CHAPTER 5
TANGENT VECTORS

In $\mathbf{R}^n$ tangent vectors can be viewed from two perspectives

(1) they capture the infinitesimal movement along a path, the direction, and
(2) they operate on functions by directional derivatives.

The first viewpoint is more familiar as a conceptual viewpoint from Calculus. If a point moves so that its position at time t is $\rho(t)$, then its velocity vector at $\rho(0)$ is $\rho'(0)$, a tangent vector. Because of the conceptual familiarity, we will begin with the first viewpoint, although there are technical difficulties to overcome. The second interpretation will be derived as a theorem. The second viewpoint is easier to generalize to a manifold. For instance, operators already form a vector space. It is the second viewpoint that ultimately plays the more important role.

Suppose M is an n-manifold. If $m \in M$, then we define a tangent vector at m as an equivalence class of paths through m. Equivalent paths will have the same derivative vector at m and so represent a tangent vector. The set of all tangent vectors at m forms the tangent space. The description and notation of tangent vectors in $\mathbf{R}^n$ from the advanced Calculus setting and in the present setting is discussed in Remark 5.9***.

Definition 5.1*.** *Suppose M is a manifold. A path is a smooth map $\rho : (-\epsilon, \epsilon) \to M$, where $\epsilon > 0$.*

As was mentioned, if $M = \mathbf{R}^n$, then $\rho'(0)$ is the velocity vector at $\rho(0)$. We also recall, from advanced Calculus, the relationship between the derivative map and the directional derivative,

$$D\rho(0)(1) = D_1\rho(0) = \rho'(0) \tag{1}$$

Definition 5.2*.** *Suppose M is a manifold and $m \in M$ A tangent vector at m is an equivalence class of paths α with $\alpha(0) = m$. Let $(\mathcal{U}, \phi)$ be a coordinate chart centered at m, two paths α and β are equivalent if $\left.\frac{d\phi\circ\alpha(t)}{dt}\right|_{t=0} = \left.\frac{d\phi\circ\beta(t)}{dt}\right|_{t=0}$.*

Denote the equivalence class of a path α by $[\alpha]$. We can picture $[\alpha]$ as the velocity vector at $\alpha(0)$.

We next observe that the equivalence class doesn't depend on the specific choice of a coordinate chart. If $(\mathcal{W}, \psi)$ is another coordinate neighborhood centered at m, then $\psi\circ\alpha = \psi\circ\phi^{-1}\circ\phi\circ\alpha$, and, we use formula (1),

$$\left.\frac{d\psi\circ\alpha(t)}{dt}\right|_{t=0} = D(\psi\circ\phi^{-1})(\phi(m))\circ D(\phi\circ\alpha)(0)(1).$$

The diffeomorphisms $\psi \circ \phi^{-1}$ and $\phi \circ \alpha$ are maps between neighborhoods in real vector spaces, so

$$\left.\frac{d\phi \circ \alpha(t)}{dt}\right|_{t=0} = \left.\frac{d\phi \circ \beta(t)}{dt}\right|_{t=0} \text{ if and only if } \left.\frac{d\psi \circ \alpha(t)}{dt}\right|_{t=0} = \left.\frac{d\psi \circ \beta(t)}{dt}\right|_{t=0}.$$

Therefore the notion of tangent vector is independent of the coordinate neighborhood. If $\rho : (-\epsilon, \epsilon) \to M$ is a path in M with $\rho(0) = m$, then $[\rho]$ is a tangent vector to M at m and is represented by the path ρ. Consistent with the notation for $\mathbf{R}^n$, we can denote $[\rho]$ by $\rho'(0)$.

Let TM_m denote the set of tangent vectors to M at m. Other common notations are M_m and T_mM.

