CHAPTER 2
MANIFOLDS

In this chapter, we address the basic notions: What is a manifold and what is a map between manifolds. Several examples are given.

An n dimensional manifold is a topological space that appears to be $\mathbf{R}^n$ near a point, i.e., locally like $\mathbf{R}^n$. Since these topological spaces appear to be locally like $\mathbf{R}^n$, we may hope to develop tools similar to those used to study $\mathbf{R}^n$ in order to study manifolds. The term manifold comes from "many fold," and it refers to the many dimensions of space that a manifold may describe.

Later in this section we will also need structure to discuss C^∞ functions on manifolds. Our notion of smooth will be C^∞, i.e., continuous partial derivatives of all orders. The terms C^∞ and smooth are usually synonymous, however, in this chapter and the next, we will use C^∞ to describe maps between real vector spaces (as in advanced Calculus) and smooth for maps between manifolds. We use this distinction since many confusing compositions occur and the reader is assumed to be familiar with C^∞ maps between real vector spaces while results on smooth maps between manifolds must be proven. We begin with the notion of a topological manifold.

Definition 2.1*.** *A topological manifold of dimension n is a second countable Hausdorff space M for which each point has a neighborhood homeomorphic to an open set in $\mathbf{R}^n$.*

This notion appears to capture the topological ideal of locally looking like $\mathbf{R}^n$, but in order to do Calculus we will need more structure. The term n-manifold is usually written for n dimensional manifold, and the dimension n is often suppressed.

Definition 2.2*.** *Let $\mathcal{U} \subset M$ be a connected open set in a topological n-manifold M, and $\phi : \mathcal{U} \to \mathbf{R}^n$ be a homeomorphism to its image $\phi(\mathcal{U})$, an open set in $\mathbf{R}^n$.*

The pair $(\mathcal{U}, \phi)$ is called a coordinate system or chart. If $x_0 \in \mathcal{U}$ and $\phi(x_0) = 0 \in \mathbf{R}^n$, then the coordinate system is centered at x_0. Call the map ϕ a coordinate map.

Definition 2.3*.** *Suppose M is a topological manifold. For $k \in \mathbf{N}$, $k = 0$, or $k = \infty$, a C^k atlas is a set of charts $\{(\mathcal{U}_i, \phi_i) \mid i \in I\}$ such that*

(1) *$\bigcup_{i\in I} \mathcal{U}_i = M$, and*
(2) *$\phi_i \circ \phi_j^{-1}$ is C^k on its domain for all $i, j \in I$.*

The k denotes the degree of differentiability. If $k = 0$, then maps are just continuous.

Definition 2.4*.** *If $\mathcal{A} = \{(\mathcal{U}_i, \phi_i) | i \in I\}$ is a C^k atlas for an n-manifold M^n and $f : \mathcal{U} \to \mathbf{R}^n$ is a homeomorphism onto its image with $\mathcal{U} \subset M$ open, then $(\mathcal{U}, f)$ is compatible with $\mathcal{A}$ if $\phi_i \circ f^{-1} : f(\mathcal{U} \cap \mathcal{U}_i) \to \phi(\mathcal{U} \cap \mathcal{U}_i)$ is C^k and $f \circ \phi_i^{-1}$ is C^k for all $i \in I$.*

Theorem 2.5*.** *If $\mathcal{A} = \{(\mathcal{U}_i, \phi_i) | i \in I\}$ is a C^k atlas for M^n then $\mathcal{A}$ is contained in a unique maximal atlas for M where atlases are ordered as sets by containment.*

Proof. Let

$$\mathcal{M} = \{(\mathcal{V}_\alpha, f_\alpha) | \mathcal{V}_\alpha \subset M \text{ is open}, f_\alpha : \mathcal{V}_\alpha \to \mathbf{R}^n \text{ is a homeomorphism onto its image, and } (\mathcal{V}_\alpha, f_\alpha) \text{ is compatible with } \mathcal{A}\}.$$

Since $\mathcal{M}$ contains all compatible charts it is the unique maximal atlas if it is an atlas. We now show that it is an atlas. If $(\mathcal{V}_\alpha, f_\alpha)$ and $(\mathcal{V}_\beta, f_\beta)$ are in $\mathcal{M}$ then we must show that $f_\beta \circ f_\alpha^{-1} : f_\alpha(\mathcal{V}_\alpha \cap \mathcal{V}_\beta) \to f_\beta(\mathcal{V}_\alpha \cap \mathcal{V}_\beta)$ is C^k. Suppose $m \in \mathcal{V}_\alpha \cap \mathcal{V}_\beta$. Take $i \in I$ such that $m \in \mathcal{U}_i$. Then on $f_\alpha(\mathcal{V}_\alpha \cap \mathcal{V}_\beta \cap \mathcal{U}_i)$, $f_\beta \circ f_\alpha^{-1} = (f_\beta \circ \phi_i^{-1}) \circ (\phi_i \circ f_\alpha^{-1})$. Since f_β and f_α are compatible with $\mathcal{A}$, $f_\beta \circ \phi_i^{-1}$ and $\phi_i \circ f_\alpha^{-1}$ are C^k on open sets of $\mathbf{R}^n$ to open sets of $\mathbf{R}^n$. Therefore $f_\beta \circ f_\alpha^{-1}$ is C^k on its domain. □

Definition 2.6*.** *A maximal C^k atlas is called a C^k differential structure.*

Definition 2.7*.** *A C^k n-manifold is a topological n-manifold M along with a C^k differential structure $\mathcal{S}$. By Theorem 2.5***, a single atlas is enough to determine the differential structure.*

The reader should note that this definition for a C^0 structure agrees with the definition of a topological manifold. A C^∞ n-manifold is also called a smooth manifold. **The word "manifold," without other adjectives, will denote a smooth manifold** as these will be the subject of the remainder of this manuscript.

