Math 7390-2 (Homework) Spring 2009

Due February 9, 2009: 1, 2, 3, 4, 5

1. Let X be a Gaussian random variable with mean 0 and variance σ^2. Show that $Ee^{tX} = e^{\frac{1}{2}\sigma^2 t^2}$ for any real number t. (The function $\varphi(t) = Ee^{tX}$ is called the moment generating function of X.)

2. Let $B(t)$ be a Brownian motion. Use Itô's formula to find a stochastic process X_t such that $B(t)^3 - 5B(t)^2 - X_t$ is a martingale.

3. Let $B_1(t)$ and $B_2(t)$ be independent Brownian motions. Check whether the market $X_t = (1, 2 + B_1(t), -t + B_1(t) + B_2(t))$ has an arbitrage.

4. Let $B_1(t)$ and $B_2(t)$ be independent Brownian motions. Check whether the market $X_t = (1, 2 + B_1(t) + B_2(t), -t - B_1(t) - B_2(t))$ has an arbitrage.

5. Let $B_1(t)$ and $B_2(t)$ be independent Brownian motions. Check whether the market $X_t = (e^t, B_1(t), B_2(t))$ has an arbitrage.

Due February 23, 2009: 6, 7, 8, 9, 10, 11

6. Show that the inclusion map $\ell_p \hookrightarrow \ell_q$ is continuous for any $1 \le p < q \le \infty$. (Hint: Prove the inequality $\|a\|_q \le \|a\|_p$ for all $a \in \ell_p$ and $1 \le p < q \le \infty$.)

7. Show that $\ell_p \ne \ell_q$ if $p \ne q$.

8. Check whether the equalities hold: $\ell_1 = \cap_{1<p\le\infty} \ell_p$, $c_0 = \cup_{1\le q<\infty} \ell_q$.

9. Let $\mu(X) < \infty$. Show that the inclusion map $L^p(X, \mu) \hookrightarrow L^q(X, \mu)$ is continuous for any $1 \le q < p \le \infty$. (Hint: Use the Hölder inequality.)

10. Let μ be the Lebesgue measure on the interval $[0, 1]$. Show that $L^p([0, 1], \mu) \ne L^q([0, 1], \mu)$ if $p \ne q$.

11. Let μ be the Lebesgue measure on the interval $[0, 1]$. Check whether the equalities hold: $L^\infty([0, 1], \mu) = \bigcap_{1\le p<\infty} L^p([0, 1], \mu)$, $L^1([0, 1], \mu) = \bigcup_{1<q\le\infty} L^q([0, 1], \mu)$.

Due March 27, 2009: 12, 13, 14, 15, 16, 17

12. Check whether $\ell_2 \subset \ell_p$, $2 < p \le \infty$, is an abstract Wiener space.

13. Check whether $L^2[0, 1] \subset L^p[0, 1]$, $1 \le p < 2$, is an abstract Wiener space.

14. For $x = (x_1, x_2, \ldots, x_n, \ldots) \in \ell_2$, define

$$\|x\| = \Big(\sum_{n=1}^{\infty} \frac{1}{n} x_n^2 \Big)^{1/2}.$$

Check whether $\|\cdot\|$ is a measurable norm.

15. Let $\mu_n \sim \exp(1)$ for all $n \geq 1$ and $\nu_n \sim \exp(\lambda_n)$, $n \geq 1$. Define

$$\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n \times \cdots,$$
$$\nu = \nu_1 \times \nu_2 \times \cdots \times \nu_n \times \cdots.$$

Find the necessary and sufficient condition on the sequence $\{\lambda_n\}$ so that $\mu \approx \nu$.

16. Let $\mu_n \sim \mathrm{N}(0,1)$ for all $n \geq 1$ and $\nu_n \sim \mathrm{N}(0, \sigma_n^2)$, $n \geq 1$. Define

$$\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n \times \cdots,$$
$$\nu = \nu_1 \times \nu_2 \times \cdots \times \nu_n \times \cdots.$$

Find the necessary and sufficient condition on the sequence $\{\sigma_n\}$ so that $\mu \approx \nu$.

17. Take an injective Hilbert-Schmidt operator T on H. Let B be the completion of H with respect to the norm $\|x\| = |Tx|$ and let μ be the Gaussian measure on B. Find the integral $\int_B \|x\|^2 \, d\mu(x)$.

,,,,,,,,,,,,,,,,,,,,,,,,,,,,,

Presentation of special topics: April 27 to May 1, 2009

1. Reiser: Construction of a Brownian motion
2. Cross: Black-Scholes model
3. Szozda: An extension of the Itô integral
4. Tao: Normalized markets
5. Chen: Completeness of a market
6. Sae-Tang: Price of an option
7. Latin: Equivalence of norms in finite dimensional spaces
8. Lodygowski: Proof of Gross' theorem