Theorem 5.3*.** *Suppose M, N, and R are manifolds.*

(1) *If $\phi : M \to N$ is a smooth map between manifolds and $m \in M$ then there is an induced map $\phi_{*m} : TM_m \to TN_{\phi(m)}$.*
(2) *If $\psi : N \to R$ is another smooth map between manifolds then $(\psi \circ \phi)_{*m} = \psi_{*\phi(m)} \circ \phi_{*m}$. This formula is called the chain rule.*
(3) *If $\phi : M \to M$ is the identity then $\phi_{*m} : TM_m \to TM_m$ is the identity. If $\phi : M \to N$ is a diffeomorphism and $m \in M$ then ϕ_{*m} is 1-1 and onto.*
(4) *TM_m is a vector space of dimension n, the dimension of M, and the induced maps are linear.*

The induced map ϕ_{*m} is defined by

$$\phi_{*m}([\alpha]) = [\phi \circ \alpha].$$

Notice that if $M = \mathbf{R}^m$, $N = \mathbf{R}^n$, then we have a natural way to identify the tangent space and the map ϕ_*. We have coordinates on the tangent space so that

$$[\phi \circ \alpha] = \left.\frac{d\phi \circ \alpha(t)}{dt}\right|_{t=0}$$

and

$$\phi_{*m}([\alpha]) = D\phi(m)(\alpha'(0)).$$

The induced map ϕ_{*m} is also commonly denoted $T\phi$ or $d\phi$. These results follow for neighborhoods in manifolds since these are manifolds too. Also note that if there is a neighborhood $\mathcal{U}$ of $m \in M$ and $\phi|_{\mathcal{U}}$ is a diffeomorphism onto a neighborhood of $\phi(m)$ then ϕ_{*m} is an isomorphism.

Proof.

(1) If $\phi : M \to N$ is a smooth map and $m \in M$ then there is an induced map $\phi_{*m} : TM_m \to TN_{\phi(m)}$ defined by $\phi_{*m}([\alpha]) = [\phi \circ \alpha]$. We need to show this map is

well-defined. Take charts $(\mathcal{U}, \theta)$ on N centered on $\phi(m)$ and $(\mathcal{W}, \psi)$ on M centered on m. If $[\alpha] = [\beta]$, then

$$\begin{aligned}
\left.\frac{d\psi \circ \alpha(t)}{dt}\right|_{t=0} &= \left.\frac{d\psi \circ \beta(t)}{dt}\right|_{t=0} \\
(\theta \circ \phi \circ \psi^{-1})_*\left(\left.\frac{d\psi \circ \alpha(t)}{dt}\right|_{t=0}\right) &= (\theta \circ \phi \circ \psi^{-1})_*\left(\left.\frac{d\psi \circ \beta(t)}{dt}\right|_{t=0}\right) \\
\left.\frac{d}{dt}(\theta \circ \phi \circ \psi^{-1} \circ \psi \circ \alpha)(t)\right|_{t=0} &= \left.\frac{d}{dt}(\theta \circ \phi \circ \psi^{-1} \circ \psi \circ \beta)(t)\right|_{t=0} \\
\left.\frac{d}{dt}(\theta \circ \phi \circ \alpha)(t)\right|_{t=0} &= \left.\frac{d}{dt}(\theta \circ \phi \circ \beta)(t)\right|_{t=0}
\end{aligned}$$

so ϕ_{*m} is well defined on equivalence classes.

(2) If $\phi : M \to N$ and $\psi : N \to R$ are smooth maps, then $(\psi \circ \phi)_{*m}([\alpha]) = [\psi \circ \phi \circ \alpha] = \psi_{*\phi(m)}([\phi \circ \alpha]) = \psi_{*\phi(m)} \circ \phi_{*m}([\alpha])$.

(3) $I_{M*m}([\alpha]) = [I_M \circ \alpha] = [\alpha]$. If $\phi \circ \phi^{-1} = I_M$ then $\phi_* \circ (\phi^{-1})_* = I_{M*} = I_{TM_m}$. Also, if $\phi^{-1} \circ \phi = I_M$, then $(\phi^{-1})_* \circ \phi_* = I_{TM_m}$. Therefore ϕ_* is a bijection and $(\phi_*)^{-1} = (\phi^{-1})_*$.