Usually the notation for the structure $\mathcal{S}$ is suppressed. However, the phrase "a manifold M" supposes that there is an unnamed differential structure $\mathcal{S}$ in the background. In particular this means that if $\mathcal{A}$ is an atlas, then $\mathcal{A} \subset \mathcal{S}$; and, if (U, ϕ) is a chart, then $(U, \phi) \in \mathcal{S}$. The differential structure contains all compatible charts. For example, if $(U, \phi) \in \mathcal{S}$ and $V \subset U$ is open, then $(V, \phi|_V) \in \mathcal{S}$.

Example 2.8.a*.** *A real n-dimensional vector space is an n-manifold.*

Let V be an n-dimensional vector space. Pick an ordered basis $v_1, \cdots, v_n$ and define charts $(V, f_{v_1,\cdots,v_n})$, where $f_{v_1,\cdots,v_n}(\sum_{i=1}^n a_i v_i) = (a_1, \cdots, a_n)$ is an isomorphism of V to $\mathbf{R}^n$. The image is all of $\mathbf{R}^n$, which is an open subset of itself. These charts are compatible since $f_{w_1,\cdots,w_n} f_{v_1,\cdots,v_n}^{-1}$ is a linear automorphism on $\mathbf{R}^n$.

Recall that every real vector space is isomorphic to $\mathbf{R}^n$, however $\mathbf{R}^n$ comes equiped with a standard ordered basis or set of coordinates. A real vector space may appear in several guises. We now mention a few instances of vector spaces that relate to matrices.

Example 2.8.b*.** *Let the set of $n \times n$ real matrices be denoted $Mat_{n\times n}$. The set $Mat_{n\times n}$ is a real vector space of dimension n^2.*

Let $e_1, e_2, \cdots, e_m$ be the standard ordered basis for $\mathbf{R}^m$. If e_{ij} is the matrix with 1 in the i-th row and j-th column and zero elsewhere, then $\{e_{ij} \mid 1 \le i, j \le n\}$ is a basis. The isomorphism of $\mathrm{Mat}_{n\times n}$ to $\mathbf{R}^{n^2}$ is determined by the linear map $f(e_{ij}) = e_{n(i-1)+j}$ for $1 \le i, j \le n$, and allows one to think of the matrix entries as coordinates in $\mathbf{R}^{n^2}$. In coordinates, if $A = (a_{ij})$, then $f(A) = (a_{11}, a_{12}, \cdots, a_{1n}, a_{21}, \cdots, a_{nn})$.

Example 2.8.c*.** *The symmetric matrices* $Sym_{n\times n} = \{A \in Mat_{n\times n} \mid A^t = A\}$ *form a vector subspace of dimension* $\dfrac{n(n+1)}{2}$.

The entries on and above the diagonal can be arbitrary. Below the diagonal, the entries are determined by symmetry, i.e., $a_{ij} = a_{ji}$.

Example 2.8.d*.** *The skew symmetric matrices* $Skew_{n\times n} = \{A \in Mat_{n\times n} \mid A^t = -A\}$ *form a vector subspace of dimension* $\dfrac{n(n-1)}{2}$.

The entries on the diagonal must be zero, since $a_{ij} = -a_{ji}$. The entries above the diagonal can be arbitrary and the entries below are determined by the antisymmetry, i.e., $a_{ij} = -a_{ji}$.

Example 2.9a*.** *The sphere* $S^n = \{x \in \mathbf{R}^{n+1} \mid |x| = 1\}$ *is an* n*-manifold.*

We construct an atlas $\{(U_1, \phi_1), (U_2, \phi_2)\}$ with the aid of a standard well-known map called stereographic projection. Let $U_1 = S^n \backslash \{(0, \cdots, 0, 1)\}$ and $U_2 = S^n \backslash \{(0, \cdots, 0, -1)\}$. Note that $U_1 \cup U_2 = S^n$. Let $\phi_1(x_1, x_2, \cdots, x_{n+1}) = (\frac{x_1}{1-x_{n+1}}, \cdots, \frac{x_n}{1-x_{n+1}})$. The map $\phi_1 : U_1 \to \mathbf{R}^n$ is called stereographic projection. The inverse map $\phi_1^{-1} : \mathbf{R}^n \to U_1$ is defined by

$$\phi_1^{-1}(y_1, \cdots, y_n) = \left(\frac{2y_1}{\sum_{i=1}^n y_i^2 + 1}, \frac{2y_2}{\sum_{i=1}^n y_i^2 + 1}, \cdots, \frac{2y_n}{\sum_{i=1}^n y_i^2 + 1}, 1 - \frac{2}{\sum_{i=1}^n y_i^2 + 1} \right).$$

Both ϕ_1 and ϕ_1^{-1} are continuous and hence ϕ_1 is a homeomorphism.

