# Math 7390-2 Homework

**Homework 1**: Due February 4, 2011

1. Let $B(t)$ be a Brownian motion. Find the joint density function of $B(s)$ and $B(t)$ for $0 < s \leq t$.

2. Let $X$ be a normal random variable with mean 0 and variance $\sigma^2$. Prove the formula

$$E(X^n) = \begin{cases} (n-1)!!\,\sigma^n, & \text{if } n \text{ is even}, \\ 0, & \text{if } n \text{ is odd}. \end{cases}$$

3. Let $\mu$ be the Gaussian measure with mean 0 and variance $\sigma^2$. Define

$$\theta(t) = \int_{-\infty}^{\infty} e^{tx}\, d\mu(x), \quad \widetilde{\theta}(t,s) = \int_{-\infty}^{\infty} e^{tx} e^{sx}\, d\mu(x). \tag{1}$$

Find a function $\rho(t)$ such that $\rho(0) = 0$, $\rho'(0) = 1$, and $\frac{\widetilde{\theta}(\rho(t),\rho(s))}{\theta(\rho(t))\theta(\rho(s))}$ is a function of the product $ts$. The resulting function

$$\varphi(t,x) = \frac{e^{\rho(t)x}}{\theta(\rho(t))}$$

is called a *generating function* for $\mu$. The corresponding orthogonal polynomials are the Hermite polynomials.

4*. Do the same thing as in Problem 3 for the Poisson measure $\mu$ with parameter $\lambda > 0$. The corresponding orthogonal polynomials are the Charlier polynomials.

5*. Let $H_n(x;\sigma^2)$ be the Hermite polynomial of degree $n$ with parameter $\sigma^2$. Find the norm $\|H_n(x;\sigma^2)\|$ with respect to the measure in Problem 3.

6*. Let $C_n(x;\lambda)$ be the Charlier polynomial of degree $n$ with parameter $\lambda$. Find the norm $\|C_n(x;\sigma^2)\|$ with respect to the measure in Problem 4.

**Homework 2**: Due February 16, 2010

7*. For $0 \leq t_1 < t_2 < \cdots < t_n$, let $\mu_{t_1,t_2,\ldots,t_n} = \delta_{(t_1,t_2,\ldots,t_n)}$, the Dirac delta measure at the point $(t_1, t_2, \ldots, t_n)$ of $I\!R^n$. Show that the collection

$$\Big\{\mu_{t_1,t_2,\ldots,t_n}\,;\, 0 \leq t_1 < t_2 < \cdots < t_n,\, n = 1, 2, \ldots,\Big\}$$

satisfies the Kolmogorov consistency condtion. Moreover, describe the corresponding probability measure on the function space $I\!R^{[0,\infty)}$.

8*. Let $X$ be a Poisson random variable with parameter $\lambda > 0$. Find the $n$-th moment $E(X^n)$ of $X$. (Hint: Use the moment generating function of $X$.)

9. Let $B(t)$ be a Brownian motion. For fixed $t$ and $s$, find the distribution function of the random variable $X = B(t) + 2B(s)$.

10. Let $B(t)$ be a Brownian motion. For fixed $t$, find the distribution of the random variable $\int_0^t B(s)\,ds$.

11*. Let $B(t)$ be a Brownian motion. For fixed $t$, find the distribution of the random variable $\int_0^t B(s)\cos(t-s)\,ds$.

12. Let $B(t)$ be a Brownian motion. Find the probability that $\left|\int_0^\pi \sin t\, dB(t)\right| > 1$.

13. Let $B(t)$ be a Brownian motion. Find the probability that $\left|\int_0^1 t^2 B(t)\,dt\right| > 2$.

14. Let $B(t)$ be a Brownian motion. Check whether $X_t = \int_0^t s^2 B(s)\,ds$ is a martingale.

15*. Let $B(t)$ be a Brownian motion. Show that $X_t = B(t)^3 - 3\int_0^t B(s)\,ds$ is a martingale.

