1. Let $H_n(x; \sigma^2)$ be the Hermite polynomial of degree $n$ with parameter $\sigma^2$. Prove the following equality

$$e^{tx - \frac{1}{2} \sigma^2 t^2} = \sum_{n=0}^{\infty} \frac{H_n(x; \sigma^2)}{n!} t^n,$$

where the left-hand side is called a generating function of the Hermite polynomials.

2. Use the above generating function of the Hermite polynomials to show that

$$H_n(x; \sigma^2) = \sum_{k=0}^{[n/2]} \binom{n}{2k} (2k-1)!!(-\sigma^2)^k x^{n-2k}.$$

3. Let $B(t)$ be a Brownian motion. Use the partitions and then take the limit to obtain the stochastic integral

$$\int_0^t B(s)^2 dB(s) = \frac{1}{3} B(t)^3 - \int_0^t B(s) ds.$$

4. Let $B(t)$ be a Brownian motion. Use the definition of conditional expectation from the elementary probability theory to evaluate

$$E[B(t)|B(1)], \quad 0 \leq t \leq 1.$$

Moreover, evaluate $E[B(t)|B(1)]$ for $t > 1$.

5. Express $\int_0^1 tB(t) \, dt$ in terms of Wiener integrals.

6. Compute $E[(\int_0^1 B(t)^2 \, dt)^2]$.

7. Find the variance of the random variable $\int_0^1 t^2 B(t) \, dt$. 
8. Express the Brownian functional $\int_0^1 B(t)^2 \, dt$ in terms of Wiener integral.

9. (Integration by parts formula for Wiener integral) Let $\theta \in L^2([a,b])$ and define a stochastic process $X(t) = \int_a^t \theta(s) \, dB(s)$, $a \leq t \leq b$. Show that for any $C^1$-function $f$ on $[a,b]$, the following equality holds:

$$\int_a^b f(t) \, dX(t) = f(t)X(t)|_a^b - \int_a^b f'(t)X(t) \, dt.$$

10. Let $f \in L^2([0,1])$ be a nonzero function and let $\tilde{f}$ be the Wiener integral of $f$, namely, $\tilde{f} = \int_0^1 f(t) \, dB(t)$. Show that the vectors $1, \tilde{f}, (\tilde{f})^2, \ldots (\tilde{f})^n$ are linearly independent for any $n \geq 1$.

11. Let $f \in L^2([a,b])$. Show that the Brownian functional $(\tilde{f})^2 - \|f\|_2^2$ is a homogeneous chaos of degree 2.

12. Express the Brownian functional $B(t) + 2B(t)^2 - 5B(t)^3$ as a sum of homogeneous chaoses.

13. Express the Brownian functional $\sin B(t)$ as a sum of homogeneous chaoses.

14. Let $\{\xi_n\}_{n=1}^\infty$ be a sequence of independent random variables with the same standard normal distribution. Show that

$$B(t) = \sum_{n=1}^{\infty} \frac{\sqrt{2}}{(n - \frac{1}{2})\pi} \left(1 - \cos \left[(n - \frac{1}{2})\pi t\right]\right) \xi_n, \quad t \geq 0$$

is a Brownian motion.
15. Let \( X = \int_a^b f(t) dB(t) \) be a Wiener integral and \( Y = \int_a^b \int_a^b g(s, t) dB(s) dB(t) \) a double Wiener-Itô integral. Show that \( E(XY) = 0. \)

16. Suppose \( f(t) \) is an adapted continuous stochastic process. Prove the equality
\[
\int_0^t B(1) f(s) dB(s) = B(1) \int_0^t f(s) dB(s) - \int_0^t f(s) ds, \quad 0 \leq t \leq 1.
\]

17. Suppose \( f(t) \) is a deterministic function in \( L^2([0, 1]) \). Show that
\[
X_t = B(t) \int_0^t f(s) dB(s) - \int_0^t f(s) ds, \quad 0 \leq t \leq 1,
\]
is a martingale with respect to the filtration given by the Brownian motion \( B(t) \).

18. Evaluate the stochastic integral
\[
\int_0^t \sin(B(1)) dB(s), \quad t \geq 0.
\]

19. Let \( f(t) \) be an adapted continuous stochastic process and \( \theta(x) \) a \( C^1 \)-function. Evaluate the stochastic integral
\[
\int_0^t \theta(B(1)) f(s) dB(s), \quad 0 \leq t \leq 1.
\]

20. Evaluate the stochastic integral
\[
\int_0^t (B(1) + B(2)) dB(s), \quad t \geq 0.
\]

21. Suppose \( X_t \) is a martingale with respect to a filtration \( \{\mathcal{F}_t; a \leq t \leq b\} \) and \( \varphi(t) \) a stochastic process being instantly independent of \( \{\mathcal{F}_t; a \leq t \leq b\} \) with \( E\varphi(t) = c \), a constant. Prove that the stochastic process \( Y_t = X_t \varphi(t) \) is a near-martingale with respect to \( \{\mathcal{F}_t; a \leq t \leq b\} \).