MATH 7390: Project report topics Spring 2002

1. **Continuous nowhere differentiable functions** (K. T. W. Weierstrass 1872)
   Investigate continuous nowhere differentiable functions. Find information from internet (try http://www.google.com) about history, motivation, and various constructions. Try the series $\sum_{n=0}^{\infty} a^n \varphi(b^n x)$ and find conditions on a, b, and φ so that this series is a continuous nowhere differentiable function.

2. **Wiener measure** (N. Wiener 1923)
   When N. Wiener introduced a measure on $C[0,1]$ in 1923, he call this space "Differential space" which refers to independent increments. This achievement is a major breakthrough in the theory of science. The measure is coined as the Wiener measure and the term differential space is replaced by Wiener space. Prove the Wiener theorem. Find some history and information about the Wiener space and the life of N. Wiener. See the references:
   - Wiener, N.: Differential space; *J. Math. Phys.* **58** (1923) 131–174
   - Wiener, N.: *Selected Papers of Nobert Wiener.* SIAM and MIT Press, 1965
   - Wiener, N.: *I am a mathematician.* MIT Press, 1964

3. **Abstract Wiener space** (L. Gross 1965)
   Let H be an infinite dimensional separable Hilbert space. The Gauss cylinder measure on H is not σ-additive. Take a weaker norm on H and let B be the completion of H with respect to this norm. The question is whether the Gauss cylinder measure has a σ-extension to B. The Gross theorem says that if the weaker norm is measurable, then the answer is affirmative. Prove this theorem. In 1971 R. M. Dudley, J. Feldman, and L. LeCam proved jointly that the converse of Gross' theorem is also true. See the references:
   - Gross, L.: Abstract Wiener spaces; *Proc. 5th Berkeley Sym. Math. Stat. Probab.* **2** (1965) 31–42
   - Kuo, H.-H.: *Gaussian Measure in Banach Spaces.* Lecture Notes in Math. vol. 463, Springer-Verlag, 1975

4. **Characteristic functions of measures on infinite dimensional spaces** (L. Gross 1963, V. V. Sazonov 1958, R. A. Minlos 1959)
   The Bochner theorem says that a function on R^n is the characteristic function of a finite measure if and only if it is continuous and positive definite. This theorem was generalized to Hilbert spaces independently by L. Gross in 1963 and V. V. Sazonov in 1959, and to nuclear spaces by R. A. Minlos in 1959. Prove these two theorems. It was shown by V. N. Sudakov and A. M. Vershik jointly in 1962 that there is no Bochner type theorem for general Banach spaces. See the references:
   - Gel'fand, I. M. and Vilenkin, N. Ya.: *Generalized Functions.* Vol. 4, English translation, Academic Press, 1964

- Gross, L.: *Harmonic Analysis on Hilbert Space.* Memoirs Amer. Math. Soc. Vol. 46, 1963
- Kuo, H.-H.: *Gaussian Measure in Banach Spaces.* Lecture Notes in Math. vol. 463, Springer-Verlag, 1975
- Sazonov, V. V.: A remark on characteristic functionals; *Teor. Veroj. i Prim.* **3** (1958) 201–205
- Sudakov, V. N. and Vershik, A. M.: Topological aspects of the theory of measures in linear spaces; *Uspekhi Mat. Nauk.* **17** (1962) 217–219

5. **Kolmogorov extension theorem** (A. N. Kolmogorv 1933)
   Kolmogorov's 1933 monograph "Foundations of the Theory of Probability" marked the beginning of modern probability theory. Here Kolmogorov showed that a stochastic process can be specified by marginal distributions. More precisely, suppose $X(t, \omega)$ is a stochastic process, then we have a family of marginal distributions. Conversely, if we have a family of probability measures satisfying consistency conditions, then these probability measures are marginal distributions of a stochastic process. Prove this theorem. See the book:
   - Lamperti, J: *Probability.* Benjamin, 1966

6. **Dichotomy of infinite product measures** (S. Kakutani 1948)
   Suppose μ_n and ν_n are equivalent measures on the real line for $n = 1, 2, \ldots$. Let $\mu = \mu_1 \times \mu_2 \times \cdots$ and $\nu = \nu_1 \times \nu_2 \times \cdots$ be the infinite product measures. It is natural to ask the question whether μ and ν are also equivalent. The surprising fact proved by S. Kakutani in 1948 says that μ and ν are either equivalent or singular to each other. Prove this theorem and give some examples. Find information on further development.
   - Kakutani, S.: On equivalence of infinite product measures; *Ann. Math.* **49** (1948) 214–224
   - Kuo, H.-H.: *Gaussian Measure in Banach Spaces.* Lecture Notes in Math. vol. 463, Springer-Verlag, 1975

7. **Construction of Brownian motion via random series** (P. Lévy 1948)
   There are several ways to construct a Brownian motion, for instance., through the Wiener space. In 1948 P. Lévy gave a very interesting construction of Brownian motion in terms of random series. T. Hida's book "Brownian motion" (Springer-Verlag 1980) has a nice presentation of Lévy's construction. Describe this construction. Moreover, relate it to Wiener space and continuous nowhere differentiable functions. See the book:
   - Hida, T.: *Brownian Motion.* Springer-Verlag, 1980

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Assignment:

1(Lohman), 2(McAllister), 3(Lee), 4(Demir), 5(Mihai), 6(Namli), 7(Wu).