Math 7366-1 Homework Spring 2021

1. The *gamma function* is defined by $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx,\ \ \mathrm{Re}(z) > 0$. Prove the following equalities:
   (1) $\Gamma(z+1) = z\,\Gamma(z)$, $\mathrm{Re}(z) > 0$.
   (2) $\Gamma(1) = 1$.
   (3) $\Gamma(n) = (n-1)!$ for any integer $n \geq 1$.
   (4) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

2. Find the dimension n so that the sphere measure $\sigma(S^{n-1})$ has the maximum value.

3. Find the dimension n so that the volume $\mathrm{V}(B^n)$ of the unit ball has the maximum value.

- In problems #4 to #7 below, we use the notation for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$:

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}, \quad 0 < p < \infty,$$
$$\|x\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}.$$

- Note that $\|\cdot\|_p$ is a norm when $1 \leq p \leq \infty$. For $0 < p < 1$, see Problem #7.

4. Let $1 \leq p < 2$. Prove that there exist positive constants a_n and b_n depending on the dimension n such that

$$a_n \|x\|_p \leq \|x\|_2 \leq b_n \|x\|_p, \quad \forall x \in \mathbb{R}^n.$$

5. Let $2 < p < \infty$. Prove that there exist positive constants c_n and d_n depending on the dimension n such that

$$c_n \|x\|_p \leq \|x\|_2 \leq d_n \|x\|_p, \quad \forall x \in \mathbb{R}^n.$$

6. Prove that there exist positive constants α_n and β_n depending on the dimension n such that

$$\alpha_n \|x\|_\infty \leq \|x\|_2 \leq \beta_n \|x\|_\infty, \quad \forall x \in \mathbb{R}^n.$$

7. Let $0 < p < 1$. Show that $\|\cdot\|_p$ is not a norm.

8. Let $0 < p < 1$. For $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, define

$$d(x, y) = |y_1 - x_1|^p + |y_2 - x_2|^p + \cdots + |y_n - x_n|^p.$$

Prove that $d(x, y)$ is a metric on $\mathbb{R}^n$.

9. Let H be an infinite dimensional Hilbert space. Does there exist a measure μ on H satisfying the following properties?
   (1) $\mu\big(b(x,r)\big) > 0$ for any $x \in H$ and $r > 0$.
   (2) $\mu(A) < \infty$ for any bounded Borel subset A of H.
   (3) μ is rotation invariant.

10. Let K be the subspace of ℓ^2 given by

$$K = \Big\{(0, x_2, x_3, \ldots, x_n, \ldots)\,;\, x_j \neq 0 \text{ for only finitely many} j'\text{s}\Big\}$$

and let ξ be the vector $(1, 1/2, \ldots, 1/n, \ldots)$ in ℓ.
   (1) Find the distance $\operatorname{dist}(\xi, K)$.
   (2) Does there exist a unique vector ξ_0 in K such that $|\xi - \xi_0| = \operatorname{dist}(\xi, K)$?

11. Let $H = \ell^2$ with norm $|\cdot|$. Define a norm $\|\cdot\|$ on H by

$$\|(a_1, a_2, \ldots, a_n, \ldots)\| = \left(\sum_{n=1}^{\infty} \frac{1}{n^2} a_n^2\right)^{1/2}.$$

Show that $\|\cdot\|$ is a measurable norm on H.

12. Define a norm $\|\cdot\|$ on ℓ^2 by

$$\|(a_1, a_2, \ldots, a_n, \ldots)\| = \left(\sum_{n=1}^{\infty} \frac{1}{n} a_n^2\right)^{1/2}.$$

Check whether $\|\cdot\|$ is measurable.

13. Let H be a separable Hilbert space with norm $|\cdot|$. Suppose T is a Hilbert–Schmidt operator on H. Prove that $\|x\| = |Tx|$, $x \in H$, is a measurable semi-norm.

14. Let H be the Hilbert space ℓ^2 with norm $\|\cdot\|_2$. Check whether the norm $\|\cdot\|_p$ on H for $2 < p \leq \infty$ is measurable on H.

15. Let H be the Hilbert space $L^2[0,1]$ with norm $\|\cdot\|_2$. Check whether the norm $\|\cdot\|_p$ on H for $1 \leq p < 2$ is measurable.

16. Suppose $\|\cdot\|$ is a measurable semi-norm on a separable Hilbert space and A is a bounded linear operator on H with respect to $\|\cdot\|$. Check whether the semi-norm

$$\|x\|_{A} = \|Ax\|, \quad x \in H,$$

is measurable.