Math 7366-1 Homework Spring 2019

1. Let $H_n(x;\sigma^2)$ be the Hermite polynomial of degree n with parameter σ^2. Prove the following equality
$$e^{tx-\frac{1}{2}\sigma^2t^2} = \sum_{n=0}^{\infty} \frac{H_n(x;\sigma^2)}{n!}\, t^n,$$
where the left-hand side is called a *generating function* of the Hermite polynomials.

2. Use the above generating function of the Hermite polynomials to show that
$$H_n(x;\sigma^2) = \sum_{k=0}^{[\![n/2]\!]} \binom{n}{2k}(2k-1)!!(-\sigma^2)^k x^{n-2k}.$$

3. Let $B(t)$ be a Brownian motion. Use the partitions and then take the limit to obtain the stochastic integral
$$\int_0^t B(s)^2\, dB(s) = \frac{1}{3}B(t)^3 - \int_0^t B(s)\, ds.$$

4. Let $B(t)$ be a Brownian motion. Use the definition of conditional expectation from the elementary probability theory to evaluate
$$E[B(t)|B(1)], \quad 0 \le t \le 1.$$
Moreover, evaluate $E[B(t)|B(1)]$ for $t > 1$.

5. Express $\int_0^1 tB(t)\, dt$ in terms of Wiener integrals.

6. Compute $E\big[\big(\int_0^1 B(t)^2\, dt\big)^2\big]$.

7. Find the variance of the random variable $\int_0^1 t^2 B(t)\, dt$.

8. Express the Brownian functional $\int_0^1 B(t)^2\,dt$ in terms of Wiener integral.

9. (Integration by parts formula for Wiener integral) Let $\theta \in L^2([a,b])$ and define a stochastic process $X(t) = \int_a^t \theta(s)\,dB(s)$, $a \le t \le b$. Show that for any C^1-function f on $[a,b]$, the following equality holds:

$$\int_a^b f(t)\,dX(t) = f(t)X(t)\Big]_a^b - \int_a^b f'(t)X(t)\,dt.$$

10. Let $f \in L^2([0,1])$ be a nonzero function and let $\tilde{f}$ be the Wiener integral of f, namely, $\tilde{f} = \int_0^1 f(t)\,dB(t)$. Show that the vectors $1,\ \tilde{f},\ (\tilde{f})^2, \dots (\tilde{f})^n$ are linearly independent for any $n \ge 1$.

11. Let $f \in L^2([a,b])$. Show that the Brownian functional $(\tilde{f})^2 - \|f\|_2^2$ is a homogeneous chaos of degree 2.

12. Express the Brownian functional $B(t) + 2B(t)^2 - 5B(t)^3$ as a sum of homogeneous chaoses.

13. Express the Brownian functional $\sin B(t)$ as a sum of homogeneous chaoses.

14. Let $\{\xi_n\}_{n=1}^{\infty}$ be a sequence of independent random variables with the same standard normal distribution. Show that

$$B(t) = \sum_{n=1}^{\infty} \frac{\sqrt{2}}{(n-\frac{1}{2})\pi}\Big(1 - \cos\big[(n-\tfrac{1}{2})\pi t\big]\Big)\xi_n, \quad t \ge 0$$

is a Brownian motion.

15. Let $X = \int_a^b f(t)\,dB(t)$ be a Wiener integral and $Y = \int_a^b \int_a^b g(s,t)\,dB(s)dB(t)$ a double Wiener-Itô integral. Show that $E(XY) = 0$.

16. Suppose $f(t)$ is an adapted continuous stochastic process. Prove the equality

$$\int_0^t B(1)f(s)\,dB(s) = B(1)\int_0^t f(s)\,dB(s) - \int_0^t f(s)\,ds, \quad 0 \le t \le 1.$$

17. Suppose $f(t)$ is a deterministic function in $L^2([0,1])$. Show that

$$X_t = B(t)\int_0^t f(s)\,dB(s) - \int_0^t f(s)\,ds, \quad 0 \le t \le 1,$$

is a martingale with respect to the filtration given by the Brownian motion $B(t)$.

18. Evaluate the stochastic integral

$$\int_0^t \sin(B(1))\,dB(s), \quad t \ge 0.$$

19. Let $f(t)$ be an adapted continuous stochastic process and $\theta(x)$ a C^1-function. Evaluate the stochastic integral

$$\int_0^t \theta(B(1))f(s)\,dB(s), \quad 0 \le t \le 1.$$

20. Evaluate the stochastic integral

$$\int_0^t (B(1) + B(2))\,dB(s), \quad t \ge 0.$$

21. Suppose X_t is a martingale with respect to a filtration $\{\mathcal{F}_t; a \le t \le b\}$ and $\varphi(t)$ a stochastic process being instantly independent of $\{\mathcal{F}_t; a \le t \le b\}$ with $E\varphi(t) = c$, a constant. Prove that the stochastic process $Y_t = X_t\varphi(t)$ is a near-martingale with respect to $\{\mathcal{F}_t; a \le t \le b\}$.