(4) Let $(\mathcal{U}, \phi)$ be a coordinate neighborhood centered at m. We first show that $T\mathbf{R}^n_{\mathbf{0}}$ is an n-dimensional vector space. Since $\mathbf{R}^n$ requires no coordinate neighborhood (i.e., it is itself), $[\alpha]$ is equivalent to $[\beta]$ if and only if $\alpha'(0) = \beta'(0)$: two paths are equivalent if they have the same derivative vector in $\mathbf{R}^n$. Every vector $\mathbf{v}$ is realized by a path $\alpha_{\mathbf{v}}$, $\alpha_{\mathbf{v}}(t) = t\mathbf{v}$. This identification gives $T\mathbf{R}^n_{\mathbf{0}}$ the vector space structure. We show that the linear structure is well defined on TM_m. The linear structure on TM_m is induced by the structure on $T\mathbf{R}^n_{\mathbf{0}}$ (where $[\alpha]+k[\beta] = [\alpha+k\beta]$ and induced maps are linear) via the coordinate maps. If $(\mathcal{V}, \psi)$ is another chart centered at m, then the structure defined by ψ and ϕ agree since $(\phi \circ \psi^{-1})_*$ is an isomorphism and $(\phi \circ \psi^{-1})_* \circ \psi_* = \phi_*$. □

We can give explicit representatives for linear combinations of paths in the tangent space TM_m. In the notation of the proof of Theorem 5.3*** part 4,

$$k[\alpha] + c[\beta] = [\phi^{-1}(k\phi \circ \alpha + c\phi \circ \beta)]$$

Note that the coordinate chart serves to move the paths into $\mathbf{R}^n$ where addition and multiplication makes sense.

Before we turn to the second interpretation of a tangent vector as a directional derivative, we pause for a philosophical comment. We first learn of functions in our grade school education . We learn to speak of the function as a whole or its value at particular points. Nevertheless, the derivative at a point does not depend on the whole function nor is it determined by the value at a single point. The derivative requires some open set about a point but any open set will do. If M is a manifold and $m \in M$, then let $\mathcal{G}_m$ be the set of functions defined on some open neighborhood of m.

Definition 5.6*.** *A function* $\ell : \mathcal{G}_m \to \mathbf{R}$ *is called a derivation if for every* $f, g \in \mathcal{G}_m$ *and* $a, b \in \mathbf{R}$,

(1) $\ell(af + bg) = a\ell(f) + b\ell(g)$ *and*
(2) $\ell(fg) = \ell(f)g(m) + f(m)\ell(g)$

Denote the space of derivations by $\mathcal{D}$. The product rule which occurs in the definition is called the Leibniz rule, just as it is in Calculus.

Proposition 5.7*.** *Elements of* TM_m *operate as derivations on* $\mathcal{G}_m$. *In fact there is a linear map* $\ell : TM_m \to \mathcal{D}$ *given by* $v \mapsto \ell_v$.

The theorem is straightforward if the manifold is $\mathbf{R}^n$. If $v \in T\mathbf{R}^n_x$, then the derivation ℓ_v is the directional derivative in the direction v, i.e., $\ell_v(f) = Df(x)(v)$. On a manifold the argument is really the same, but more technical as the directions are more difficult to represent. We will see in Theorem 5.8*** that the derivations are exactly the directional derivatives.

Proof. If $\alpha : ((-\epsilon, \epsilon), \{0\}) \to (M, \{m\})$ represents v then define $\ell_v(f) = \left.\dfrac{df \circ \alpha(t)}{dt}\right|_{t=0}$. The fact that ℓ_v is a linear functional and the Leibniz rule follow from these properties of the derivative.

To show that ℓ is a linear map requires calculation. Suppose $(\mathcal{U}, \phi)$ is a coordinate chart centered at m. If $[\alpha]$ and $[\beta]$ are equivalence classes that represent tangent vectors in TM_m and $c, k \in \mathbf{R}$, then $\phi^{-1}((k\phi\alpha(t) + c\phi\beta(t)))$ represents $k[\alpha] + c[\beta]$. Hence,

