Math 7366-1 Homework Spring 2020

1. Prove that the following statements are equivalent.
   (a) $X(t)$ is a stochastic process satisfying conditions (2) and (3) in the definition of a Brownian motion.
   (b) $X(t)$ is a stochastic process with marginal distributions given by

$$\mu_{t_1,t_2,\ldots,t_n}(A) = \frac{1}{\sqrt{(2\pi)^n t_1(t_2-t_1)\cdots(t_n-t_{n-1})}}$$
$$\times \int_A \exp\Big[-\frac{1}{2}\Big(\frac{x_1^2}{t_1}+\frac{(x_2-x_1)^2}{t_2-t_1}+\cdots+\frac{(x_n-x_{n-1})^2}{t_n-t_{n-1}}\Big)\Big]\,dx_1dx_2\cdots dx_n$$

   for any $0 < t_1 < t_2 < \cdots < t_n$ and $n \geq 1$.
   (c) $X(t)$ is a stochastic process such that for any $0 < t_1 < t_2 < \cdots < t_n$ and any real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$,

$$E\exp\Big[i\Big(\lambda_1 X(t_1)+\lambda_2\big(X(t_2)-X(t_1)\big)+\cdots+\lambda_n\big(X(t_n)-X(t_{n-1})\big)\Big)\Big]$$
$$= \exp\Big[-\frac{1}{2}\Big(\lambda_1^2 t_1+\lambda_2^2(t_2-t_1)+\cdots+\lambda_n^2(t_n-t_{n-1})\Big)\Big]$$

2. Suppose$f$ is a continuously differentiable function on a finite closed interval $[a, b]$. Prove that $V_f < \infty$ and $Q_f = 0$ on $[a, b]$.

3. Suppose X is a Gaussian random variable with mean 0 and variance σ^2. Evaluate $E|X^n|$.

4. Let $B(t)$ be a Brownian motion. Show that

$$\sum_{i=1}^{n} \big|B(t_i) - B(t_{i-1})\big|^3 \longrightarrow 0$$

   in $L^2(\Omega)$ as $\|\Delta_n\| \to 0$.

5. Let X and Y be independent random variables. Show that for any Borel measurable functions f and g, the random variables $f(X)$ and $g(Y)$ are also independent.

6. Suppose X and Y be independent random variables. Prove that the σ-fields $\sigma(X)$ and $\sigma(Y)$ are independent.

7. Let $B(t)$ be a Brownian motion. Use the definition of Itô integral given in class to obtain the stochastic integral

$$\int_0^t B(s)^2\,dB(s) = \frac{1}{3}B(t)^3 - \int_0^t B(s)\,ds.$$

8. Express $\int_0^1 (t^2+t)B(t)\,dt$ in terms of Wiener integrals.

9. Find the mean and variance of $\int_0^1 B(t)^2\,dt$.

10. Suppose $f(t)$ and $g(t)$ are stochastic processes in $L^2_{\mathrm{ad}}([a,b]\times\Omega)$ and assume that

$$\int_a^b f(t)\,dB(t) = \int_a^b g(t)\,dB(t), \quad \text{almost surely}$$

What is the relationship between $f(t)$ and $g(t)$?

11. Prove that

$$\sum_{i=1}^{n} \big|B(t_i) - B(t_{i-1})\big|(t_i - t_{i-1})$$

converges to 0 in $L^2(\Omega)$ as $\|\Delta\| \to 0$.

12. Let μ be the probability measure on the real line with $\mu(\{1\}) = \mu(\{2\}) = 1/2$. Apply the Gram–Schmidt orthogonalization procedure to the sequence $\{1, x, x^2, x^3, \ldots\}$ of monomials (as far as you can since the whole sequence is not linearly independent) to get an orthogonal basis for $L^2(\mu)$.

13. Do the same thing as that stated in Problem 12 for the probability measure ν given by $\nu(\{1\}) = \nu(\{2\}) = \nu(\{3\}) = 1/3$.

14. Let $f(t)$ be a deterministic C^1-function on $[a,b]$ and $X(t)$ an Itô process. Prove the following integration by parts formula:

$$\int_a^b f(t)\,dX(t) = f(t)X(t)\Big]_a^b - \int_a^b f'(t)X(t)\,dt.$$

15. Evaluate the stochastic integral

$$\int_0^t (B(1) + B(2))\, dB(s), \quad t \geq 0.$$

16. Evaluate the stochastic integral

$$\int_0^t \sin(B(1))\, dB(s), \quad t \geq 0.$$

17. Suppose $f(t)$ is an adapted continuous stochastic process. Prove the equality

$$\int_0^t B(1) f(s)\, dB(s) = B(1) \int_0^t f(s)\, dB(s) - \int_0^t f(s)\, ds, \quad 0 \leq t \leq 1.$$

18. Suppose $f(t)$ is a deterministic function in $L^2([0,1])$. Show that

$$X_t = B(t) \int_0^t f(s)\, dB(s) - \int_0^t f(s)\, ds, \quad 0 \leq t \leq 1,$$

is a martingale with respect to the filtration given by the Brownian motion $B(t)$.

19. Let $f(t)$ be an adapted continuous stochastic process and $\theta(x)$ a C^1-function. Evaluate the stochastic integral

$$\int_0^t \theta(B(1)) f(s)\, dB(s), \quad 0 \leq t \leq 1.$$

20. Let $f \in L^2([a,b])$. Show that the Brownian functional $(\tilde{f})^2 - \|f\|_2^2$ is a homogeneous chaos of degree 2.

21. Express the Brownian functional $B(t) + 2B(t)^2 - 5B(t)^3$ as a sum of homogeneous chaoses.