Math 7390-1 (Homework) Spring 2007

1. Let g be a monotonically increasing function on a finite closed interval $[a, b]$. Show that any continuous function f on $[a, b]$ is Riemann–Stieltjes integrable with respect to g.

2. Let $a < c < b$ and $g = 1_{[c,b]}$, i.e., $g(x) = 0$ for $a \leq x < c$ and $g(x) = 1$ for $c \leq x \leq b$. Show that a bounded function f on $[a, b]$ is Riemann-Stieltjes integrable with respect to g in the sense of Equation (1.1.1) if and only if f is continuous at c.

3. Let $C(x)$ be the Cantor function on $[0, 1]$. Evaluate the Riemann-Stieltjes integrals $\int_0^1 x\, dC(x)$ and $\int_0^1 x^2\, dC(x)$.

4. For each $n \geq 1$ let X_n be a Gaussian random variable with mean μ_n and variance σ_n^2. Suppose the sequence X_n converges to X in $L^2(\Omega)$. Show that the limits $\mu = \lim_{n\to\infty} \mu_n$ and $\sigma^2 = \lim_{n\to\infty} \sigma_n^2$ exist and that X is a Gaussian random variable with mean μ and variance σ^2.

5. Let $B(t)$ be a Brownian motion. Show that $E|B(s) - B(t)|^4 = 3|s - t|^2$.

6. Show that the marginal distribution of a Brownian motion $B(t)$ for $0 < t_1 < t_2 < \cdots t_n$ is given by

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\begin{aligned}
&P\{B(t_1) \leq a_1, B(t_2) \leq a_2, \ldots, B(t_n) \leq a_n\} \\
&\quad = \frac{1}{\sqrt{(2\pi)^n t_1 (t_2 - t_1) \cdots (t_n - t_{n-1})}} \int_{-\infty}^{a_n} \cdots \int_{-\infty}^{a_1} \\
&\qquad \exp\Big[-\frac{1}{2}\Big(\frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1} + \cdots \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}} \Big) \Big]\, dx_1 dx_2 \cdots dx_n.
\end{aligned}
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7. Let $B(t)$ be a Brownian motion. For fixed t and s, find the distribution function of the random variable $X = B(t) + B(s)$.

8. Let $B(t)$ be a Brownian motion. Find constants a and b so that $X(t) = \int_0^t \big(a + b\frac{u}{t}\big)\, dB(u)$ is also a Brownian motion.

9. Check whether $X_t = B(2t) - B(t)$ is a Brownian motion.

10. Let $B(t)$ be a Brownian motion. Find the distribution of the Wiener integral $X_t = \int_0^t e^{t-s}\, dB(s)$. Check whether X_t is a martingale.

11. Let $B(t)$ be a Brownian motion. Find the distribution of $\int_0^t B(s)\, ds$. Check whether $Y_t = \int_0^t B(s)\, ds$ is a martingale.

12. Let $B(t)$ be a Brownian motion. Show that $X_t = \frac{1}{3}B(t)^3 - \int_0^t B(s)\,ds$ is a martingale.

13. Let $B(t)$ be a Brownian motion. Find the means and variances of the stochastic integral $\int_a^b |B(t)|\,dB(t)$.

14. Let $B(t)$ be a Brownian motion. Find the means and variances of the stochastic integral $\int_a^b \big(\operatorname{sgn} B(t)\big)\,dB(t)$.

15. Let $B(t)$ be a Brownian motion. Show that $f(t) = e^{B(t)^2}$ does not belong to $L^2([0,1]\times\Omega)$.

16. For a partition $\Delta = \{a = t_0 < t_1 < \cdots < t_n = b\}$, define

$$M_\Delta = \sum_{j=0}^{n-1} B\Big(\frac{t_j + t_{j+1}}{2}\Big)\big(B(t_{j+1}) - B(t_j)\big).$$

Find $\lim_{\|\Delta\|\to 0} M_\Delta$ in $L^2(\Omega)$.

