1. Let $B(t)$ be a Brownian motion. Show that $E|B(s) - B(t)|^4 = 3|s - t|^2$.

2. Show that the marginal distribution of a Brownian motion $B(t)$ for $0 < t_1 < t_2 < \cdots < t_n$ is given by

$$P\{B(t_1) \leq a_1, B(t_2) \leq a_2, \ldots, B(t_n) \leq a_n\}$$

$$= \frac{1}{\sqrt{(2\pi)^n t_1 (t_2 - t_1) \cdots (t_n - t_{n-1})}} \int_{-\infty}^{a_1} \cdots \int_{-\infty}^{a_n} \exp \left[ -\frac{1}{2} \left( \frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1} + \cdots + \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}} \right) \right] dx_1 dx_2 \cdots dx_n.$$ 

3. Let $B(t)$ be a Brownian motion. For fixed $t$ and $s$, find the distribution function of the random variable $X = B(t) + B(s)$.

4. Let $B(t)$ be a Brownian motion. Show that $\lim_{t \to 0^+} t B(1/t) = 0$ almost surely. Define $W(0) = 0$ and $W(t) = t B(1/t)$ for $t > 0$. Prove that $W(t)$ is a Brownian motion.

5. Let $B(t)$ be a Brownian motion. Find constants $a$ and $b$ so that $X(t) = \int_0^t (a + b u^2) dB(u)$ is also a Brownian motion.

6. Let $B(t)$ be a Brownian motion. Show that $X(t) = \int_0^t (2t - u) dB(u)$ and $Y(t) = \int_0^t (3t - 4u) dB(u)$ are Gaussian processes with mean function 0 and covariance function $3s^2 t - \frac{2}{3} s^3$ for $s \leq t$.

(Remark: The process $X(t)$ is canonical, while $Y(t)$ is not canonical.)

7. For each $n \geq 1$ let $X_n$ be a Gaussian random variable with mean $\mu_n$ and variance $\sigma_n^2$. Suppose the sequence $X_n$ converges to $X$ in $L^2(\Omega)$. Show that the limits $\mu = \lim_{n \to \infty} \mu_n$ and $\sigma^2 = \lim_{n \to \infty} \sigma_n^2$ exist and that $X$ is a Gaussian random variable with mean $\mu$ and variance $\sigma^2$.

8. Let $B(t)$ be a Brownian motion. Find the distribution of the Wiener integral $X_t = \int_0^t e^{t-s} dB(s)$. Check whether $X_t$ is a martingale.

9. Let $B(t)$ be a Brownian motion. Find the distribution of $\int_0^t B(s) ds$. Check whether $Y_t = \int_0^t B(s) ds$ is a martingale.

10. Find the distribution of the integral $\int_0^1 s B(s) ds$. More generally, find the distribution of the integral $\int_0^1 s^n B(s) ds$.

11. Let $B(t)$ be a Brownian motion. Check that $X_t = \frac{1}{3} B(t)^3 - \int_0^t B(s) ds$ is a martingale.
12. Let $B(t)$ be a Brownian motion and let $0 < s \leq t \leq u \leq v$. Show that the random variables $\frac{1}{t} B(t) - \frac{1}{s} B(s)$ and $aB(u) + bB(v)$ are independent for any $a, b \in \mathbb{R}$.

13. Let $B(t)$ be a Brownian motion and let $0 < s \leq t \leq u \leq v$. Show that the random variables $aB(s) + bB(t)$ and $\frac{1}{v} B(v) - \frac{1}{u} B(u)$ are independent for any $a, b \in \mathbb{R}$ satisfying the condition $as + bt = 0$.

14. Let $B(t)$ be a Brownian motion. Find the means and variances of the stochastic integrals $\int_a^b |B(t)| dB(t)$ and $\int_a^b (\text{sgn} B(t)) dB(t)$.