The second coordinate chart (U_2, ϕ_2), stereographic projection from the south pole, is given by $\phi_2 = -\phi_1 \circ (-1)$ where (-1) is multiplication by -1 on the sphere. Since multiplication by -1 is a homeomorphism of the sphere to itself (its inverse is itself), the map $\phi_2 : U_2 \to \mathbf{R}^n$ is a homeomorphism.

Checking the compatability conditions, we have $\phi_2 \circ \phi_1^{-1}(y_1, \cdots, y_n) = \dfrac{1}{\sum_{i=1}^n y_i^2}(y_1, \cdots, y_n)$ and $\phi_2 \circ \phi_1^{-1} = \phi_1 \circ \phi_2^{-1}$. Hence, S^n is shown to be an n-manifold.

Example 2.9b*.** *Another atlas for the sphere* S^n.

We use $2(n+1)$ coordinate charts to construct this atlas. For each $i \in \{1, \cdots, n+1\}$ let $U_{i,+} = \{(x_1, \cdots, x_{n+1}) \mid x_i > 0\}$ and $U_{i,-} = \{(x_1, \cdots, x_{n+1}) \mid x_i > 0\}$. Define $\phi_{i,+} : U_{i,+} \to \mathbf{R}^n$ by $\phi_{i,+}(x_1, \cdots, x_{n+1}) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1})$ and $\phi_{i,+} : U_{i,-} \to \mathbf{R}^n$ by $\phi_{i,-}(x_1, \cdots, x_{n+1}) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1})$. The coordinate x_i is a function of $x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1}$ on the sets $U_{i,+}$ and $U_{i,-}$.

The atlases in Examples 2.9a*** and 2.9b*** are compatible and give the same differential structure. See Exercise 1***.

Example 2.10*.** *Suppose* $U_1 \subset \mathbf{R}^n$ *and* $U_2 \subset \mathbf{R}^m$ *are open sets. If* $f : U_1 \to U_2$ *is a* C^∞ *function, then the graph of f,*

$$G_f = \{(x, y) \in \mathbf{R}^{n+m} \mid y = f(x)\}$$

is a manifold.

There is only one coordinate neighborhood required. Let $\pi : U_1 \times U_2 \to U_1$ be the projection $\pi(x, y) = x$ and let $i_f : U_1 \to U_1 \times U_2$ be defined by $i_f(x) = (x, f(x))$. The one coordinate neighborhood is $(G_f, \pi|_{G_f})$. Both π and i_f are C^∞ maps. The composites are $\pi|_{G_f} \circ i_f = I_{U_1}$ and $i_f \circ \pi|_{G_f} = I_{G_f}$. Hence, $\pi|_{G_f}$ is a homeomorphism.

Proposition 2.11*.** *An open subset of an n-manifold is an n-manifold.*

Proof. Suppose M is an n-manifold and $U \subset M$ is an open subset. If (ϕ, V) is a chart of M, then $(\phi|_{V\cap U}, V \cap U)$ is a chart for U. □

While the next example uses Proposition 2.11***, it is just an open subset of $\mathbf{R}^{n^2}$.

Example 2.12*.** $GL(n, \mathbf{R}) = \{n \times n$ *nonsingular matrices*$\}$ *is an n^2-manifold.*

Consider the function $\det : \mathrm{Mat}_{n\times n} \to \mathbf{R}$. We use the usual coordinates on $\mathrm{Mat}_{n\times n}$, the entries as was described in the first example. In these coordinates,

$$\det(A) = \sum_{\sigma\in S_n} (-1)^{\mathrm{sign}\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

This function is an n-th degree polynomial. Hence, it is a continuous map. The set $\det^{-1}(\mathbf{R} \setminus 0)$ is an open set, the set of nonsingular matrices, $GL(n, \mathbf{R})$.

If a vector space V does not have a natural choice of basis or we do not wish to focus on the choice of basis, then we write its set of automorphisms as $GL(V)$. By picking a basis for V, it becomes $GL(n, \mathbf{R})$.

It is useful to see that the basic operations in $GL(n, \mathbf{R})$ are C^∞.

Proposition 2.13*.** *The following maps are C^∞:*

(1) $GL(n, \mathbf{R}) \times GL(n, \mathbf{R}) \to GL(n, \mathbf{R})$ *by* $(A, B) \mapsto AB$
(2) $GL(n, \mathbf{R}) \to GL(n, \mathbf{R})$ *by* $A \mapsto A^T$
(3) $GL(n, \mathbf{R}) \to GL(n, \mathbf{R})$ *by* $A \mapsto A^{-1}$

Proof. Recall that $GL(n, \mathbf{R})$ is an open subset of $\mathbf{R}^{n^2}$ and that $GL(n, \mathbf{R}) \times GL(n, \mathbf{R})$ is an open subset of $\mathbf{R}^{n^2} \times \mathbf{R}^{n^2}$. The notion of C^∞ is from advanced Calculus.

