Due September 25, 2008: 1, 2, 3, 4, 5, 6, 7

1. Let $C(x)$ be the Cantor function on $[0,1]$. Evaluate the Riemann-Stieltjes integrals $\int_0^1 x\,dC(x)$ and $\int_0^1 x^2\,dC(x)$.

2. Let $X_n, n \geq 1$, be a Gaussian random variable with mean μ_n and variance σ_n^2. Assume that the sequence $\{X_n\}$ converges to a random variable X in $L^2(\Omega)$. Prove that $\mu = \lim_{n\to\infty} \mu_n$ and $\sigma^2 = \lim_{n\to\infty} \sigma_n^2$ exist. Moreover, show that X is a Gaussian random variable with mean μ and variance σ^2.

3. Let $B(t)$ be a Brownian motion. Find $E|B(s) - B(t)|^n$ for $n = 1, 2, \ldots$.

4. Let $B(t)$ be a Brownian motion and $0 < s < t < u$. Find the distribution of the random variable $X = B(s) + B(t) + B(u)$.

5. Check whether $X_t = B(2t) - B(t)$ is a Brownian motion.

6. Let $0 < s < t$. Show that the joint distribution of $B(s)$ and $B(t)$ is given by

$$P(B(s) \leq x, B(t) \leq y) = \frac{1}{2\pi\sqrt{s(t-s)}} \int_{u=-\infty}^{u=x} \int_{v=-\infty}^{v=y} e^{-\frac{1}{2}\left(\frac{u^2}{s} + \frac{(v-u)^2}{t-s}\right)}\,dudv.$$

7. Let $0 < s < t$. Use the distribution in Problem #6 to find the conditional expectations $E[B(t)|B(s) = x]$ and $E[B(s)|B(t) = y]$.

Due November 13, 2008: 8, 9, 10, 11, 12, 13, 14, 15

8. Let $B(t)$ be a Brownian motion. Find the distribution of $\int_0^t B(s)\,ds$. Check whether $Y_t = \int_0^t B(s)\,ds$ is a martingale.

9. Let $B(t)$ be a Brownian motion. Show that $X_t = \frac{1}{3}B(t)^3 - \int_0^t B(s)\,ds$ is a martingale.

10. Let $B(t)$ be a Brownian motion. Find the means and variances of the stochastic integral $\int_a^b \big(\operatorname{sgn} B(t)\big)\,dB(t)$.

11. For a partition $\Delta = \{a = t_0 < t_1 < \cdots < t_n = b\}$, define

$$M_\Delta = \sum_{j=0}^{n-1} B\Big(\frac{t_j + t_{j+1}}{2}\Big)\big(B(t_{j+1}) - B(t_j)\big).$$

Find $\lim_{\|\Delta\|\to 0} M_\Delta$ in $L^2(\Omega)$.

12. Let $X_t = B(1)B(t), 0 \leq t \leq 1$.

(a) Show that X_t is not a martingale with respect to the filtration $\mathcal{F}_t = \sigma\{B(s); s \leq t\}$.

(b) For $0 \leq s \leq t \leq 1$, find $E[X_t|\mathcal{F}_s]$.

13. Let $X_{\lambda,\varepsilon} = \varepsilon^{-\lambda} \int_0^1 e^{-B(t)^2/2\varepsilon} \, dB(t)$. Show that $X_{\lambda,\varepsilon} \to 0$ in $L^2(\Omega)$ as $\varepsilon \downarrow 0$ if and only if $\lambda < \frac{1}{4}$. Check whether $\lim_{\varepsilon \downarrow 0} X_{\frac{1}{4},\varepsilon}$ exits in $L^2(\Omega)$.

14. Let $Y_{\lambda,\varepsilon} = \varepsilon^{-\lambda} \int_0^\varepsilon e^{-B(t)^2/2\varepsilon} \, dB(t)$. Show that $Y_{\lambda,\varepsilon} \to 0$ in $L^2(\Omega)$ as $\varepsilon \downarrow 0$ if and only if $\lambda < \frac{1}{2}$. Check whether $\lim_{\varepsilon \downarrow 0} Y_{\frac{1}{2},\varepsilon}$ exits in $L^2(\Omega)$.

15. Let $Z_{\lambda,\varepsilon} = \varepsilon^{-\lambda} \int_0^{\varepsilon^2} e^{-B(t)^2/2\varepsilon} \, dB(t)$. Show that $Z_{\lambda,\varepsilon} \to 0$ in $L^2(\Omega)$ as $\varepsilon \downarrow 0$ if and only if $\lambda < 1$. Check whether $\lim_{\varepsilon \downarrow 0} Z_{1,\varepsilon}$ exits in $L^2(\Omega)$.