15. Let $B(t)$ be a Brownian motion. Show that $f(t) = e^{B(t)^2}$ does not belong to $L^2([0, 1] \times \Omega)$.

16. For a partition $\Delta = \{a = t_0 < t_1 < \cdots < t_n = b\}$, define

$$M_{\Delta} = \sum_{j=0}^{n-1} B\left(\frac{t_j + t_{j+1}}{2}\right)\left(B(t_{j+1}) - B(t_j)\right).$$

Find $\lim_{\|\Delta\| \to 0} M_{\Delta}$ in $L^2(\Omega)$.

17. Let $X = \int_a^b B(t) dB(t)$. Find the distribution function of the random variable $X$.

18. Let $X = \int_a^b \left[\sin(B(t)) + \cos(B(t))\right] dB(t)$. Find the variance of the random variable $X$.

19. Let $f \in L^2([a, b])$ and $X_t = X_a + \int_a^t f(s) dB(s)$. Show that

$$\int_a^b f(t)X_t dB(t) = \frac{1}{2} \left(X_b^2 - X_a^2 - \int_a^b f(t)^2 dt\right).$$

20. Let $X_t = B(1)B(t), 0 \leq t \leq 1$.

(a) Show that $X_t$ is not a martingale with respect to the filtration $\mathcal{F}_t = \sigma\{B(s); s \leq t\}$.

(b) For $0 \leq s \leq t \leq 1$, find $E[X_t | \mathcal{F}_s]$.

21. Show that $X_t = e^{B(t)} - 1 - \frac{1}{2} \int_0^t e^{B(s)} ds$ is a martingale.

22. Show that $X_t = e^{B(t) - \frac{1}{2} t}$ is a martingale.

23. Let $f(t)$ be nonanticipating and $\int_a^b E|f(t)|^2 dt < \infty$. Show that

$$\left(\int_a^t f(s) dB(s)\right)^2 - \int_a^t f(s)^2 ds$$

is a martingale and find its Doob-Meyer qv-process.
24. Let $\lambda \in \mathbb{R}$. Show that $M(t) = e^{\lambda B(t) - \lambda^2 t/2}$ is a martingale and its Doob-Meyer qv-process is given by

$$\langle M \rangle_t = \lambda^2 \int_0^t e^{2\lambda B(u) - \lambda^2 u} \, du.$$

25. Find the quadratic variation of a Poisson process $N(t)$ with parameter $\lambda > 0$.

26. Let $f \in L^2([a, b])$. Find the quadratic variation and the Doob-Meyer qv-process of the martingale $\int_a^t f(s) \, dB(s)$.

27. Let $s \leq t$. Show that

$$E\{B(t)^3 \mid \mathcal{F}_s\} = 3(t-s)B(s) + B(s)^3. \quad (*)$$

28. Use Equation $(*)$ to derive a martingale.

29. Use the Itô formula to show that $X_t = B(t)^3 - 3tB(t)$ is a martingale and to derive the Doob-Meyer qv-process of $X_t$.

30. Let a market be given by

$$dX_0(t) = 0, \quad X_0(0) = 1,$$

$$dX_1(t) = 2 \, dt + dB_1(t) + dB_2(t),$$

$$dX_2(t) = -dt - dB_1(t) - dB_2(t),$$

where $B_1(t)$ and $B_2(t)$ are two independent Brownian motions. Check whether the market $X(t) = (X_0(t), X_1(t), X_2(t)), 0 \leq t \leq T$, has an arbitrage.

31. Let $B_1(t)$ and $B_2(t)$ be two independent Brownian motions. Check whether the market $X(t) = (e^t, B_1(t), B_2(t)), 0 \leq t \leq T$, has an arbitrage.

32. Let a market be given by

$$dX_0(t) = 0, \quad X_0(0) = 1,$$

$$dX_1(t) = 2 \, dt + dB_1(t) + dB_2(t),$$

where $B_1(t)$ and $B_2(t)$ are two independent Brownian motions. Check whether the market $X(t) = (X_0(t), X_1(t)), 0 \leq t \leq T$, is complete.