The first map, multiplication, is a quadratic polynomial in each coordinate as the i, j entry of AB is $\sum_{k=1}^{n} a_{ik} b_{kj}$.

The second map, transpose, is just a reordering of coordinates. In fact, transpose is a linear map.

The third map, inverse, is a rational function of the entries of A. The numerator is the determinant of a minor of A and the denominator is $\det A$, a polynomial that is nonzero on $GL(n, \mathbf{R})$. □

Proposition 2.14*.** *The product of an n-manifold and an m-manifold is an $(n + m)$-manifold.*

Proof. Suppose M is an m-manifold with atlas $\mathcal{A}_M = \{(U_i, \phi_i) \mid i \in I\}$ and N is an n-manifold with atlas $\mathcal{A}_N = \{(V_j, \psi_j) \mid j \in J\}$. An atlas for $M \times N$ is

$$\mathcal{A} = \{(U_i \times V_j, \phi_i \times \psi_j) \mid i \in I, j \in J\},$$

where $\phi_i \times \psi_j(x,y) = (\phi_i(x), \psi_j(y)) \in \mathbf{R}^n \times \mathbf{R}^m$ for $(x,y) \in U_i \times V_j$.

It is easy to see that $\mathcal{A}$ is an atlas, since the union $\bigcup_{(i,j)\in I\times J} U_i \times V_j = M \times N$, and it is easy to check compatibility. If $(U_{i_1}, \phi_{i_1}), (U_{i_2}, \phi_{i_2}) \in \mathcal{A}_M$ and $(V_{j_1}, \psi_{j_1}), (V_{j_2}, \psi_{j_2}) \in \mathcal{A}_N$, then

$$(\phi_{i_1} \times \psi_{j_1}) \circ (\phi_{i_2} \times \psi_{j_2})^{-1} = (\phi_{i_1} \circ (\phi_{i_2})^{-1}) \times (\psi_{i_1} \circ (\psi_{i_2})^{-1}) \tag{1}$$

on the set $(\phi_{i_2} \times \psi_{j_2})((U_{i_1} \times V_{j_1}) \cap (U_{i_2} \times V_{j_2})) = \phi_{i_2}(U_{i_1} \cap U_{i_2}) \times \psi_{i_2}(V_{j_1} \cap V_{j_2})$ an open set in $\mathbf{R}^m \times \mathbf{R}^n$. Since $\mathcal{A}_M$ and $\mathcal{A}_N$ are atlases, the right side of (1) is C^∞, hence the left side is also, and $\mathcal{A}$ is an atlas. $\square$

When can a manifold be pieced together from abstract charts? The question is of philosophical and practical interest.

Theorem 2.15*.** *Let X be a set. Suppose $\mathcal{A} = \{(\mathcal{U}_i, \phi_i) | i \in I\}$ satisfies*

(1) *$\mathcal{U}_i \subset X$ for each $i \in I$.*
(2) *$\bigcup_{i\in I} \mathcal{U}_i = X$.*
(3) *$\phi_i : \mathcal{U}_i \to \phi_i(\mathcal{U}_i) \subset \mathbf{R}^n$ is a bijection for all i.*
(4) *$\phi_i(\mathcal{U}_i),\ \phi_i(\mathcal{U}_i \cap \mathcal{U}_j) \subset \mathbf{R}^n$ are open for all $i, j \in I$.*
(5) *$\phi_j \phi_i^{-1} : \phi_i(\mathcal{U}_i \cap \mathcal{U}_j) \to \phi_j(\mathcal{U}_i \cap \mathcal{U}_j)$ is C^∞ for all $i, j \in I$.*

Then there is a unique topology on X such that each $\mathcal{U}_i$ is open and each ϕ_i is a homeomorphism. If the topology is second countable and Hausdorff then X is an n-manifold and $\mathcal{A}$ is an atlas.

Remark 2.16*.** *If for every $x, y \in X$ there are $i, j \in I$ with $x \in \mathcal{U}_i$, $y \in \mathcal{U}_j$ and $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$ or there is an $i \in I$ with $x, y \in \mathcal{U}_i$, then X is Hausdorff.*

If the open cover $\{\mathcal{U}_i | i \in I\}$ has a countable subcover, then X is second countable since the countable union of second countable spaces is a second countable space.

