Math 7380 Homework Fall 2005

1. Show that the marginal distribution of a Brownian motion $B(t)$ for $0 < t_1 < t_2 < \cdots t_n$ is given by

$$
\begin{aligned}
&P\{B(t_1) \le a_1, B(t_2) \le a_2, \ldots, B(t_n) \le a_n\} \\
&= \frac{1}{\sqrt{(2\pi)^n t_1(t_2 - t_1) \cdots (t_n - t_{n-1})}} \int_{-\infty}^{a_n} \cdots \int_{-\infty}^{a_1} \\
&\quad \exp\Big[-\frac{1}{2}\Big(\frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1} + \cdots \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}\Big)\Big]\, dx_1 dx_2 \cdots dx_n.
\end{aligned}
$$

2. Let $B(t)$ be a Brownian motion. Show that $\lim_{t\to 0^+} tB(1/t) = 0$ almost surely. Define $W(0) = 0$ and $W(t) = tB(1/t)$ for $t > 0$. Prove that $W(t)$ is a Brownian motion.
3. Let $B(t)$ be a Brownian motion. Find all constants a and b so that $X(t) = \int_0^t \big(a + b\frac{u}{t}\big)\, dB(u)$ is also a Brownian motion.
4. Let $B(t)$ be a Brownian motion. Show that both $X(t) = \int_0^t (2t - u)\, dB(u)$ and $Y(t) = \int_0^t (3t - 4u)\, dB(u)$ are Gaussian processes with mean function 0 and the same covariance function $3s^2t - \frac{2}{3}s^3$ for $s \le t$.
5. Let $B(t) = (B_1(t), \ldots, B_n(t))$ be an $\mathbb{R}^n$-valued Brownian motion. Find the density functions of $R(t) = |B(t)|$ and $S(t) = |B(t)|^2$.
6. Let $f(x, y)$ be the joint density function of random variables X and Y. The *marginal density function* of Y is given by $f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$. The *conditional density function* of X given $Y = y$ is defined by $f_{X|Y}(x|y) = f(x, y)/f_Y(y)$. The *conditional expectation* of X given $Y = y$ is defined by $E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx$. Let $\sigma(Y)$ be the σ-field generated by Y. Prove that

$$E[X|\sigma(Y)] = \theta(Y),$$

where θ is the function $\theta(y) = E[X|Y = y]$.
7. Let $B(t)$ be a Brownian motion. Find the distribution of the Wiener integral $X_t = \int_0^t e^{t-s}\, dB(s)$. Check whether X_t is a martingale.
8. Let $B(t)$ be a Brownian motion. Check that $X_t = \frac{1}{3}B(t)^3 - \int_0^t B(s)\, ds$ is a martingale.
9. For a partition $\Delta = \{a = t_0 < t_1 < \cdots < t_n = b\}$, define

$$M_\Delta = \sum_{j=0}^{n-1} B\Big(\frac{t_j + t_{j+1}}{2}\Big)\big(B(t_{j+1}) - B(t_j)\big).$$

Find $\lim_{\|\Delta\|\to 0} M_\Delta$ in $L^2(\Omega)$.

10. Find the variance of the random variable $X = \int_a^b \sqrt{t}\,\sin(B(t))\,dB(t)$.

11. Let $X_t = B(1)B(t), 0 \le t \le 1$.
   (a) Check whether the random variable $B(1)$ is measurable with respect to the σ-field $\mathcal{F}_t = \sigma\{B(s); s \le t\}$ for any $0 \le t < 1$.
   (b) Show that for each $0 \le t < 1$, the random variable X_t is not measurable with respect to $\mathcal{F}_t$ given in (a).
   (c) For $0 \le s \le t \le 1$, find $E[X_t \,|\, \mathcal{F}_s]$.

