CHAPTER 3
SUBMANIFOLDS

One basic notion that was not addressed in Chapter 1 is the concept of containment: When one manifold is a subset on a second manifold. This notion is subtle in a manner the reader may have experienced in general topology. We give an example.

Example 3.1. *Let $\mathbf{Q}_d$ be the rational numbers with the discrete topology and $\mathbf{R}$ the usual real numbers. The inclusion $\iota : \mathbf{Q}_d \to \mathbf{R}$ is a one-to-one continuous map.*

Is $\mathbf{Q}_d$ a subspace of $\mathbf{R}$? The usual answer is no, it is only a subspace if ι is a homeomorphism to its image, $\iota(\mathbf{Q}_d) = \mathbf{Q}$. This map is not a homeomorphism to its image since the topology on $\mathbf{Q}$, the subspace topology, is not discrete. the same issue arises in manifold theory. In fact $\mathbf{Q}_d$ is a 0-dimensional manifold and ι is a smooth map. We, however, will not refer to $\mathbf{Q}_d$ as a submanifold of $\mathbf{R}$, i.e., submanifold is not quite the correct relationship of $\mathbf{Q}_d$ to $\mathbf{R}$. This relationship will be studied in Chapter 6*** (immersed submanifold). For the notion "N is a submanifold of M," we require that N inherits its differential structure from M. Some authors refer to this relationship as "embedded submanifold."

We give the definition of submanifold in Definition 3.2***. At first glance, it may appear to be overly restrictive, however, it turns out not to be the case. It is analogous to the notion of subspace topology. This assertion is justified in a later chapter, Theorem 6.3***.

Definition 3.2*.** *Suppose $m > n$ and write $\mathbf{R}^m = \mathbf{R}^n \times \mathbf{R}^{m-n}$. Let M be an m-manifold and $N \subset M$. Suppose that for each $x \in N$ there is a chart of M*

$$(U, \phi) \text{ centered at } x \text{ such that } \phi^{-1}(\mathbf{R}^n \times \{0\}) = U \cap N \tag{1}$$

Then N is an n-dimensional submanifold of M.

Charts that satisfy property (1) from Definition 3.2*** are called slice charts or slice coordinate neighborhoods.

We next observe that if $N \subset M$ is an n-dimensional submanifold, then N is an n-manifold.

Proposition 3.3*.** *Suppose that N is an n-dimensional submanifold of the m-manifold M. Then N is an n-manifold and*

$$\begin{aligned}\mathcal{A} = &\{(U_x \cap N, \phi|_{U \cap N}) \mid (U_x, \phi) \text{ is a chart of } M \text{ centered at } x \in N \text{ such} \\ &\quad \text{that } \phi^{-1}(\mathbf{R}^n \times \{0\}) = U \cap N\}\end{aligned}$$

is an atlas for N.

Proof. We first show that N is a topological manifold in the induced topology on N. The topology on N is the subspace topology. Therefore, N is Hausdorff and second countable

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since M is Hausdorff and second countable. Also since the topology on N is the subspace topology, $U_x \cap N$ is an open set in N. The set $\phi(U_x \cap N) = \phi(U_x) \cap (\mathbf{R}^n \times \{0\})$, since (U_x, ϕ) is a slice neighborhood. The map ϕ is a homeomorphism to its image, so $\phi|_{U_x \cap N}$ is a homeomorphism to its image. Now, since there is a neighborhood $U_x \cap N$ around any $x \in N$, the requirements of Definition 2.1*** are shown and N is a topological manifold.

To show that N is an n-manifold, we must show that the charts satisfy the compatibility condition of Definition 2.3*** item 2. Suppose (U_y, ψ) is another coordinate neighborhood in the potential atlas $\mathcal{A}$. Then

$$\left(\phi|_{U_x \cap N}(\psi|_{U_y \cap N}))^{-1}\right)|_{\psi(U_x \cap U_y \cap N)} = (\phi\psi^{-1})|_{\psi(U_x \cap U_y \cap N)}$$

is C^∞ since it is the composition of the inclusion of $\mathbf{R}^n$ into $\mathbf{R}^m$ composed with $\phi\psi^{-1}$, two C^∞ functions. This completes the proof. $\square$