*Proof of Theorem 2.15***.* For each $x \in M$ we give a neighborhood basis. Induce a topology on $\mathcal{U}_i$ by taking $\mathcal{O} \subset \mathcal{U}_i$ is open if and only if $\mathcal{O} = \phi_i^{-1}(W)$ for $W \subset \mathbf{R}^n$ open. The map ϕ_i is then a homeormorphism. We must check that this gives a well defined neighborhood basis for a topology. Suppose $x \in \mathcal{U}_i \cap \mathcal{U}_j$. The neighborhood basis is then defined by both ϕ_i and ϕ_j. Since $\phi_j \phi_i^{-1}$ is a homeomorphism of open sets, $\phi_j = (\phi_j \phi_i^{-1}) \circ \phi_i$ defines a neighborhood basis of x in a manner consistent with ϕ_i. This construction defines a neighborhood basis of each point and so a topology on X. This is the only topology with each ϕ_i a homeomorphism as the basis determines the topology. We now see that $\mathcal{A}$ satisfies the conditions for an atlas: $\{\mathcal{U}_i | i \in I\}$ is an open cover by 2, ϕ_i is a homeomorphism by 3 and the construction above, and the compatibility condition is 5. If X is Hausdorff and second countable, then X is an n-manifold. $\square$

Now that we know the definition of a manifold, the next basic concept is a map between manifolds.

Definition 2.17*.** *Suppose $f : M^m \to N^n$ is a function between manifolds. If for all charts $(\mathcal{U}, \phi)$ and $(\mathcal{W}, \psi)$ in the differential structures of M and N respectively, $\psi \circ f \circ \phi^{-1}$ is C^∞ on its domain, then f is a smooth map or function.*

The reader should note that Definition 2.17*** agrees with the notion of a C^∞ map of functions from Calculus, i.e., if $O \subset \mathbf{R}^m$, then $f : O \to \mathbf{R}^n$ is C^∞ as defined in Calculus if and only if it is smooth as a map between manifolds.

In order to check if a function between manifolds is smooth, one does not have to check every chart in a differential structure. It is enough to check one chart about each point as is proved in the following theorem.

Proposition 2.18*.** *Let $f : M^m \to N^n$ be a function between manifolds. Further suppose that for each $x \in M$ there are charts $(\mathcal{U}_x, \phi_x)$ about x for M and $(\mathcal{W}_{f(x)}, \psi_{f(x)})$ about $f(x)$ for N such that $\psi_{f(x)} \circ f \circ \phi_x^{-1}$ is C^∞ on its domain. Then f is smooth.*

Proof. Suppose that (U, ϕ) and (W, ψ) are in the differential structure for M and N respectively. We wish to show that $\psi \circ f \circ \phi^{-1}$ is C^∞ on its domain, $\phi(f^{-1}(W) \cap U)$. Take a point in the domain of $\psi \circ f \circ \phi^{-1}$, say $\phi(x)$ for $x \in f^{-1}(W) \cap U$. Then $\phi(f^{-1}(W) \cap U \cap f^{-1}(W_{f(x)}) \cap U_x)$ is an open neighborhood of $\phi(x)$ in $\mathbf{R}^m$, and on this open set,

$$\psi \circ f \circ \phi^{-1} = (\psi \circ \psi_{f(x)}^{-1}) \circ (\psi_{f(x)} \circ f \circ \phi_x^{-1}) \circ (\phi_x \circ \phi^{-1})$$

The compositions in parentheses are maps between real spaces. The first and third are C^∞ since $(W_{f(x)}, \psi_{f(x)})$, (W, ψ) and (U_x, ϕ_x), (U, ϕ) are compatible pairs of charts. The second composition is C^∞ by the hypothesis of the theorem. Hence, $\psi \circ f \circ \phi^{-1}$ is C^∞, so by Definition 2.17***, f is smooth. □

Proposition 2.19*.** *The composition of smooth functions is a smooth function. Suppose $f : M^m \to N^n$ and $g : N^n \to K^k$ are smooth functions between manifolds. Then $g \circ f : M^m \to K^k$ is a smooth function.*

Proof. Suppose $p \in M$. Suppose that (U, ϕ), (V, ψ), and (W, φ) are chart on M, N, K respectively; and $p \in U$, $f(p) \in V$, and $g(f(p)) \in W$. The composite $\varphi \circ g \circ f \circ \phi^{-1}$ is defined on a neighborhood of p, and we need to show that this composite is C^∞ on some neighborhood of p. The composite function

$$\varphi \circ g \circ \psi^{-1} \circ \psi \circ f \circ \phi^{-1} = \varphi \circ g \circ f \circ \phi^{-1}$$

on some neighborhood of p. Note that $(\varphi \circ g \circ \psi^{-1}) \circ (\psi \circ f \circ \phi^{-1})$ is the same function and it is C^∞ since each of the functions in parentheses is C^∞ because both f and g are smooth. □

The notion of equivalence between differential manifolds is diffeomorphism.

Definition 2.20*.** *Suppose M and N are differential manifolds. If there is a smooth map $f : M \to N$ with a smooth inverse $f^{-1} : N \to M$, then*

(1) *f is called a diffeomorphism, and*
(2) *M and N are diffeomorphic.*

We make the following simple but useful observation.

Proposition 2.21*.** *Suppose that M is an n-manifold, $U \subset M$, $\phi : U \to \mathbf{R}^n$, and $\phi(U)$ is open in $\mathbf{R}^n$. The pair (U, ϕ) is a chart of the manifold M if and only if ϕ is a diffeomorphism.*

Proposition 2.22*.** *If M is an n-manifold, then the set of diffeomorphisms of M is a group under composition.*

Let $\mathrm{Diff}(M)$ denote the group of diffeomorphisms of M.

