X_0 = 1$, with α being a real number. The solution is given by the exponential Brownian motion $X_t = e^{B(t) - \frac{1}{2}t + \alpha t}$. Compute the mean $E(X_t)$ and the variance $\mathrm{var}(X_t)$. Moreover, discuss the limits of $E(X_t)$ and $\mathrm{var}(X_t)$ as $t \to \infty$ to see how the solution X_t depends on the parameter α.

Homework (3)

11. Suppose X is a nonnegative random variable. Prove that $P\{X > 0\} > 0$ if and only if $E(X) > 0$.

12. Solve the linear stochastic differential equation

$$dX_t = \big\{\alpha(t)X_t + \beta(t)\big\}\, dB(t) + \big\{\rho(t)X_t + \mu(t)\big\}\, dt.$$

13. Show that the portfolio $\theta(t) = \big(\int_0^t B(s)^2\, ds - tB(t)^2,\ B(t)^2\big)$ is an arbitrage for the market $X(t) = (1, t)$.

14. Let $X(t) = \big(1, B(t)\big)$ be a market. Find $\theta_0(t)$ so that the portfolio $\theta(t) = \big(\theta_0(t), B(t)\big)$ is self-financing. Is the resulting $\theta(t)$ admissible for the market $X(t)$? Is it an arbitrage for $X(t)$?

15. Evaluate the expectation $E \exp\big[\frac{1}{2}\int_0^1 B(t)^2\, dt\big] = (\cos 1)^{-1/2}$.
Hint: Modify the proof of Kac's formula on pages 48-50 of my 1975 book and use the following identity

$$\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{(2n-1)^2}\right) = \cos\frac{\pi x}{2}. \tag{1}$$

On page 50 the following identity is used:

$$\prod_{n=1}^{\infty}\left(1 + \frac{x^2}{(2n-1)^2}\right) = \cosh\frac{\pi x}{2}. \tag{2}$$

It looks like Equation (2) can be obtained by replacing x with ix in Equation (1) and vice versa. Thus the expectation in this homework problem can be informally obtained by replacing $\alpha = -1/2$ in Kac's formula.

Homework (4)

16. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for H and let T be an operator on H given by $Te_n = a_n e_n$, $n \geq 1$, with $\sum_{n=1}^{\infty} a_n^2 < \infty$. (Such an operator with eigenvalues being square summable is called a Hilbert–Schmidt operator on H.) Show that the semi-norm $\|x\| = |Tx|$, $x \in H$, is measurable.

17. Let $i : C' \hookrightarrow C$ be the classical Wiener space and let $\widetilde{C} \hookrightarrow C' \hookrightarrow C$ be the associated triple. Show that the space $\widetilde{C}$ consists of those functions f in C satisfying the conditions that $f'(1) = 0$, f' is absolutely continuous, and f'' is bounded.

18. Check whether $\ell^2 \hookrightarrow \ell^p$, $2 < p < \infty$, is an abstract Wiener space.

19. Check whether $L^2[0,1] \hookrightarrow L^p[0,1]$, $1 \leq p < 2$, is an abstract Wiener space.

20. For $x = (x_1, x_2, \ldots, x_n, \ldots) \in \ell^2$, define

$$\|x\| = \Big(\sum_{n=1}^{\infty} \frac{1}{n} x_n^2 \Big)^{1/2}.$$

    Check whether $\| \cdot \|$ is a measurable norm.