Theorem 3.4*.** *Suppose $\mathcal{O} \subset \mathbf{R}^{n+m}$ is an open set and $f : \mathcal{O} \to \mathbf{R}^m$ is a C^∞ map. Let $q \in \mathbf{R}^m$ and $M = f^{-1}(q)$. If $Df(x)$ has rank m for all $x \in M$, then M is an n-dimensional submanifold of $\mathbf{R}^{n+m}$.*

Proof. For each $x \in M$, there is a neighborhood of x, $\mathcal{U}_x \subset \mathbf{R}^{n+m}$; an open set $U_1 \subset \mathbf{R}^n$; an open set $U_2 \subset \mathbf{R}^m$; and a diffeomorphism $H : \mathcal{U}_x \to U_1 \times U_2$ such that $H(M \cap \mathcal{U}_x) = U_1 \times \{0\}$ as was laid out in the Rank Theorem. The pair $(\mathcal{U}_x, H)$ is a chart for the smooth manifold $\mathbf{R}^{n+m}$ since H is a diffeormorphism, Proposition 2.21***. This chart satisfies the required property given in Definition 3.2***. Hence, M is an n-dimensional submanifold of $\mathbf{R}^{n+m}$. $\square$

Example 3.5*.** *S^n is a submanifold of $\mathbf{R}^{n+1}$.*

Let $l : \mathbf{R}^{n+1} \to \mathbf{R}$ by $l(x_1, x_2, \cdots, x_{n+1}) = \sqrt{x_1^2 + \cdots x_{n+1}^2}$. Then $S^n = l^{-1}(1)$. The map l is C^∞ on $\mathbf{R}^{n+1} \setminus \mathbf{0}$. In fact, each partial derivative is a rational function of $x_1, \cdots, x_{n+1}$, and l, i.e., $\frac{\partial l}{\partial x_i} = \frac{x_i}{l}$. Therefore the partial derivative of a rational function of $x_1, \cdots, x_{n+1}, l$ is another such function and l is C^∞. To check that the rank of $D_{\mathbf{v}}l(\mathbf{x})$ is one, it is enough to show that some directional derivative is not zero. Hence, for $\mathbf{x} \in S^n$, we compute,

$$\begin{aligned} D_{\mathbf{x}}l(\mathbf{x}) &= \frac{d}{dt}\, l(\mathbf{x} + t\mathbf{x})|_{t=0} \\ &= \frac{d}{dt}\, (1+t)\sqrt{x_1^2 + \cdots x_{n+1}^2}\bigg|_{t=0} \\ &= \frac{d}{dt}\, (1+t)|_{t=0} \\ &= 1. \end{aligned}$$

Since l has rank 1 on S^n, Theorem 3.4*** applies.

Proposition 3.3*** along with the Rank Theorem*** gives instructions for computing an atlas. The atlas essentially comes from the Implicit Function Theorem. The charts include the atlas in Example 2.9b***. The reader should check this fact.

Example 3.6*.** *Let* $SL(n,\mathbf{R}) = \{A \in Mat_{n\times n} \mid detA = 1\}$. *Then*

(1) *If* $A \in SL(n,\mathbf{R})$, *then* $D det(A)$ *has rank 1.*
(2) $SL(n,\mathbf{R})$ *is an* (n^2-1)*-manifold.*

First, $\det : \mathbf{R}^{n^2} \to \mathbf{R}$ is a polynomial (as shown in Example 2.12***) and so C^∞. To show that $D\det(A)$ has rank 1 it is only necessary to show that some directional derivative $D_B\det(A)$ is nonzero. We compute for $A \in SL(n,\mathbf{R})$,

$$\begin{aligned} D_A\det(A) &= \frac{d}{dt}\ \det(A+tA)\big|_{t=0} \\ &= \frac{d}{dt}\ (1+t)^n\det(A)\big|_{t=0} \\ &= \frac{d}{dt}\ (1+t)^n\big|_{t=0} \\ &= n. \end{aligned}$$

Hence item 1 is shown.