Proof. The identity map on M, the map I_M, is a smooth map, which we easily check. If $(U.\psi)$ is any coordinate chart for M, then $\phi \circ I_M \circ \phi^{-1}$ is just the identity on $\phi(U) \subset \mathbf{R}^n$. By Proposition 2.18***, I_M is smooth.

A diffeomorphism f has a smooth inverse by its definition.

The composition of two diffeomorphisms is again a diffeomorphism. If f and g are diffeomorphisms, the composition is smooth by Proposition 2.19***. The inverse of $f \circ g$ is $g^{-1} \circ f^{-1}$.

Finally note that the composition of functions is associative. Hence, $\mathrm{Diff}(M)$ is a group. □

We now consider a method of constructing manifolds. Those students who have already learned of covering spaces will recognize the construction, although no background is assumed in these notes.

The construction involves a subgroup G of the diffeomorphisms of an n-manifold M. We can consider the quotient space M/G defined by the equivalence relation on M that $x \sim y$ if and only if $y = g(x)$ for some $g \in G$. Notice that $\sim$ is an equivalence relation so that M/G makes sense as a topological space:

(1) If $x \sim x$ since $x = I_M(x)$
(2) If $x \sim y$ then $y = g(x)$ and $x = g^{-1}(y)$, so $y \sim x$
(3) If $x \sim y$ and $y \sim z$, then $y = g(x)$ and $z = h(y)$, so $z = h \circ g(x)$ and $x \sim z$

We know that M/G is a topological space under the quotient topology and that the quotient map $\pi : M \to M/G$ is continuous. Can we guarantee it is a manifold? In general, the answer is no, but we do have the following theorem.

Theorem 2.23*.** *Suppose M is an n-manifold, and G is a finite subgroup of $\mathrm{Diff}(M)$. Suppose that G satisfies one of the following, either (a) or (b):*
(a) If for some $x \in M$ and $g \in G$, $g(x) = x$, then g is the identity, or
(b) There is atlas $\mathcal{A}$ for M such that if $(U, \phi) \in \mathcal{A}$,
(1) *then $(g(U), \phi \circ g^{-1}) \in \mathcal{A}$, and*
(2) *$h(U) \cap g(U) = \emptyset$ for all $g, h \in G$, $g \neq h$.*

Then M/G is an n-manifold and the quotient map $\pi : M \to M/G$ is a smooth map.

The condition "if for some $x \in M$ and $g \in G$, $g(x) = x$, then g is the identity," says that no g, other that the identity can fix a point. The group G is said to operate without fixed points. Before proving Theorem 2.23***, we first prove the following lemma.

Lemma 2.24*.** *Suppose M is an n-manifold, and G is a finite subgroup of $\mathrm{Diff}(M)$. Suppose that G satisfies the following property: If for some $x \in M$ and $g \in G$, $g(x) = x$, then g is the identity. Then there is atlas $\mathcal{A}$ for M such that if $(U, \phi) \in \mathcal{A}$,*
(1) *then $(g(U), \phi \circ g^{-1}) \in \mathcal{A}$, and*

(2) $h(U) \cap g(U) = \emptyset$ *for all* $g, h \in G$, $g \neq h$.

Proof. We first show that for every $g \in G$ and $x \in M$ there is an open set O_g, a neighborhood of $g(x)$, such that

$$\text{if } g, h \in G \text{ and } g \neq h \text{ then } O_g \cap O_h = \emptyset. \tag{1}$$

Fix $x \in M$. For each pair $g(x), h(x)$ there are open sets O_{gh} and neighborhood of $g(x)$ and O_{hg} and neighborhood of $h(x)$ such that $O_{gh} \cap O_{hg} = \emptyset$. These sets exist since M is Hausdorff. Let $O_g = \bigcap_{h \in G} O_{gh}$. The intersection is finite, so O_g is open. Given $g, h \in G$, $g \neq h$, then $O_g \subset O_{gh}$ and $O_h \subset O_{hg}$, so

$$O_g \cap O_h \subset O_{gh} \cap O_{hg} = \emptyset.$$

The next step is to show there is a neighborhood of x, call it O, such that

$$g(O) \cap h(O) = \emptyset \text{ for any } g \neq h.$$

We have $g(x) \in O_g$, so $g^{-1}(O_g)$ is a neighborhood of x. Let

$$O = \bigcap_{g \in G} g^{-1}(O_g).$$

Now, $g(O) \subset g(g^{-1}(O_g)) = O_g$, and therefore, $g(O) \cap h(O) \subset O_g \cap O_h = \emptyset$.