12. Show that $X_t = e^{B(t)} - 1 - \frac{1}{2}\int_0^t e^{B(s)}\,ds$ is a martingale.

13. Show that $X_t = e^{B(t) - \frac{1}{2}t}$ is a martingale.

14. Let $X_\varepsilon = \int_0^1 \varepsilon^{-\lambda} e^{-B(t)^2/2\varepsilon}\,dB(t)$. Show that $X_\varepsilon \to 0$ in $L^2(\Omega)$ as $\varepsilon \downarrow 0$ if and only if $0 < \lambda < 1/4$.

15. Let $Y_\varepsilon = \int_0^\varepsilon \varepsilon^{-\lambda} e^{-B(t)^2/2\varepsilon}\,dB(t)$. Show that $Y_\varepsilon \to 0$ in $L^2(\Omega)$ as $\varepsilon \downarrow 0$ if and only if $0 < \lambda < 1/2$.

16. Let $Z_\varepsilon = \int_0^{\varepsilon/2} \varepsilon^{-\lambda} e^{-B(t)^2/2\varepsilon}\,dB(t)$. Show that $Z_\varepsilon \to 0$ in $L^2(\Omega)$ as $\varepsilon \downarrow 0$ if and only if $0 < \lambda < 1/2$.

17. Find the quadratic variation of a Poisson process $N(t)$ with parameter $\lambda > 0$.

18. Suppose $\lambda \in \mathbb{R}$. Prove that $M(t) = e^{\lambda B(t) - \lambda^2 t/2}$ is a martingale and the compensator of $M(t)^2$ is given by

$$\langle M \rangle_t = \lambda^2 \int_0^t e^{2\lambda B(u) - \lambda^2 u}\,du.$$

19. Let $f \in L^2[a,b]$ and $M(t) = \int_a^t f(s)\,dB(s)$. Find the quadratic variation process $[M]_t$ of $M(t)$ and the compensator $\langle M \rangle_t$ of $M(t)^2$.

20. Let $s \le t$. Show that

$$E\big\{B(t)^3 \,\big|\, \mathcal{F}_s\big\} = 3(t-s)B(s) + B(s)^3. \tag{1}$$

21. Use Equation (1) to derive a martingale $X_t = B(t)^3 - 3tB(t)$. Find the quadratic variation process $[X]_t$ and the compensator $\langle X \rangle_t$.

22. Let $B(t)$ be a Brownian motion. Find all deterministic functions $\rho(t)$ such that $e^{B(t)+\rho(t)}$ is a martingale.

23. Let $B_1(t)$ and $B_2(t)$ be two independent Brownian motions and let $\Delta_n = \{t_0, t_1, \ldots, t_{n-1}, t_n\}$ be a partition of $[a,b]$. Show that

$$\sum_{i=1}^n \big(B_1(t_i) - B_1(t_{i-1})\big)\big(B_2(t_i) - B_2(t_{i-1})\big) \longrightarrow 0$$

in $L^2(\Omega)$ as $\|\Delta_n\| = \max_{1 \le i \le n}(t_i - t_{i-1})$ tends to 0.

24. Suppose $X_1, \ldots, X_n$ are random variables such that for all $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$,

$$Ee^{i(\lambda_1 X_1 + \cdots + \lambda_n X_n)} = e^{i(\lambda_1 \mu_1 + \cdots + \lambda_n \mu_n) - \frac{1}{2}(\lambda_1^2 \sigma_1^2 + \cdots + \lambda_n^2 \sigma_n^2)},$$

where $\mu_j \in \mathbb{R}$ and $\sigma_j > 0$ for $j = 1, \ldots, n$. Prove that the random variables $X_1, \ldots, X_n$ are independent and normally distributed.

25. Check whether $X(t) = \int_0^t \operatorname{sgn}(B(s) - s)\, dB(s)$ is a Brownian motion.

26. Let $B(t)$ be a Brownian motion with respect to a probability measure P. Find the density function of $B\left(\frac{2}{3}\right)$ with respect to the probability measure $dQ = e^{B(1) - \frac{1}{2}}\, dP$.

27. Let $B(t)$ be a Brownian motion with respect to a probability measure P. Find a probability measure with respect to which the stochastic process $W(t) = B(t) + t - t^3$, $0 \le t \le 2$, is a Brownian motion.