To see item 2, we use Theorem 3.4***. The set $GL(n,\mathbf{R}) \subset \mathbf{R}^{n^2-1}\times\mathbf{R}$ is an open set and $\det : GL(n,\mathbf{R}) \to \mathbf{R}$ is a C^∞ map. The set $\det^{-1}(1) = SL(n,\mathbf{R})$, and $D\det(A)$ has rank 1 for each $A \in SL(n,\mathbf{R})$. Therefore, by Theorem 3.4***, $SL(n,\mathbf{R})$ is an (n^2-1)-manifold.

Example 3.7*.** *Let* $O(n,\mathbf{R}) = \{A \in Mat_{n\times n} \mid AA^T = I\}$ *and let* $SO(n,\mathbf{R}) = \{A \in Mat_{n\times n} \mid AA^T = I$ *and* $detA = 1\}$. *Also let* $f : Mat_{n\times n} \to Mat_{n\times n}$ *by* $f(A) = AA^T$

(1) $f(A) \in Sym_{n\times n}$ *and* f *is* C^∞.
(2) *If* $A \in O(n,\mathbf{R})$, *then* $Df(A)$ *has rank* $\dfrac{n(n+1)}{2}$.
(3) $O(n,\mathbf{R})$ *and* $SO(n,\mathbf{R})$ *are* $\dfrac{n(n-1)}{2}$*-manifolds.*

First note that $AA^T = I$ implies that $(\det(A))^2 = 1$ so A is invertible.

The map f is a composition the identity cross the transpose and multiplication. These maps are C^∞ maps by Proposition 2.13***. Since f is the composition of C^∞ maps, it is C^∞. Since $f(A) = AA^T$ and $(AA^T)^T = AA^T$, $f(A) \in \mathrm{Sym}_{n\times n}$. This argument shows item 1.

To show item 2, it is enough to show that $Df(A)$ is surjective since $\dim\mathrm{Sym}_{n\times n} = \dfrac{n(n+1)}{2}$. Now,

$$\begin{aligned} Df(A)(M) =& \frac{d}{dt}\ (A+tM)(A+tM)^T\big|_{t=0} \\ &= AM^T + MA^T \\ &= AM^T + (AM^T)^T \end{aligned}$$

Hence $Df(A)$ is a composition of two maps

$$\begin{aligned} \mathrm{Mat}_{n\times n} &\to \mathrm{Mat}_{n\times n} \\ M &\mapsto MA^T \end{aligned} \quad \text{and} \quad \begin{aligned} \mathrm{Mat}_{n\times n} &\to \mathrm{Sym}_{n\times n} \\ X &\mapsto X + X^T \end{aligned}$$

The first map is onto if A^T is invertible, which it is if $A \in f^{-1}(I)$. The second map is onto, since, if $Y \in \mathrm{Sym}_{n\times n}$ then $Y = Y^T$ and $Y = \frac{1}{2}Y + (\frac{1}{2}Y)^T$, i.e., if $X = \frac{1}{2}Y$, then $Y = X + X^T$. Therefore, $Df(A)$ is a surjection and has rank $\dfrac{n(n+1)}{2}$.

The third item follows from Theorem 3.4***. Since $f^{-1}(I) = O(n,\mathbf{R})$, $O(n,\mathbf{R})$ is a manifold of dimension $n^2 - \dfrac{n(n+1)}{2} = \dfrac{n(n-1)}{2}$. The determinant function is continuous on $\mathrm{Mat}_{n\times n}$ and $O(n,\mathbf{R}) \subset \mathrm{Mat}_{n\times n}$. If $AA^T = I$, then $\det A = \pm 1$. Therefore $SO(n,\mathbf{R})$ is an open subset of $O(n,\mathbf{R})$ and hence a manifold of dimension $\dfrac{n(n-1)}{2}$, by Proposition 2.11***.

The next example requires notation and a lemma.

Lemma 3.8*.** *Suppose that n is even and let J be the $n \times n$ matrix*

$$J = \begin{pmatrix} 0_{n/2\times n/2} & -I_{n/2\times n/2} \\ I_{n/2\times n/2} & 0_{n/2\times n/2} \end{pmatrix}$$

Then $J^T = -J = J^{-1}$.