**Homework 3**: Due March 18, 2011

16*. For a partition $\Delta_n = \{a = t_0 < t_1 < \cdots < t_n = b\}$, define

$$M_{\Delta_n} = \sum_{i=1}^{n} B\Big(\frac{t_{i-1}+t_i}{2}\Big)\big(B(t_i) - B(t_{i-1})\big).$$

Find $\lim_{\|\Delta_n\|\to 0} M_{\Delta_n}$ in $L^2(\Omega)$.

17. Let $f(t) = B\big(\frac{1}{2}\big)1_{[1/2,1]}(t)$. Show that $\int_0^1 f(t)\,dB(t)$ is not a Gaussian random variable.

18. Let $X = \int_a^b |B(t)|\,dB(t)$. Find the variance of the random variable $X$.

19. The *signum function* is defined by $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(x) = x/|x|$ if $x \neq 0$. Let $X_t = \int_0^t \operatorname{sgn}(B(s))\,dB(s)$. Show that for $s < t$, the random variable $X_t - X_s$ has mean 0 and variance $t - s$.

20. Find the variance of the random variable $X = \int_a^b \sqrt{t}\, e^{B(t)}\,dB(t)$.

21*. Show that $X_t = e^{B(t)} - 1 - \frac{1}{2}\int_0^t e^{B(s)}\,ds$ is a martingale.

22. Let $X_{\lambda,\varepsilon} = \varepsilon^{-\lambda}\int_0^1 e^{-B(t)^2/2\varepsilon}\,dB(t)$. Show that $X_{\lambda,\varepsilon} \to 0$ in $L^2(\Omega)$ as $\varepsilon \downarrow 0$ if and only if $\lambda < \frac{1}{4}$. Check whether $\lim_{\varepsilon\downarrow 0} X_{\frac{1}{4},\varepsilon}$ exits in $L^2(\Omega)$.

23*. Let $Y_{\lambda,\varepsilon} = \varepsilon^{-\lambda}\int_0^{\varepsilon} e^{-B(t)^2/2\varepsilon}\,dB(t)$. Show that $Y_{\lambda,\varepsilon} \to 0$ in $L^2(\Omega)$ as $\varepsilon \downarrow 0$ if and only if $\lambda < \frac{1}{2}$. Check whether $\lim_{\varepsilon\downarrow 0} Y_{\frac{1}{2},\varepsilon}$ exits in $L^2(\Omega)$.

24. Let $Z_{\lambda,\varepsilon} = \varepsilon^{-\lambda}\int_0^{\varepsilon^2} e^{-B(t)^2/2\varepsilon}\,dB(t)$. Show that $Z_{\lambda,\varepsilon} \to 0$ in $L^2(\Omega)$ as $\varepsilon \downarrow 0$ if and only if $\lambda < 1$. Check whether $\lim_{\varepsilon\downarrow 0} Z_{1,\varepsilon}$ exits in $L^2(\Omega)$.

**Homework 4**: Due April 1, 2011

25. Let $B(t)$ be a Brownian motion. Find all deterministic functions $\rho(t)$ such that the stochastic process $X(t) = e^{B(t)+\rho(t)}$ is a martingale. (Note: Do not use Itô's formula)

26*. Let $B(t)$ be a Brownian motion and $h(t)$ a deterministic function in $L^2(0,\infty)$. Find all deterministic functions $\rho(t)$ such that the stochastic process $X(t) = e^{\int_0^t h(s)\,dB(s)+\rho(t)}$ is a martingale. (Note: You may use Itô's formula)

27. Let $f(t,x) = H_n(x;t)$ be the Hermite polynomial of degree $n$ with parameter $t$.
    (a) Apply Itô's formula to the function $f(t,x)$ to show that $M_n(t) = H_n(B(t);t)$ is a martingale.
    (b) Apply Itô's formula again to the function $f(t,x)^2$ to show that the compensator of $M_n(t)^2$ is given by $\langle M_n\rangle_t = n^2 \int_0^t H_{n-1}(B(s);s)^2\,ds$.