$$\begin{aligned}
\ell_{k[\alpha]+c[\beta]}(f) &= \left.\frac{df(\phi^{-1}((k\phi\alpha(t) + c\phi\beta(t))))}{dt}\right|_{t=0} \\
&= f_*\phi_*^{-1}\left(\left.\frac{d(k\phi\alpha(t) + c\phi\beta(t))}{dt}\right|_{t=0}\right) \\
&= f_*\phi_*^{-1}\left(k\left.\frac{d(\phi\alpha(t)}{dt}\right|_{t=0} + c\left.\frac{d(\phi\beta(t))}{dt}\right|_{t=0}\right) \\
&= kf_*\phi_*^{-1}\left(\left.\frac{d\phi\alpha(t)}{dt}\right|_{t=0}\right) + cf_*\phi_*^{-1}\left(\left.\frac{d\phi\beta(t)}{dt}\right|_{t=0}\right) \\
&= k\left.\frac{df(\phi^{-1}(\phi(\alpha(t))))}{dt}\right|_{t=0} + c\left.\frac{df(\phi^{-1}(\phi(\beta(t))))}{dt}\right|_{t=0} \\
&= k\left.\frac{df((\alpha(t)))}{dt}\right|_{t=0} + c\left.\frac{df((\beta(t)))}{dt}\right|_{t=0} \\
&= k\ell_{[\alpha]}(f) + c\ell_{[\beta]}(f)
\end{aligned}$$

Lines 3 and 4 respectively follow from the linearity of the derivative and the total derivative map. Therefore ℓ is linear. $\square$

The second interpretation of tangent vectors is given in the following Theorem.

Theorem 5.8*.** *The linear map $\ell : TM_m \to \mathcal{D}$ given by $v \mapsto \ell_v$ is an isomorphism. The elements of TM_m are the derivations on $\mathcal{G}_m$.*

Proof. We first note two properties on derivations.

$$\text{If } f(m) = g(m) = 0, \text{ then } \ell(fg) = 0 \tag{1}$$

Since $\ell(fg) = f(m)\ell(g) + g(m)\ell(f) = 0 + 0$.

$$\text{If } k \text{ is a constant, then } \ell(k) = 0 \tag{2}$$

Since $\ell(k) = k\ell(1) = k(\ell(1) + \ell(1)) = 2k\ell(1)$, $\ell(k) = 2\ell(k)$ and $\ell(k) = 0$.

We now observe that ℓ is one-to-one. Let $(\mathcal{U}, \phi)$ be a coordinate chart centered at m. Suppose $v \neq 0$ is a tangent vector. We will show that $\ell_v \neq 0$. Let $\phi_*(v) = w_1 \in \mathbf{R}^n$. Note that $w_1 \neq 0$. Then $[\phi^{-1}(tw_1)] = v$ where t is the real variable. Let $w_1, \cdots, w_n$ be a basis for $\mathbf{R}^n$ and $\pi(\sum_{i=1}^n a_i w_i) = a_1$. Then

$$\begin{aligned}
\ell_v(\pi \circ \phi) &= \ell_{[\phi^{-1}(tw_1)]}(\pi \circ \phi) \\
&= \left.\frac{d\pi(\phi(\phi^{-1}(tw_1)))}{dt}\right|_{t=0} \\
&= \left.\frac{dtw_1}{dt}\right|_{t=0} \\
&= w_1.
\end{aligned}$$

Next we argue that ℓ is onto. Let $(\mathcal{U}, \phi)$ be a coordinate chart centered at m and let e_i for $i = 1, \cdots, n$ be the standard basis for $\mathbf{R}^n$. We consider the path $t \mapsto \phi^{-1}(te_i)$ and compute some useful values of ℓ, i.e., the partial derivatives.

$$\begin{aligned}
\ell_{[\phi^{-1}(te_i)]}(f) &= \left.\frac{df\phi^{-1}(te_i)}{dt}\right|_{t=0} \\
&= \left.\frac{\partial f\phi^{-1}}{\partial x_i}\right|_{\vec{0}}
\end{aligned}$$