Proof. Just compute the transpose, the negative, and the inverse. □

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Example 3.9*.** *Suppose that n is even and let $Sp(n,\mathbf{R}) = \{A \in Mat_{n\times n} \mid AJA^T = J\}$. Also let $f : Mat_{n\times n} \to Mat_{n\times n}$ by $f(A) = AJA^T$*

1. *$f(A) \in Skew_{n\times n}$ and f is C^∞.*
2. *If $A \in Sp(n,\mathbf{R})$, then $Df(A)$ has rank $\dfrac{n(n-1)}{2}$.*
3. *$Sp(n,\mathbf{R})$ is an $\dfrac{n(n+1)}{2}$-manifold.*

First note that $AJA^T = J$ implies that $(\det(A))^2 = 1$ so A is invertible.

The map f is a composition the identity cross the transpose and multiplication. These maps are C^∞ maps by Proposition 2.13***. Since f is the composition of C^∞ maps, it is C^∞. Since $f(A) = AJA^T$ and $(AJA^T)^T = AJ^TA^T = -AJA^T$ by Lemma 3.8***, $f(A) \in \mathrm{Skew}_{n\times n}$. This argument shows item 1.

To show item 2, it is enough to show that $Df(A)$ is surjective since $\dim \mathrm{Skew}_{n\times n} = \dfrac{n(n-1)}{2}$. Now,

$$\begin{aligned} Df(A)(M) =& \frac{d}{dt}\,(A+tM)J(A+tM)^T\Big|_{t=0} \\ &= AJM^T + MJA^T \\ &= AJM^T - (AJM^T)^T \end{aligned}$$

Hence $Df(A)$ is a composition of two maps

$$\begin{aligned} \mathrm{Mat}_{n\times n} &\to \mathrm{Mat}_{n\times n} \\ M &\mapsto MJA^T \end{aligned} \quad \text{and} \quad \begin{aligned} \mathrm{Mat}_{n\times n} &\to \mathrm{Sym}_{n\times n} \\ X &\mapsto X - X^T \end{aligned}$$

The first map is onto if JA^T is invertible, which it is if $A \in f^{-1}(J)$. The second map is onto, since, if $Y \in \mathrm{Skew}_{n\times n}$ then $Y = -Y^T$ and $Y = \frac{1}{2}Y - (\frac{1}{2}Y)^T$, i.e., if $X = \frac{1}{2}Y$, then $Y = X - X^T$. Therefore, $Df(A)$ is a surjection and has rank $\dfrac{n(n-1)}{2}$.

The third item follows from Theorem 3.4***. Since $f^{-1}(I) = Sp(n, \mathbf{R})$, $Sp(n, \mathbf{R})$ is a manifold of dimension $n^2 - \dfrac{n(n-1)}{2} = \dfrac{n(n+1)}{2}$.

Proposition 3.10*.** *Suppose that N is an n-dimensional submanifold of the m-manifold M. Suppose that U is an open neighborhood of N in M and $g : U \to P$ is a smooth map to a manifold P. Then, $g|_N : N \to P$ is a smooth map.*

Proof. Suppose $x \in N$ is an arbitrary point and (O, η) is a chart of P about $g(x)$. By Proposition 2.18***, it is enough to check that there is a chart of N, (V, ψ), with $x \in V$ and $\eta \circ g|_M \circ \psi^{-1}$ is C^∞. By the definition of submanifold, Definition 3.2***, we can always find a product chart (of M) centered at x. Suppose the chart is (W, ϕ), so we can take $V = W \cap N$ and $\psi = \phi|_{W\cap N}$. Then

$$\eta \circ g|_M \circ \psi^{-1} = \eta \circ g|_M \circ (\phi^{-1})|_{\mathbf{R}^n \times \{0\} \cap \phi(W)},$$

the restriction of a C^∞ function of m variables to its first n variables by setting the last $m - n$ variables to zero. This map is C^∞. $\square$

Example 3.11*.**
Suppose that M is either of $O(n)$, $SO(n)$, $SL(n, \mathbf{R})$, or $Sp(n, \mathbf{R})$. Then multiplication

$$M \times M \to M$$
$$(A, B) \mapsto AB$$

and inverse

$$M \to M$$
$$A \mapsto A^{-1}$$

are smooth maps.