We now produce the atlas. Take a chart (U_x, ϕ_x) about x with $U_x \subset O$. Then $(g(U_x), \phi_x \circ g^{-1})$ is a chart about $g(x)$. Note that $h(U) \cap g(U) \subset h(O) \cap g(O) = \emptyset$. Hence, the atlas $\{(g(U_x), \phi_x \circ g^{-1}) \mid x \in M, g \in G\}$ satisfies the required properties. □

*Proof of Theorem 2.23***.* We first construct the coordinate charts. Take $\mathcal{A}$ an atlas that satisfies the two items in Lemma 2.24*** and take $(U, \phi) \in \mathcal{A}$. The quotient map π is a continuous and open map. By the second item in Lemma 2.24***, π is also one-to-one on U. Therefore $\pi|_U$ is a homeomorphism. Denote its inverse by $i_U : \pi(U) \to U$. Let $\Phi = \phi \circ i_U$, then the pair $(\pi(U), \Phi)$ is a coordinate chart for M/G.

We have constructed charts about any point and it remains to show that these charts are compatible. Suppose $z \in M/G$ and $z \in \pi(V) \cap \pi(U)$ where $(V, \psi) \in \mathcal{A}$ and $(\pi(V), \Psi)$ is a chart of M/G constructed as in the previous paragraph. We wish to show that

$$\Psi \circ \Phi^{-1} : \Phi(\pi(U) \cap \pi(V)) \to \Psi(\pi(U) \cap \pi(V))$$

is a diffeomorphism. Since it as an inverse (of the same form) we only have to show it is C^∞. Now, for some $y \in M$, $\pi^{-1}(z) = \{g(y) \mid g \in G\}$ a set of $|G|$ points. There are two group elements $g, h \in G$ with $g(y) \in V$ and $h(y) \in U$. Let O be a neighborhood of $h(y)$ in $hg^{-1}(V) \cap U$. A neighborhood of $g(y)$ in $V \cap gh^{-1}(U)$ is $gh^{-1}(O)$. The map $\Psi \circ \Phi^{-1}$ on the open set $\pi(O)$ is

$$\begin{aligned} \Psi \circ \Phi^{-1} &= (\psi \circ i_V)(\phi \circ i_U)^{-1} \\ &= \psi \circ (i_V \circ \pi|_U) \circ \phi^{-1} \\ &= \psi \circ (gh^{-1}) \circ \phi^{-1} \end{aligned} \tag{1}$$

since $gh^{-1} : O \to gh^{-1}(O)$ is a diffeomorphism, so is the composite (1). This completes the proof that M/G is a manifold.

That π is a smooth map is almost a tautology. Given $x \in M$ take $(\pi(U), i_U)$ a chart in $\mathcal{A}$ about x and $(\pi(U), \Phi)$ the chart about $\pi(x)$ constructed above in the first paragraph. The map which we are required to show is C^∞ is $\Phi \circ \pi \circ \phi^{-1} = \phi \circ (i_U \circ \pi|_U) \circ \phi^{-1}$ which is just the identity on $\phi(U)$. Therefore π is smooth. □

We give two examples.

Example 2.25*.** *Real Projective Space* $\mathbf{RP}^n$.

The n-sphere, $S^n \subset \mathbf{R}^{n+1}$ admits the action of the group $\mathbf{Z}_2 = \{1, -1\}$. Multiplication by 1 is the identity which is smooth, Theorem 2.22***. We check -1 using the charts from Example 2.9b***. Multiplication by -1 is a map

$$-1 : U_{i,+} \to U_{i,-} \text{ and } -1 : U_{i,-} \to U_{i,+}.$$

Now, both $\phi_{i,+} \circ (-1) \circ \phi_{i,-}^{-1}$ and $\phi_{i,-} \circ (-1) \circ \phi_{i,+}^{-1}$ are the same as maps from $B_1(0) \to B_1(0)$. They are $(y_1, \cdots, y_n) \mapsto (-y_1, \cdots, -y_n)$ which is C^∞ on $\mathbf{R}^n$. Hence, $\{1, -1\} \subset \text{Diff}(M)$. Therefore, by Theorem 2.23***, $S^n/\mathbf{Z}_2$ is an n-manifold. It is called real projective space. Denote it $\mathbf{RP}^n$.

Let $q : S^n \to \mathbf{RP}^n$ be the quotient map. Notice that if $U \subset S^n$ that is entirely in some hemisphere of S^n then $q(U)$ can serve as a neighborhood for a chart since $U \cap -U = \emptyset$.

Example 2.26*.** *Configuration Space.*

Imagine that n-particles are moving in $\mathbf{R}^m$. These particles are ideal particles in that they are points, i.e., they have no diameter. What are the possible arrangements? If the particles are labeled, they we can write the arrangement as an n-tuple, $(x_1, x_2, \cdots, x_n)$. Since no two particles can occupy the same point in space, the manifold that describes such arrangements is $\{(x_1, x_2, \cdots, x_n) \mid x_i \neq x_j \text{ for } i \neq j\}$, i.e., $\mathbf{R}^{mn}$ with several subspaces removed. It is a manifold since it is an open subset of $\mathbf{R}^{mn}$. It is ordered configuration space. Denote it C_{nm}

If the n-particles are not labeled, then we can only know the location of the n pariciles and not which particle occupies which location, e.g., we cannot distinguish between (x, y, z) and (y, x, z). The symmetric group S_n acts on C_{nm} by $\sigma \cdot (x_1, x_2, \cdots, x_n) = (x_{\sigma^{-1}1}, x_{\sigma^{-1}2}, \cdots, x_{\sigma^{-1}n})$. The space C_{nm}/S_n is configuration space.