Let $x_i(a_1, \cdots, a_n) = a_i$. Suppose $\mathbf{d}$ is any derivation. We will need to name certain values. Let $\mathbf{d}(x_i \circ \phi) = a_i$. These are just fixed numbers. Suppose f is C^∞ on a neighborhood of m. Taylor's Theorem says that for p in a neighborhood of $\vec{0} \in \mathbf{R}^n$,

$$f \circ \phi^{-1}(p) = f \circ \phi^{-1}(\vec{0}) + \sum_{i=1}^n \left.\frac{\partial f \circ \phi^{-1}}{\partial x_i}\right|_{\vec{0}} x_i(p) + \sum_{i,j=1}^n R_{ij}(p) x_i(p) x_j(p)$$

where $R_{ij}(p) = \int_0^1 (t-1) \left.\frac{\partial^2 f \circ \phi^{-1}}{\partial x_i \partial x_j}\right|_{tp} dt$ are C^∞ functions. So,

$$f = f(m) + \sum_{i=1}^n \left.\frac{\partial f \circ \phi^{-1}}{\partial x_i}\right|_{\vec{0}} x_i \circ \phi + \sum_{i,j=1}^n (R_{ij} \circ \phi) \cdot (x_i \circ \phi) \cdot (x_j \circ \phi).$$

We now apply $\mathbf{d}$. By (2), $\mathbf{d}(f(m)) = 0$. Since $x_j \circ \phi(m) = 0$, the terms $\mathbf{d}((R_{ij} \circ \phi) \cdot (x_i \circ \phi) \cdot (x_j \circ \phi)) = 0$ by (1). Also, $\mathbf{d}(\left.\frac{\partial f \circ \phi^{-1}}{\partial x_i}\right|_{\vec{0}} x_i \circ \phi) = a_i \ell_{[\phi^{-1}(te_i)]}(f)$. Hence, $\mathbf{d} = \ell_{\sum_{i=1}^n a_i [\phi^{-1}(te_i)]}$, and ℓ is onto. $\square$

Remark 5.9*.** *Tangent vectors to points in* $\mathbf{R}^n$.

The usual coordinates on $\mathbf{R}^n$ give rise to standard coordinates on $T_p\mathbf{R}^n$. Let $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ with the only nonzero entry in the i-th spot. The path in $\mathbf{R}^n$ defined by $\alpha_i(t) = te_i + p$ is a path with $\alpha_i(0) = p$. Its equivalence class $[\alpha_i]$ is a vector in $T_p\mathbf{R}^n$ and we denote it $\left.\frac{\partial}{\partial x_i}\right|_p$. In Advanced Calculus, the ordered basis $\left.\frac{\partial}{\partial x_1}\right|_p, \cdots, \left.\frac{\partial}{\partial x_n}\right|_p$ is the usual basis in which the Jacobian matrix is usually written and sets up a natural isomorphism $T_p\mathbf{R}^n \cong \mathbf{R}^n$. The reader should notice that the isomorphism is only natural because $\mathbf{R}^n$ has a natual basis and is not just an abstract n-dimensional vector space. If ρ is a path in $\mathbf{R}^n$, then $\rho'(0) \in T_{\rho(0)}\mathbf{R}^n$ via this isomorphism. This notation is also consistant with the operator notation (the second interpretation) since,

$$\begin{aligned}\left.\frac{\partial}{\partial x_i}\right|_p (f) &= [f \circ \alpha_i] \\ &= \left.\frac{d}{dt} f(te_i + p)\right|_{t=0} \\ &= \left.\frac{\partial f}{\partial x_i}\right|_{x=p} \in \mathbf{R}^n \cong T_p\mathbf{R}^n\end{aligned}$$

In the first line, the tangent vector $\left.\frac{\partial}{\partial x_i}\right|_p$ operates via the second interpretation on the function f.

Example 5.10*.** *TM_x for M an n-dimensional submanifold of* $\mathbf{R}^k$.

Suppose $M \subset \mathbf{R}^k$ is a submanifold and $i : M \to \mathbf{R}^k$ is the inclusion. Take (U_x, ϕ) a slice coordinate neighborhood system for $\mathbf{R}^k$ centered at x as specified in the definition of a submanifold, Definition 3.2***, $\phi : U_x \to U_1 \times U_2$. Under the natural coordinates of $T\mathbf{R}^k_x \cong \mathbf{R}^k$, $TM_x = \phi(U_1 \times \{0\}) \subset \mathbf{R}^k$ and i_{*x} has rank n.