Just as M is a submanifold of $GL(n, \mathbf{R})$, $M \times M$ is also a submanifold of $GL(n, \mathbf{R}) \times GL(n, \mathbf{R})$. Both $GL(n, \mathbf{R})$ and $GL(n, \mathbf{R}) \times GL(n, \mathbf{R})$ are open subsets of real spaces. Multiplication and inverse are C^∞ functions as explained in Example 2.13***. By Proposition 3.10***, these maps are smooth maps between manifolds.

Exercises

Exercise 1*.** *Suppose $\mathcal{O} \subset \mathbf{R}^{n+m}$ is an open set, $f : \mathcal{O} \to \mathbf{R}^m$ is a C^∞ map, $q \in \mathbf{R}^m$, and $M = f^{-1}(q)$. Further suppose $\mathcal{U} \subset \mathbf{R}^{s+k}$ is an open set, $g : \mathcal{U} \to \mathbf{R}^k$ is a C^∞ map, $p \in \mathbf{R}^k$, and $n = g^{-1}(p)$. If $Df(x)$ has rank m for all $x \in M$ and $Dg(x)$ has rank k for all $x \in N$, then show that M and N are manifolds; and, $(f \times g)^{-1}(q,p)$ is the manifold $M \times N$.*

Exercise 2*.** *Show that the atlas for S^n constructed in Example 3.5*** does include the $2(n+1)$ charts constructed in Example 2.9b***.*

Exercise 3*.** *Let M be defined by*

$$M = \{(w, x, y, z) \in \mathbf{R}^4 \mid w^3 + x^2 + y^3 + z^2 = 0 \text{ and } ye^z = we^x + 2\}.$$

Show that M is a two dimensional submanifold of $\mathbf{R}^4$.

The reader may wish to compare the following Exercise to Example 2.10***.

Exercise 4*.** *Suppose $U_1 \subset \mathbf{R}^n$ and $U_2 \subset \mathbf{R}^m$ are open sets. If $f : U_1 \to U_2$ is a C^∞ function, then show the graph of f,*

$$G_f = \{(x, y) \in \mathbf{R}^{n+m} \mid y = f(x)\}$$

is a submanifold of $\mathbf{R}^{n+m}$.

In the exercise below, we use that $\mathbf{R}^4 \cong \mathbf{C}^2$ under the isomorphism of real vector spaces, $(w, x, y, z) \mapsto (w + xi, y + zi)$.

Exercise 5*.** a. *Suppose p and q are relatively prime integers. Let $\omega = e^{\frac{2\pi i}{p}}$. Show that $\tau : S^3 \to S^3$ by $\tau(z_1, z_2) = (\omega z_1, \omega^q z_2)$ is a smooth map. Let $G = \{\tau, \tau^2, \cdots, \tau^p\}$. Show that G is a subgroup of Diff(S^3).*

b. *Show that S^3 is $\{(z_1, z_2) \in \mathbf{C}^2 \mid z_1\bar{z}_1 + z_2\bar{z}_2 = 1\}$, and a submanifold of $\mathbf{C}^2$.*

c. *Show that S^3/G is a 3-manifold. It is denoted $L(p, q)$ and is called a lens space.*

Warmup Exercises, Chapter 3

Exercise 1*. *Let* $F : \mathbf{R}^2 \to \mathbf{R}$ *be defined by*

$$F(x, y) = x^3 + xy + y^3.$$

Which level sets are embedded submanifolds of $\mathbf{R}$.

Exercise 2*. *Consider the map* $F : \mathbf{R}^4 \to \mathbf{R}^2$ *defined by*

$$F(x_1, x_2, x_3, x_4) = (x_1^2 + x_2, x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_2).$$

Show that F *restricted to* $M = F^{-1}(0, 1)$ *has rank 2 at every point of* M.

Exercise 3*. *Which level sets of*

$$f(x_1, \dots, x_{n+1}) = x_1 x_2 \cdots x_{n+1} + 1$$

*are submanifolds, according to Theorem 3.4****.