Math 7366-1 Homework

1. Let $f(t)$ be a continuously differentiable function on $[a, b]$. Prove that the following equality holds almost surely for $\omega \in \Omega$:

$$\left(\int_a^b f(t)\, dB(t)\right)(\omega) = f(t)B(t,\omega)\Big]_a^b - \int_a^b B(t,\omega) f'(t)\, dt,$$

where the integral in the left-hand side is a Wiener integral, while the one in the right-hand side is a Riemann integral.

2. Let X be a normal random variable with mean 0 and variance σ^2. Evaluate the moments $E(X^n)$ of X for $n \geq 1$. You can use any method.

3. Let $B(t)$ be a Brownian motion and $\mathcal{F}_s = \sigma\{B(u); u \leq s\}$.
   (a) Compute $E[B(t)^3 \,|\, \mathcal{F}_s]$ for $0 \leq s \leq t$.
   (b) Deduce from (a) a martingale out of $B(t)^3$.

4. Let $B(t)$ be a Brownian motion. Use the definition of the Itô integral to compute $\int_a^b B(t)^2\, dB(t)$.

5. Let $B(t)$ be a Brownian motion. Find the variances of the random variables:
   (a) $\int_a^b B(t)^3\, dB(t)$, (b) $\int_a^b e^{B(t)}\, dB(t)$, (c) $\int_a^b \operatorname{sgn}(B(t))\, dB(t)$

,,

6. Let $B(t)$ be a Brownian motion. Find the probability that $\left|\int_0^\pi \sin t\, dB(t)\right| > 1$.

7. Let $B(t)$ be a Brownian motion. Find the probability that $\left|\int_0^1 t^2 B(t)\, dt\right| > 2$.

8. Let $X = \int_a^b |B(t)|\, dB(t)$. Find the mean and variance of the random variable X.

9. Find the mean and variance of the random variable $X = \int_a^b \sqrt{t}\, e^{B(t)}\, dB(t)$.

10. Let $B(t)$ be a Brownian motion. Find all deterministic functions $\rho(t)$ such that the stochastic process $X(t) = e^{B(t)+\rho(t)}$ is a martingale. (Do not use Itô's formula)

11. Let $B(t)$ be a Brownian motion and $\Delta_n = \{t_0, t_1, \ldots, t_{n-1}, t_n\}$ a partition of $[a, b]$. Prove that

$$\sum_{i=1}^n (t_i - t_{i-1})\big(B(t_i) - B(t_{i-1})\big) \longrightarrow 0$$

in $L^2(\Omega)$ as $\|\Delta_n\| = \max_{1\leq i\leq n}(t_i - t_{i-1})$ tends to 0.

For the problems 12–22 below, use the *Hermite polynomials* $H_n(x;\rho)$ defined as follows. For each x, expand the function $e^{tx-\frac{1}{2}\rho t^2}$ as a power series in t, namely

$$e^{tx-\frac{1}{2}\rho t^2} = \sum_{n=0}^{\infty} \frac{H_n(x;\rho)}{n!}\, t^n,$$

where $H_n(x;\rho)/n!$ is the coefficient of t^n.

12. Multiply the power series of e^{tx} and $e^{-\frac{1}{2}\rho t^2}$ in t together and compare the coefficients of t^n to find $H_n(x;\rho)$ for $n = 0, 1, 2, 3, 4$.

13. Find the general formula for $H_n(x;\rho)$.

14. Prove the equality
$$H_n(x;\rho) = (-\rho)^n e^{x^2/2\rho}\, D_x^n e^{-x^2/2\rho}.$$

15. Let μ_ρ be the Gaussian measure with mean 0 and variance ρ. Prove the orthogonality of the Hermite polynomials with respect to μ_ρ:
$$\int_{-\infty}^{\infty} H_n(x;\rho) H_m(x;\rho)\, d\mu_\rho(x) = \delta_{nm} n! \rho^n,$$
where δ_{nm} is the Kronecker delta.

16. Prove the equality
$$H_{n+1}(x;\rho) = xH_n(x;\rho) - \rho n H_{n-1}(x;\rho).$$

17. Prove the equality
$$D_x H_n(x;\rho) = nH_{n-1}(x;\rho).$$

18. Prove the equality
$$(-\rho D_x^2 + xD_x)H_n(x;\rho) = nH_n(x;\rho).$$

19. Prove the equality
$$\frac{\partial}{\partial \rho} H_n(x;\rho) = -\frac{1}{2}\frac{\partial^2}{\partial x^2} H_n(x;\rho).$$

20. Prove the equality
$$\int_a^b H_n(B(t);t)\, dB(t) = \frac{1}{n+1}\Big(H_{n+1}(B(b);b) - H_{n+1}(B(a);a)\Big).$$

21. Show that $M_n(t) = H_n(B(t);t)$, $t \geq 0$, is a martingale.

22. Let $M_n(t) = H_n(B(t);t)$, $t \geq 0$. Show that $\langle M_n\rangle_t = n^2 \int_0^t H_{n-1}(B(s);s)^2\, ds$.