We show that C_{nm}/S_n is an mn-manifold. We use Theorem 2.23***. The manifold M is C_{nm} and the subgroup of the diffeomorphisms is S_n the permutation group. Each permutation is the restriction of a linear map on $\mathbf{R}^{mn}$ to C_{nm}, and so each permutation is C^∞, i.e., smooth. The inverse of a permutation is again a permutation. Hence, S_n is a subgroup of Diff(C_{nm}). If σ fixes $\mathbf{x} = (x_1, \cdots, x_n)$, then $x_i = x_{\sigma(i)}$ for $i = 1, \cdots, n$. If $\sigma(i) = j$ and $j \neq i$, then $x_i = x_j$ for an $i \neq j$. Therefore $\mathbf{x}$ is not in C_{nm}. If $\sigma(i) = i$ for all $i = 1, \cdots, n$, then σ is the identity. We can now apply Theorem 2.23***. By Theorem 2.23***, C_{nm}/S_n is an nm-manifold.

In physics, the term configuration space is used for to describe the space of physical configurations of a mechanical system. If $m = 3$, then Example 2.26*** is the configuration space of n particles in ordinary 3-dimensional space.

Exercises

Exercise 1*.** *Verify the calculations of Example 2.9a***. Show the two atlases given for S^n in Example 2.9a*** and Example 2.9b*** give the same differential structure and so may be merged.*

Exercise 2*.** *$S^1 \times S^1$ is a 2-manifold, $S^2 \times S^1$ is a 3-manifold,and $S^2 \times S^1 \times S^1$ is a 4-manifold.*

Of course these all follow from Proposition 2.13***. The reader should note, however, that there is an ambiguity in $S^2 \times S^1 \times S^1$, is it $(S^2 \times S^1) \times S^1$ or $S^2 \times (S^1 \times S^1)$? The reader should show that the atlases are compatible and so these are the same manifold.

There is also a second approach that is sometimes used to define smooth functions. In this approach, one first defines a smooth function for $f : M \to \mathbf{R}$ only. The statement of the next exercise would be a defintion in some textbooks, e.g., Warner and Helgason, but for us, it is a proposition.

Exercise 3*.** *Show that a function $f : M^m \to N^n$ between manifolds is smooth if and only if for all open sets $\mathcal{U} \subset N$ and all smooth functions $g : \mathcal{U} \to \mathbf{R}$, $g \circ f$ is smooth on its domain.*

Exercise 4*.** *Consider $\mathbf{R}$ with the following three atlases:*

(1) $\mathcal{A}_1 = \{f \mid f(x) = x\}$
(2) $\mathcal{A}_2 = \{f \mid f(x) = x^3\}$
(3) $\mathcal{A}_3 = \{f \mid f(x) = x^3 + x\}$

Which of these atlases determines the same differential structure. Which of the manifolds are diffeomorphic?

Exercise 5*.** *Let M, N, and Q be manifolds.*

(1) *Show that the projections $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ are smooth.*
(2) *Show that $f : Q \to M \times N$ is smooth iff $\pi_i f$ is smooth for $i = 1, 2$.*
(3) *Show for $b \in N$ that the inclusion $x \mapsto (x, b) : M \to M \times N$ is smooth.*

The following is a difficult exercise.

Exercise 6*.** *Prove that the set of all $n \times n$ matrices of rank k (where $k < n$) is a smooth manifold. What is its dimension?*

If this is too hard, then prove that the set of all $n \times n$ matrices of rank 1 is a smooth manifold of dimension $2n - 1$.

Warmup Exercises, Chapter 2

Exercise 1*. *Suppose that* $\{(U_i, \phi) : i \in I\}$ *is an atlas for* M*. Argue that a chart* (V, ψ) *is compatible with the atlas if for each* $x \in V$*, there exists an open set* W*,* $x \in W \subseteq V$ *and an* $i_x \in I$ *such that* $\phi_{i_x} \circ (\psi|_W)^{-1}$ *and* $\psi|_W \circ \phi_{i_x}^{-1}$ *are* C^∞*.*

Exercise 2*. *Suppose that* $\{(U_i, \phi) : i \in I\}$ *is an atlas for* M*,* $J \subseteq I$ *and* $\bigcup_{i \in J} U_i = M$*. Argue that* $\{(U_i, \phi) : i \in I\}$ *is an atlas that generates the same differentiable structure on* M*.*

Exercise 3*. *Let* $\phi : U \to \mathbf{R}^n$ *be a chart for a smooth manifold* M *and let* V *be a nonempty open subset of* U*. Argue that* $\phi|_V : V \to \mathbf{R}^n$ *is also a chart in the differentiable structure of* M*.*

Exercise 4*. *Let* U *be a nonempty open subset of a manifold* M*. Show that the charts of* M *with domain contained in* U *form a differentiable structure on* U*. Show that the restriction of any chart on* M *to* U *belongs to this differentiable structure.*

Exercise 5*. *Prove Proposition 2.21***.*