To see these facts, note that $\phi \circ i \circ (\phi|_{U_x \cap M})^{-1} : U_1 \times \{0\} \to U_1 \times U_2$ is the inclusion. So, $\operatorname{rank}(i_*) = \operatorname{rank}((\phi \circ i \circ \phi|_{U_x \cap M})_*) = n$. Under the identification $T\mathbf{R}^k_x \cong \mathbf{R}^k$, $\phi_{*x}(\mathbf{R}^n \times \{0\}) = D\phi(x)(\mathbf{R}^n \times \{0\}) \subset \mathbf{R}^k$. This is the usual picture of the tangent space as a subspace of $\mathbf{R}^k$ (i.e., shifted to the origin) that is taught in advanced Calculus.

Example 5.11*.** *TS^n_x for $S^n \subset \mathbf{R}^{n+1}$, the n-sphere.*

This is a special case of Example 5.10***. Suppose $(x_1, \cdots, x_{n+1}) \in S^n$, i.e., $\sum_{i=1}^{n+1} x_i^2 = 1$. One of the x_i must be nonzero, we assume that $x_{n+1} > 0$. The other cases are analogous. The inclusion from the Implicit Function Theorem is $\phi|_{\mathbf{R}^n}(x_1, \cdots, x_n) = (x_1, \cdots, x_n, \sqrt{1 - \sum_{i=1}^n x_i^2})$ so

$$D\phi|_{\mathbf{R}^n}(x_1, \cdots, x_n)(v_1, \cdots, v_n) = (v_1, \cdots, v_n, \frac{\sum_{i=1}^n -x_i v_i}{\sqrt{1 - \sum_{i=1}^n x_i^2})}).$$

Since $x_{n+1} > 0$, $x_{n+1} = \sqrt{1 - \sum_{i=1}^n x_i^2)}$ and the tangent space is

$$\begin{aligned} T_{(x_1,\cdots,x_{n+1})}S^n &= \{(v_1, \cdots, v_n, \frac{\sum_{i=1}^n -x_i v_i}{x_{n+1}}) \mid v_i \in \mathbf{R}\} \\ &= \{(w_1, \cdots, w_{n+1}) \mid \sum_{i=1}^{n+1} w_i x_i = 0\} \end{aligned}$$

Example 5.12*.** *Recall that $O(n) \subset Mat_{n\times n} = \mathbf{R}^{n^2}$ is a submanifold of dimension $\frac{n(n-1)}{2}$ which was shown in Example 3.7***. Then, we claim,*

$$X \in T_A O(n) \subset Mat_{n\times n}$$

if and only if XA^{-1} is skew.

This computation is a continuation of Example 3.7***. Suppose $A \in O(n)$. Since $O(n) = f^{-1}(I)$, $T_A O(n) \subset \mathrm{Ker}(Df(A))$. The dimension of the kernel and the dimension of $T_A O(n)$ are both $\frac{n(n-1)}{2}$. Therefore $T_A O(n) = \mathrm{Ker}(Df(A))$. It is enough to show that $\mathrm{Ker}(Df(A)) \subset \{X \mid XA^{-1}$ is skew$\}$ since the dimension of $\{X \mid XA^{-1}$ is skew$\}$ is the dimension of $\mathrm{Skew}_{n\times n} = \frac{n(n-1)}{2}$ (from Example 2.8d***). So it is enough to show that XA^{-1} is skew.

Again, from Example 3.7***, $Df(A)(X) = AX^T + XA^T$. If $Df(A)(X) = 0$, then $AX^T = -XA^T$. Since $A \in O(n)$, $A^{-1} = A^T$. So,

$$(XA^{-1})^T = (XA^T)^T = AX^T = -XA^T = -XA^{-1}$$

Therefore XA^{-1} is skew.

Example 5.13*.** *Recall that $Sp(n, \mathbf{R}) \subset Mat_{n\times n} = \mathbf{R}^{n^2}$ is a submanifold of dimension $\frac{n(n+1)}{2}$ which was shown in Example 3.9***. Then, we claim,*

$$X \in T_A Sp(n, \mathbf{R}) \subset Mat_{n\times n}$$

if and only if JXA^{-1} is symmetric.

This computation is a continuation of Example 3.9***. Suppose $A \in Sp(n, \mathbf{R})$.

