Math 7390 (Homework) Spring 2004

1. Let $B(t)$ be a Brownian motion. Show that $E|B(s) - B(t)|^4 = 3|s - t|^2$.
2. Show that the marginal distribution of a Brownian motion $B(t)$ for $0 < t_1 < t_2 < \cdots t_n$ is given by

$$P\{B(t_1) \le a_1, B(t_2) \le a_2, \ldots, B(t_n) \le a_n\}$$
$$= \frac{1}{\sqrt{(2\pi)^n t_1(t_2 - t_1)\cdots(t_n - t_{n-1})}} \int_{-\infty}^{a_n} \cdots \int_{-\infty}^{a_1}$$
$$\exp\Big[-\frac{1}{2}\Big(\frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1} + \cdots \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}\Big)\Big]\, dx_1 dx_2 \cdots dx_n.$$

3. Let $B(t)$ be a Brownian motion. For fixed t and s, find the distribution function of the random variable $X = B(t) + B(s)$.
4. Let $B(t)$ be a Brownian motion. Show that $\lim_{t\to 0^+} tB(1/t) = 0$ almost surely. Define $W(0) = 0$ and $W(t) = tB(1/t)$ for $t > 0$. Prove that $W(t)$ is a Brownian motion.
5. Let $B(t)$ be a Brownian motion. Find constants a and b so that $X(t) = \int_0^t \big(a + b\frac{u}{t}\big)\, dB(u)$ is also a Brownian motion.
6. Let $B(t)$ be a Brownian motion. Show that $X(t) = \int_0^t (2t - u)\, dB(u)$ and $Y(t) = \int_0^t (3t-4u)\, dB(u)$ are Gaussian processes with mean function 0 and covariance function $3s^2 t - \frac{2}{3}s^3$ for $s \le t$.

   (Remark: The process $X(t)$ is canonical, while $Y(t)$ is not canonical.)
7. For each $n \ge 1$ let X_n be a Gaussian random variable with mean μ_n and variance σ_n^2. Suppose the sequence X_n converges to X in $L^2(\Omega)$. Show that the limits $\mu = \lim_{n\to\infty} \mu_n$ and $\sigma^2 = \lim_{n\to\infty} \sigma_n^2$ exist and that X is a Gaussian random variable with mean μ and variance σ^2.
8. Let $B(t)$ be a Brownian motion. Find the distribution of the Wiener integral $X_t = \int_0^t e^{t-s}\, dB(s)$. Check whether X_t is a martingale.
9. Let $B(t)$ be a Brownian motion. Find the distribution of $\int_0^t B(s)\, ds$. Check whether $Y_t = \int_0^t B(s)\, ds$ is a martingale.
10. Find the distribution of the integral $\int_0^1 sB(s)\, ds$. More generally, find the distribution of the integral $\int_0^1 s^n B(s)\, ds$.
11. Let $B(t)$ be a Brownian motion. Check that $X_t = \frac{1}{3}B(t)^3 - \int_0^t B(s)\, ds$ is a martingale.

12. Let $B(t)$ be a Brownian motion and let $0 < s \leq t \leq u \leq v$. Show that the random variables $\frac{1}{t}B(t) - \frac{1}{s}B(s)$ and $aB(u) + bB(v)$ are independent for any $a, b \in \mathbb{R}$.

13. Let $B(t)$ be a Brownian motion and let $0 < s \leq t \leq u \leq v$. Show that the random variables $aB(s)+bB(t)$ and $\frac{1}{v}B(v)-\frac{1}{u}B(u)$ are independent for any $a, b \in \mathbb{R}$ satisfying the condition $as + bt = 0$.

14. Let $B(t)$ be a Brownian motion. Find the means and variances of the stochastic integrals $\int_a^b |B(t)|\, dB(t)$ and $\int_a^b \big(\operatorname{sgn} B(t)\big)\, dB(t)$.

