Math 7390-2 **Homework** **Spring 2003**

1. Prove the Kolmogorov extension theorem.
   (see the book "Probability" by J. Lamperti, W. A. Benjamin, Inc., 1966)

2. Prove the Kolmogorov continuity theorem.
   (see my notes from last semester.)

3. Let $H_n(x;\sigma^2)$ be the Hermite polynomial of degree n with parameter σ^2, namely,

$$H_n(x;\sigma^2) = (-\sigma^2)^n e^{x^2/2\sigma^2} D_x^n e^{-x^2/2\sigma^2}.$$

   Prove the following identities:

   (a) $e^{tx-\frac{1}{2}\sigma^2 t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x;\sigma^2)$ (generating function)

   (b) $H_{n+1}(x;\sigma^2) - xH_n(x;\sigma^2) + \sigma^2 n H_{n-1}(x;\sigma^2) = 0,\ n \geq 1.$

   (c) $\frac{d}{dx} H_n(x;\sigma^2) = nH_{n-1}(x;\sigma^2),\ n \geq 1.$

   (d) $\Big(\sigma^2 \frac{d^2}{dx^2} - x\frac{d}{dx} + n\Big) H_n(x;\sigma^2) = 0,\ n \geq 0.$

4. Show that $\left\{ \frac{H_n(x;\sigma^2)}{\sqrt{n!}\sigma^n} \,\middle|\, n = 0, 1, 2, \ldots, n, \ldots \right\}$ is an orthonormal basis for the Hilbert space $L^2\big(\mathbb{R}, \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}\, dx\big)$.

5. Let P_n be the orthogonal projection onto the space of homogeneous chaos of order n. Suppose f is a nonzero function in $L^2(a,b)$. Show that

$$P_n\big(I(f)^n\big) = H_n\big(I(f); \|f\|^2\big),$$

   where $I(f)$ is the Wiener integral of f.

6. Prove the martingale convergence theorem.
   (see either one of the books: (1) K. L. Chung "A Course in Probability Theory"; or (2) R. M. Dudley "Real Analysis and Probability")

7. Read the paper by N. Wiener "The homogeneous chaos", *Amer. J. Math.* **60** (1938) 897–936

8. Read the paper by K. Itô "Multiple Wiener integral", *M. Math. Soc. Japan* **3** (1951) 157–169

9. Let H be a real Hilbert space with norm $|\cdot|$. Show that $|\cdot|$ is not a measurable norm when $\dim(H) = \infty$.

10. Let H be a real Hilbert space with norm $|\cdot|$ and let T be an injective Hilbert-Schmidt operator on H. Show that the norm $\|x\| = |Tx|$, $x \in H$, is measurable.

11. Let $Y(t) = \int_0^t s\, dB(s)$. Find a function $\alpha(t)$ so that $Z(t) = Y(\alpha(t))$ is a Brownian motion.

12. Solve the linear stochastic differential equation

$$dX_t = \big\{\alpha(t)X_t + \beta(t)\big\}\, dB(t) + \big\{\rho(t)X_t + \mu(t)\big\}\, dt.$$

13. Show that the portfolio $\theta(t) = \big(\int_0^t B(s)^2\, ds - tB(t)^2,\, B(t)^2 \big)$ is an arbitrage for the market $X(t) = (1, t)$.

14. Let $X(t) = \big(1, B(t)\big)$ be a market. Find $\theta_0(t)$ so that the portfolio $\theta(t) = \big(\theta_0(t), B(t)\big)$ is self-financing. Is the resulting $\theta(t)$ admissible for the market $X(t)$? Is it an arbitrage for $X(t)$?

15. Evaluate the expectation $E \exp\big[\frac{1}{2}\int_0^1 B(t)^2\, dt\big] = (\cos 1)^{-1/2}$.
Hint: Modify the proof of Kac's formula on pages 48-50 of my 1975 book and use the following identity

$$\prod_{n=1}^{\infty} \left(1 - \frac{x^2}{(2n-1)^2}\right) = \cos \frac{\pi x}{2}. \tag{1}$$

On page 50 the following identity is used:

$$\prod_{n=1}^{\infty} \left(1 + \frac{x^2}{(2n-1)^2}\right) = \cosh \frac{\pi x}{2}. \tag{2}$$

It looks like Equation (2) can be obtained by replacing x with ix in Equation (1) and vice versa. Thus the expectation in this homework problem can be informally obtained by replacing $\alpha = -1/2$ in Kac's formula.