Since $Sp(n, \mathbf{R}) = f^{-1}(J)$, $T_A Sp(n, \mathbf{R}) \subset \mathrm{Ker}(Df(A))$. The dimension of the kernel and the dimension of $T_A Sp(n, \mathbf{R})$ are both $\frac{n(n+1)}{2}$. Therefore $T_A Sp(n, \mathbf{R}) = \mathrm{Ker}(Df(A))$. It is enough to show that $\mathrm{Ker}(Df(A)) \subset \{X \mid JXA^{-1}$ is symmetric$\}$ since the dimension of $\{X \mid JXA^{-1}$ is symmetric $\}$ is the dimension of $\mathrm{Sym}_{n\times n} = \frac{n(n+1)}{2}$ (from Example 2.8c***). So it is enough to show that JXA^{-1} is symmetric.

Again, from Example 3.9***, $Df(A)(X) = AJX^T + XJA^T$. If $Df(A)(X) = 0$, then $-AJX^T = XJA^T$. Since $A \in Sp(n, \mathbf{R})$, $A^{-1} = JA^T J^T$. So,

$$\begin{aligned} (JXA^{-1})^T = (JXJA^T J^T)^T = (-JAJX^T J^T)^T &= -JXJ^T A^T J^T \\ &= JXJA^T J^T \text{as } J^T = -J \text{ by Lemma 3.8***} \\ &= JXA^{-1} \end{aligned}$$

Therefore XA^{-1} is symmetric.

Remark 5.14*.** *Notation for Tangent vectors*

The space $\mathbf{R}^n$ comes equipped with a canonical basis $e_1, \cdots, e_n$ which allows us to pick a canonical basis for $T\mathbf{R}^n_x$. For an n-manifold M, TM_p doesn't have a natural basis. We can give coordinates on TM_p in terms of a chart. Suppose that (U, ϕ) is a chart for a neighborhood of $p \in U \subset M$. Write $\phi = (\phi_1, \cdots, \phi_n)$ in terms of the coordinates on $\mathbf{R}^n$. Hence, $\phi_i = x_i \circ \phi$. We can import the coordinates $T\mathbf{R}^n_{\phi(p)}$. Let

$$\left.\frac{\partial}{\partial \phi_i}\right|_p = \phi_*^{-1}(\left.\frac{\partial}{\partial x_i}\right|_{\phi(p)})$$

As a path $\left.\frac{\partial}{\partial \phi_i}\right|_p$ is the equivalence class of $\phi^{-1}(te_i + \phi(p))$. As an operator,

$$\left.\frac{\partial}{\partial \phi_i}\right|_p (f) = \left.\frac{\partial f \circ \phi^{-1}}{\partial x_i}\right|_{\phi(p)}.$$

Exercises

Exercise 1*.** *Suppose* $F : \mathbf{R}^4 \to \mathbf{R}^2$ *by*

$$F((w,x,y,z)) = (wxyz, x^2y^2).$$

Compute F_* *and be explicit in exhibiting the bases in the notation used in Remark 5.9***. Where is* F *singular?*

The reader may wish to review Example 2.10** and Exercise 4*** from chapter 3 for the following exercise.

Exercise 2*.** *Let* $g((x,y)) = x^2 + y^2$ *and* $h((x,y)) = x^3 + y^2$. *Denote by* G_g *and* G_h *the graphs of* g *and* h *which are submanifolds of* $\mathbf{R}^3$. *Let* $F : G_g \to G_h$ *by*

$$F : ((x,y,z)) = (x^3, xyz, x^9 + x^2y).$$

*The reader may wish to review Example 2.10** and Exercise *** from chapter 3.*

a. Explicitly compute the derivative F_* *and be clear with your notation and bases.*

b. Find the points of G_g *where* F *is singular. What is the rank of* F_{*p} *for the various singular points* $p \in G_g$.

Exercise 3*.** *Let* $F : \mathbf{R}^3 \to S^3$ *be defined by*

$$F((\theta,\phi,\eta)) = (\sin\eta\sin\phi\cos\theta, \sin\eta\sin\phi\sin\theta, \sin\eta\cos\phi, \cos\eta).$$

Use the charts from stereographic projection to compute F_* *in terms of the bases discussed in Remark 5.9*** and Remark 5.14***.*