CHAPTER 1
BACKGROUND

The prerequisites for this course are a background in general topology, advanced calculus (analysis), and linear algebra.

The background in general topology is a knowledge of the notions of a topology as given in a one semester or quarter course: the relative topology, the quotient topology, compact sets and topological spaces, second countable topological spaces, and Hausdorff spaces.

At several points in the notes connections are made to algebraic topology, e.g., the fundamental group, but algebraic topology is not required in the logical dependence of these notes.

The assumed analysis background is a course in Calculus of several variables that includes the The Inverse Function Theorem, The Implicit Function Theorem, The Change of Variables Formula from integration, and The Mean Value Theorem for Integrals. In addition, we will also use the theorem of Existence and Uniqueness of solutions to first order differential equations. We now state and discuss these Theorems as well as setting some notation.

Suppose $O \subset \mathbf{R}^m$ and $f : O \to \mathbf{R}^n$. If $\mathbf{p} \in O$ and $\mathbf{v} \in \mathbf{R}^m$, then the directional derivative of f at $\mathbf{p}$ in the direction $\mathbf{v}$ is

$$D_{\mathbf{v}} f(\mathbf{p}) = \lim_{h \to 0} \frac{f(\mathbf{p} + h\mathbf{v}) - f(\mathbf{p})}{h}, \quad \text{if the limit exists.}$$

Some authors require that $\mathbf{v}$ is a unit vector. This assumption is almost universal in calculus books, but not in analysis books or literature. We do not assume that $\mathbf{v}$ is a unit vector.

If f is differentiable, then the derivative at $\mathbf{p}$ is the linear map $Df(\mathbf{p}) : \mathbf{R}^m \to \mathbf{R}^n$ defined by

$$Df(\mathbf{p})(\mathbf{v}) = D_{\mathbf{v}} f(\mathbf{p}).$$

The matrix for $Df(\mathbf{p})$, in the standard basis, is the Jacobian matrix $Jf(\mathbf{p})$ and

$$Jf(\mathbf{p}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}$$

The basic theorems that we assume from a background of analysis are: The Inverse Function Theorem, The Implicit Function Theorem, The Change of Variables Formula from integration, and The Mean Value Theorem for Integrals. In addition, we will also use the theorem of existence and uniqueness of solutions to first order differential equations. We now state and discuss these Theorems.

Inverse Function Theorem.

Implicit Function Theorem. *Let $f : \mathbf{R}^{n+m} \to \mathbf{R}^m$ be a C^∞ function. Write points in $\mathbf{R}^{n+m} = \mathbf{R}^n \times \mathbf{R}^m$ as $(t_1, \cdots, t_n, x_1, \cdots, x_m) = (\mathbf{t}; \mathbf{x})$ and $f = (f_1, \cdots, f_m)$. Let $f(\mathbf{t}_0; \mathbf{x}_0) = p$ for $p \in \mathbf{R}^m$. Suppose that $det(\frac{\partial f_j}{\partial x_i}\Big|_{(\mathbf{t}_0;\mathbf{x}_0)}) = det(Df|_{\mathbf{R}^m}(\mathbf{t}_0; \mathbf{x}_0)) \neq 0$. Then there is a n-dimensional open set T_0 containing $\mathbf{t}_0$, an m-dimensional open set X_0 containing $\mathbf{x}_0$, and a unique function $g : T_0 \to \mathbf{R}^m$ such that*

(1) *g is C^∞*
(2) *$g(\mathbf{t}_0) = \mathbf{x}_0$, and*
(3) *for $(\mathbf{t}; \mathbf{x}) \in T_0 \times X_0$, $f(\mathbf{t}; \mathbf{x}) = p$ if and only if $g(\mathbf{t}) = \mathbf{x}$.*

There is a corollary of the Implicit Function Theorem that will, for us, be the useful form of the Implicit Function Theorem. It is sometimes called the Rank Theorem.

The Rank Theorem. *Suppose $O \subset \mathbf{R}^{n+m}$ is an open set and $f : O \to \mathbf{R}^m$ is C^∞. Suppose $p \in \mathbf{R}^m$, $q \in O$ and $f(q) = p$. If $Df(q)$ has rank m, then there is an open set $U \subset O$, $q \in U$; open sets $U_1 \subset \mathbf{R}^n$, and $U_2 \subset \mathbf{R}^m$;and, a diffeomorphism $H : U \to U_1 \times U_2$ such that $y \in U \cap f^{-1}(p)$ if and only if $H(y) \in U_1 \times \{0\}$.*

Proof. The kernel of $Df(q)$ is n-dimensional, so let $\mathrm{Ker}(Df(q)) = \mathrm{span}\{\mathbf{v}_1, \cdots, \mathbf{v}_n\} = \mathbf{R}^n$ and let $\mathrm{span}\{\mathbf{v}_{n+1}, \cdots, \mathbf{v}_{n+m}\} = \mathbf{R}^m$, so that $\mathbf{v}_{n+1}, \cdots, \mathbf{v}_{n+m}$ is an ordered basis of $\mathbf{R}^{n+m} = \mathbf{R}^n \times \mathbf{R}^m$. Let $K : \mathbf{R}^{n+m} \to \mathbf{R}^{n+m}$ be defined by $K(y) = \sum_{i=1}^{n+m} y_i \mathbf{v}_i + q$ for each $y = (y_1, \cdots, y_{n+m}) \in \mathbf{R}^{n+m}$. The map K is a C^∞ diffeomorphism, it is an isomorphism and a translation. Also $Df \circ K(0)(\mathbf{R}^m) = Df(q) \circ DK(0)(\mathbf{R}^m) = Df(q)(\mathrm{span}\{\mathbf{v}_{n+1}, \cdots, \mathbf{v}_{n+m}\}) = \mathbf{R}^m$, and so $\det(Df \circ K(0)|_{\mathbf{R}^m}) \neq 0$. We have that $K^{-1}(O)$ is an open set, $f \circ K : K^{-1}(O) \to \mathbf{R}^m$ is a C^∞ function, $f \circ K(0) = p$, and $\det(Df \circ K(0)|_{\mathbf{R}^m}) \neq 0$, and hence we can apply the Implicit Function Theorem. We conclude that there are open sets $T \subset \mathbf{R}^n$, $X \subset \mathbf{R}^m$, $T \times X \subset K^{-1}(O)$, and a C^∞ function $g : T \to X$ such that $f \circ K(t; x) = p$ if and only if $g(t) = x$, i.e., $(f \circ K)^{-1}(q)$ is the graph of g.

Take an open set $U_1 \subset T$ with $0 \in U_1$ and an open set $U_2 \subset X$ such that $O_1 = \{(t, x) \mid x \in g(t) + U_2\} \subset T \times X$. It is an exercise in general topology that this can always be arranged (one first takes U_1 so that its closure is a compact subset of T). Let $h : O_1 \to U_1 \times U_2$ by $h(t, x) = (t, x - g(t))$. The map h is a C^∞ diffeomorphism, its inverse is $(t, x) \mapsto (t, x + g(t))$. Under this diffeomorphism, $G_g \cap O_1$ maps to $U_1 \times \{0\}$.

To complete the proof, take $H = h \circ K^{-1}$ and $U = h \circ K^{-1}(O_1)$. Then $H : U \to U_1 \times U_2$ is a C^∞ diffeomorphism and $x \in U \cap f^{-1}(p)$ if and only if $H(x) \in U_1 \times \{0\}$. □

Change of Variables Formula.

Mean Value Theorem for Integrals.