Math 7390 (Homework) Spring, 2002

1. (§1.2) Find the characteristic functionals of the following continuous distributions: (1) Normal, (2) Uniform on an interval, (3) Exponential, (4) Gamma, (5) Cauchy.

2. (§1.2) Find the characteristic functionals of the following discrete distributions: (1) Binomial, (2) Poisson, (3) Geometric, (4) Negative binomial.

3. (§1.3) Consider the random variable $X(\omega) = \omega(t) + \omega(s)$, $\omega \in C[0,1]$ (the Wiener space). Find the distribution function of X.

4. (§1.4) Let H be an infinite dimensional Hilbert space with norm $|\cdot|$. Show that $|\cdot|$ is not a measurable norm.

5. (§1.4) Let H be an infinite dimensional Hilbert space with norm $|\cdot|$ and T a Hilbert-Schmidt operator of H. Show that $\|x\| = |Tx|$ is a measurable semi-norm.

6. (§1.5) Let n be the canonical normal weak distribution on a HIlbert space H. Show that if h and k are orthogonal in H, then $n(h)$ and $n(k)$ are independent random variables.

7. (§1.5) Let n be the canonical normal weak distribution on a HIlbert space H and $\{e_n\}$ an orthonormal basis for H. Can the random series $\sum_{n=1}^{\infty} n(e_n)^2$ possibly converge to a random variable in some sense?

8. (§1.6) Find the characteristic functional of the standard Gaussian measure on $\mathbb{R}^n$.

9. (§1.6) Let H be a real Hilbert space with norm $|\cdot|$. Use the result in Problem 8 to show that the function $\varphi(x) = \exp\big[-\frac{1}{2}|x|^2\big]$, $x \in H$, is positive definfite.

10. (§1.7) Let μ_n and ν_n be the exponential distributions with parameters λ_n and ρ_n, respectively. Let $\mu = \mu_1 \times \cdots \times \mu_n \times \cdots$ and $\nu = \nu_1 \times \cdots \times \nu_n \times \cdots$. Find condition on the parameters so that μ and ν are equivalent measures.

11. (§2.2) Show that the sum of two independent Gaussian random variables is also a Gaussian random variable.

12. (§2.2) Suppose the joint distribution function of X and Y is given by

$$f(x,y) = \frac{\sqrt{3}}{4\pi}\, e^{-\frac{1}{2}(x^2 - xy + y^2)}, \quad (x,y) \in \mathbb{R}^2.$$

Show that (a) both X and Y are Gaussian with mean 0 and variance 4/3, (b) X and Y are not independent, and (c) for any $a, b \in \mathbb{R}$ the random variable $aX + bY$ is Gaussian.

13. (§2.2) Let $\{X_n\}$ be a sequence of Gaussian random variables. Suppose X_n converges to X in $L^2(\Omega)$. Show that X is also a Gaussian random variable.

14. (§2.2) Let $B(t)$ be a Brownian motion. Find constants a and b so that $X(t) = \int_0^t \left(a + b\frac{u}{t}\right) dB(u)$ is also a Brownian motion.
(Answer: $a = 2, b = -3$ and $a = -2, b = 3$.)

15. (§2.2) Let $B(t)$ be a Brownian motion. Find constants a, b, and c so that $X(t) = \int_0^t \left(a + b\frac{u}{t} + c\frac{u^2}{t^2}\right) dB(u)$ is also a Brownian motion.
(Answer: $a = 3, b = -12, c = 10$ and $a = -3, b = 12, c = -10$)

16. (§2.2) Let $B(t)$ be a Brownian motion. Show that for any n there exist nonzero constants $a_0, a_1, \ldots, a_n$ so that $X(t) = \int_0^t \left(a_0 + a_1\frac{u}{t} + a_2\frac{u^2}{t^2} + \cdots + a_n\frac{u^n}{t^n}\right) dB(u)$ is also a Brownian motion.

17. (§2.2) Let $B(t)$ be a Brownian motion. Show that both $X(t) = \int_0^t (2t - u)\, dB(u)$ and $Y(t) = \int_0^t (3t - 4u)\, dB(u)$ are Gaussian processes with mean function 0 and covariance function $3s^2t - \frac{2}{3}s^3$ for $s \le t$.
(Remark: The process $X(t)$ is canonical, while $Y(t)$ is not canonical.)

18. (§2.3) Let $B(t)$ be a Brownian motion. Find the distribution of $X_t = \int_0^t e^{t-s}\, dB(s)$. Check whether X_t is a martingale.

19. (§2.3) Let $B(t)$ be a Brownian motion. Find the distribution of $Y_t = \int_0^t B(s)\, ds$. Check whether Y_t is a martingale.

20. (§2.4) Let $B(t)$ be a Brownian motion. Check that $X_t = \frac{1}{3}B(t)^3 - \int_0^t B(s)\, ds$ is a martingale.

21. (§2.5) Let $B(t)$ be a Brownian motion. Find the covariance of $\int_a^b |B(t)|\, dB(t)$ and $\int_a^b \big(\operatorname{sgn} B(t)\big)\, dB(t)$.

22. (§2.6) Let $B(t)$ be a Brownian motion. Show that $X_t = \int_0^t e^{B(s)}\, dB(s)$ is a martingale.

23. (§3.1) Let $B(t)$ be a Brownian motion. Show that $f(t) = e^{B(t)^2}$ does not belong to $L^2([0,1] \times \Omega)$.

24. (§3.2) Let $B(t)$ be a Brownian motion. Show that $X_t = \int_0^t e^{B(s)^2}\, dB(s)$ is a local martingale.

25. (§3.3) A function φ on $\mathbb{R}$ is called *convex* if the inequality

$$\varphi(\alpha x + \beta y) \le \alpha\varphi(x) + \beta\varphi(y)$$

holds for any $\alpha, \beta \ge 0$, $\alpha + \beta = 1$ and $x, y \in \mathbb{R}$. Show that if φ is convex, then the inequality

$$\varphi(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n) \le \alpha_1\varphi(x_1) + \alpha_2\varphi(x_2) + \cdots \alpha_n\varphi(x_n)$$

holds for any $n \ge 3$, $\alpha_1, \alpha_2, \ldots, \alpha_n \ge 0$, $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$ and $x_1, x_2, \ldots, x_n \in \mathbb{R}$.