Math 7380-2 (Homework) Fall 2011

1. Let $B_1(t)$ and $B_2(t)$ be independent Brownian motions. Check whether the market $X_t = (1, 2 + B_1(t), -t + B_1(t) + B_2(t))$ has an arbitrage.
2. Let $B_1(t)$ and $B_2(t)$ be independent Brownian motions. Check whether the market $X_t = (1, 2 + B_1(t) + B_2(t), -t - B_1(t) - B_2(t))$ has an arbitrage.
3. Let $B_1(t)$ and $B_2(t)$ be independent Brownian motions. Check whether the market $X_t = (e^t, B_1(t), B_2(t))$ has an arbitrage.
4. Show that the inclusion map $\ell_p \hookrightarrow \ell_q$ is continuous for any $1 \le p < q \le \infty$. (Hint: Prove the inequality $\|a\|_q \le \|a\|_p$ for all $a \in \ell_p$ and $1 \le p < q \le \infty$.)
5. Show that $\ell_p \ne \ell_q$ if $p \ne q$.
6. Check whether the equalities hold: $\ell_1 = \cap_{1<p\le\infty} \ell_p$, $\ell_\infty = \cup_{1\le q<\infty} \ell_q$.
7. Let $\mu(X) < \infty$. Show that the inclusion map $L^p(X, \mu) \hookrightarrow L^q(X, \mu)$ is continuous for any $1 \le q < p \le \infty$. (Hint: Use the Hölder inequality.)
8. Let μ be the Lebesgue measure on the interval $[0, 1]$. Show that $L^p([0, 1], \mu) \ne L^q([0, 1], \mu)$ if $p \ne q$.
9. Let μ be the Lebesgue measure on the interval $[0, 1]$. Check whether the equalities hold: $L^\infty([0, 1], \mu) = \bigcap_{1\le p<\infty} L^p([0, 1], \mu)$, $L^1([0, 1], \mu) = \bigcup_{1<q\le\infty} L^q([0, 1], \mu)$.
10. Let F consist of sequences with finitely many nonzero entries. Show that F is dense in ℓ_p for any $1 \le p < \infty$.
11. Let F be as given in Problem 10 and let c_0 consist of sequences converging to 0 with the supremum norm. Show that F is dense in c_0.
12. Let F be as given in Problem 10 and let c consist of convergent sequences with the supremum norm. Show that F is not dense in c.
13. Let c_0 be as given in Problem 11. Check whether the equalities $c_0 = \cup_{1\le q<\infty} \ell_q$ holds.
14. Let d be a metric on a set X. Show that $\tilde{d}(x, y) = \frac{d(x,y)}{1+d(x,y)}$ is also a metric on X. Moreover, show that d and $\tilde{d}$ generate the same topology.
15. Let $T : \ell_2 \to \ell_2$ be the linear map

$$T(a_1, a_2, \ldots, a_n, \ldots) = (\lambda_1 a_1, \lambda_2 a_2, \ldots, \lambda_n a_n, \ldots).$$

Prove that T is a compact operator if and only if $\lim_{n\to\infty} \lambda_n = 0$.