28. Let $B_1(t)$ and $B_2(t)$ be independent Brownian motions and $\Delta_n = \{t_0, t_1, \ldots, t_{n-1}, t_n\}$ a partition of $[a,b]$. Show that

$$\sum_{i=1}^{n} \big(B_1(t_i) - B_1(t_{i-1})\big)\big(B_2(t_i) - B_2(t_{i-1})\big) \longrightarrow 0$$

in $L^2(\Omega)$ as $\|\Delta_n\| = \max_{1\le i\le n}(t_i - t_{i-1})$ tends to 0.

29. Let $N(t)$ be a Poisson process with paramter $\lambda > 0$ and $\Delta_n = \{t_0, t_1, \ldots, t_{n-1}, t_n\}$ a partition of $[a,b]$. Prove that

$$\sum_{i=1}^{n} (t_i - t_{i-1})\big(N(t_i) - N(t_{i-1})\big) \longrightarrow 0$$

in $L^2(\Omega)$ and almost surely as $\|\Delta_n\| = \max_{1\le i\le n}(t_i - t_{i-1})$ tends to 0.

30*. Check whether the assertion in Problem 28 remains true for the compensated Poisson process $\widetilde{N}(t)$.

**Homework 5**: Due April 15, 2011

31*. Check whether $X(t) = \int_0^t \operatorname{sgn}(B(s) - s)\, dB(s)$ is a Brownian motion.

32. Prove that if $h$ satisfies the Novikov condition, then $h$ belongs to $L^2_{\mathrm{ad}}([a,b] \times \Omega)$.

33. Let $B(t)$ be a Brownian motion with respect to $P$. Find a probability measure $Q$ with respect to which $W(t) = B(t) + t - t^3$, $0 \le t \le 2$, is a Brownian motion.

34. Let $B(t)$ be a Brownian motion with respect to $P$. Find a probability measure $Q$ with respect to which $W(t) = B(t) + \int_0^t \min\{1, |B(s)|\}\, ds$, $0 \le t \le 3$, is a Brownian motion.

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**Homework 6**: Due April 25, 2011

35. Evaluate the stochastic integral $\int_0^t \arctan B(s)\, dB(s)$.

36. Let $B(t)$, $t \ge 0$, be a Brownian motion with respect to $P$. Show that the stochastic process $B(t) - t$, $0 \le t \le 2$, is not a Brownian motion with respect to the probability measure $dQ = e^{B(1) - \frac{1}{2}}\, dP$.

37. Let $B(t)$ be a Brownian motion with respect to $P$. Can the stochastic process

$$W(t) = B(t) - \int_0^t \frac{1}{1 + B(s)^2}\, ds, \quad 0 \le t \le 10,$$

be a Brownian motion with respect to some probability measure $Q$? If so, find $Q$.

38*. Evaluate the double Wiener–Itô integral $\int_0^1 \int_0^1 t\, dB(s)\, dB(t)$.

39. Find the homogeneous chaos expansion for the function $B(t)^4 + 2B(t)^3$.

40. Find the homogeneous chaos expansion for the function $B(t)e^{B(t)}$.

41. Find the Wiener–Itô decomposition of the function $\big(B(4) - B(1)\big)\big(B(3) - B(2)\big)$.

## Homework Presentation

1. Monday (5/2/11): Abeynanda, Babaee, Heider, Jeon

   4*, 7*, 16*, 26*

2. Wednesday (5/4/11): Bandyopadhyay, Peng, Yang, Zhang

   5*, 11*, 21*, 31*

3. Friday (5/6/11): Huang, Kang, Smyser (in my office), Wang

   6*, 15*, 23*, 38*