15. Let $B(t)$ be a Brownian motion. Show that $f(t) = e^{B(t)^2}$ does not belong to $L^2([0,1] \times \Omega)$.

16. For a partition $\Delta = \{a = t_0 < t_1 < \cdots < t_n = b\}$, define

$$M_\Delta = \sum_{j=0}^{n-1} B\Big(\frac{t_j + t_{j+1}}{2}\Big)\big(B(t_{j+1}) - B(t_j)\big).$$

Find $\lim_{\|\Delta\| \to 0} M_\Delta$ in $L^2(\Omega)$.

17. Let $X = \int_0^1 B(t)\, dB(t)$. Find the distribution function of the random variable X.

18. Let $X = \int_a^b \big[\sin(B(t)) + \cos(B(t))\big]\, dB(t)$. Find the variance of the random variable X.

19. Let $f \in L^2([a,b])$ and $X_t = X_a + \int_a^t f(s)\, dB(s)$. Show that

$$\int_a^b f(t) X_t\, dB(t) = \frac{1}{2}\Big(X_b^2 - X_a^2 - \int_a^b f(t)^2\, dt\Big).$$

20. Let $X_t = B(1)B(t), 0 \leq t \leq 1$.
(a) Show that X_t is not a martingale with respect to the filtration $\mathcal{F}_t = \sigma\{B(s); s \leq t\}$.
(b) For $0 \leq s \leq t \leq 1$, find $E[X_t | \mathcal{F}_s]$.

21. Show that $X_t = e^{B(t)} - 1 - \frac{1}{2}\int_0^t e^{B(s)}\, ds$ is a martingale.

22. Show that $X_t = e^{B(t) - \frac{1}{2}t}$ is a martingale.

23. Let $f(t)$ be nonanticipating and $\int_a^b E|f(t)|^2\, dt < \infty$. Show that

$$\Big(\int_a^t f(s)\, dB(s)\Big)^2 - \int_a^t f(s)^2\, ds$$

is a martingale and find its Doob-Meyer qv-process.

24. Let $\lambda \in \mathbb{R}$. Show that $M(t) = e^{\lambda B(t) - \lambda^2 t/2}$ is a martingale and its Doob-Meyer qv-process is given by

$$\langle M \rangle_t = \lambda^2 \int_0^t e^{2\lambda B(u) - \lambda^2 u}\, du.$$

25. Find the quadratic variation of a Poisson process $N(t)$ with parameter $\lambda > 0$.

26. Let $f \in L^2([a,b])$. Find the quadratic variation and the Doob-Meyer qv-process of the martingale $\int_a^t f(s)\, dB(s)$.

27. Let $s \leq t$. Show that

$$E\big\{B(t)^3 \,\big|\, \mathcal{F}_s\big\} = 3(t-s)B(s) + B(s)^3. \tag{$*$}$$

28. Use Equation $(*)$ to derive a martingale.

29. Use the Itô formula to show that $X_t = B(t)^3 - 3tB(t)$ is a martingale and to derive the Doob-Meyer qv-process of X_t.

30. Let a market be given by

$$\begin{aligned} dX_0(t) &= 0, \quad X_0(0) = 1, \\ dX_1(t) &= 2\, dt + dB_1(t) + dB_2(t), \\ dX_2(t) &= -dt - dB_1(t) - dB_2(t), \end{aligned}$$

where $B_1(t)$ and $B_2(t)$ are two independent Brownian motions. Check whether the market $X(t) = \big(X_0(t), X_1(t), X_2(t)\big), 0 \leq t \leq T$, has an arbitrage.

31. Let $B_1(t)$ and $B_2(t)$ be two independent Brownian motions. Check whether the market $X(t) = \big(e^t, B_1(t), B_2(t)\big), 0 \leq t \leq T$, has an arbitrage.

32. Let a market be given by

$$\begin{aligned} dX_0(t) &= 0, \quad X_0(0) = 1, \\ dX_1(t) &= 2\, dt + dB_1(t) + dB_2(t), \end{aligned}$$

where $B_1(t)$ and $B_2(t)$ are two independent Brownian motions. Check whether the market $X(t) = \big(X_0(t), X_1(t)\big), 0 \leq t \leq T$, is complete.