1. Let $B(t)$ be a Brownian motion. For fixed t and s, find the distribution function of the random variable $X = B(t) + B(s)$.
2. Let $B(t)$ be a Brownian motion. Find constants a and b such that $X(t) = \int_0^t \left(a + b\frac{u}{t}\right) dB(u)$ is also a Brownian motion.
3. Let $B(t)$ be a Brownian motion. Find the distribution of $\int_0^t e^{t-s}\, dB(s)$. Check whether $X_t = \int_0^t e^{t-s}\, dB(s)$ is a martingale.
4. Let $B(t)$ be a Brownian motion. Find the distribution of $\int_0^t B(s)\, ds$. Check whether $Y_t = \int_0^t B(s)\, ds$ is a martingale.
5. Let $X = \int_0^1 B(t)\, dB(t)$. Find the distribution function of the random variable X.
6. Let $X = \int_a^b |B(t)|\, dB(t)$. Find the variance of the random variable X.
7. The *signum function* is defined by $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(x) = x/|x|$ if $x \neq 0$. Let $X_t = \int_0^t \operatorname{sgn}(B(s))\, dB(s)$. Show that for $s < t$, the random variable $X_t - X_s$ has mean 0 and variance $t - s$.
8. Find the variance of the random variable $X = \int_a^b \sqrt{t}\, e^{B(t)}\, dB(t)$.
9. Show that $X_t = e^{B(t) - \frac{1}{2}t}$ is a martingale.
10. Let $B(t)$ be a Brownian motion. Find all deterministic functions $\rho(t)$ so that $e^{B(t)+\rho(t)}$ is a martingale.
11. Let $B_1(t)$ and $B_2(t)$ be two independent Brownian motions and let $\Delta_n = \{t_0, t_1, \ldots, t_{n-1}, t_n\}$ be a partition of a finite interval $[a, b]$. Show that
$$\sum_{i=1}^{n} \big(B_1(t_i) - B_1(t_{i-1})\big)\big(B_2(t_i) - B_2(t_{i-1})\big) \longrightarrow 0$$
in $L^2(\Omega)$ as $\|\Delta_n\| = \max_{1 \le i \le n}(t_i - t_{i-1})$ tends to 0.
12. Evaluate the stochastic integral $\int_0^t \arctan B(s)\, dB(s)$.
13. Check whether $X(t) = \int_0^t \operatorname{sgn}(B(s) - s)\, dB(s)$ is a Brownian motion.
14. Let $B(t)$ be a Brownian motion with respect to a probability measure P. Find a probability measure with respect to which the stochastic process $W(t) = B(t) + t - t^3$, $0 \le t \le 2$, is a Brownian motion.
15. Let $B(t)$ be a Brownian motion with respect to a probability measure P. Find a probability measure with respect to which the stochastic process $W(t) = B(t) + \int_0^t \min\{1, B(s)\}\, ds$, $0 \le t \le 3$, is a Brownian motion.