17. Let $X = \int_0^1 B(t)\,dB(t)$. Find the distribution function of the random variable X.

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18. Let $X_t = B(1)B(t), 0 \le t \le 1$.
(a) Show that X_t is not a martingale with respect to the filtration $\mathcal{F}_t = \sigma\{B(s); s \le t\}$.
(b) For $0 \le s \le t \le 1$, find $E[X_t|\mathcal{F}_s]$.

19. Show that $X_t = e^{B(t)} - 1 - \frac{1}{2}\int_0^t e^{B(s)}\,ds$ is a martingale.

20. Let $X_{\lambda,\varepsilon} = \varepsilon^{-\lambda}\int_0^1 e^{-B(t)^2/2\varepsilon}\,dB(t)$. Show that $X_{\lambda,\varepsilon} \to 0$ in $L^2(\Omega)$ as $\varepsilon \downarrow 0$ if and only if $\lambda < \frac{1}{4}$. Check whether $\lim_{\varepsilon\downarrow 0} X_{\frac{1}{4},\varepsilon}$ exits in $L^2(\Omega)$.

21. Let $Y_{\lambda,\varepsilon} = \varepsilon^{-\lambda}\int_0^\varepsilon e^{-B(t)^2/2\varepsilon}\,dB(t)$. Show that $Y_{\lambda,\varepsilon} \to 0$ in $L^2(\Omega)$ as $\varepsilon \downarrow 0$ if and only if $\lambda < \frac{1}{2}$. Check whether $\lim_{\varepsilon\downarrow 0} Y_{\frac{1}{2},\varepsilon}$ exits in $L^2(\Omega)$.

22. Let $Z_{\lambda,\varepsilon} = \varepsilon^{-\lambda}\int_0^{\varepsilon^2} e^{-B(t)^2/2\varepsilon}\,dB(t)$. Show that $Z_{\lambda,\varepsilon} \to 0$ in $L^2(\Omega)$ as $\varepsilon \downarrow 0$ if and only if $\lambda < 1$. Check whether $\lim_{\varepsilon\downarrow 0} Z_{1,\varepsilon}$ exits in $L^2(\Omega)$.

23. Find the quadratic variation process $[N]_t$ of a Poisson process $N(t)$ with parameter $\lambda > 0$.

24. Suppose $\lambda \in \mathbb{R}$. Prove that $M(t) = e^{\lambda B(t) - \lambda^2 t/2}$ is a martingale and the compensator of $M(t)^2$ is given by

$$\langle M\rangle_t = \lambda^2 \int_0^t e^{2\lambda B(u) - \lambda^2 u}\,du.$$

25. Let $f \in L^2[a,b]$ and $M(t) = \int_a^t f(s)\,dB(s)$. Find the quadratic variation process $[M]_t$ of $M(t)$ and the compensator $\langle M\rangle_t$ of $M(t)^2$.

26. Let $f(t)$ be a function in $L^2[a,b]$. Show that

$$M(t) = \left(\int_a^t f(s)\,dB(s)\right)^2 - \int_a^t f(s)^2\,ds$$

is a martingale and find the compensator of $M(t)^2$.

27. Let $s \le t$. Show that

$$E\{B(t)^3 \,|\, \mathcal{F}_s\} = 3(t-s)B(s) + B(s)^3. \tag{$*$}$$

28. Use Equation $(*)$ to derive a martingale $X_t = B(t)^3 - 3tB(t)$. Find the quadratic variation process $[X]_t$ and the compensator $\langle X\rangle_t$.

29. Let $f(t)$ be adapted and $\int_a^b |f(t)|^2\,dt < \infty$ almost surely. Show that

$$M(t) = \left(\int_a^t f(s)\,dB(s)\right)^2 - \int_a^t f(s)^2\,ds$$

is a local martingale.

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