# CHAPTER 6
# IMMERSIONS AND EMBEDDINGS

In this chapter we turn to inclusion maps of one manifold to another. If $f : N \to M$ is an inclusion, then the image should also be a manifold. In chapter 3, we saw one situation where a subset of $f(N) \subset M$ inherited the structure of a manifold: when each point of $f(N)$ had a slice coordinate neighborhood of $M$. In this chapter, we show that is the only way it can happen if $f(N)$ is to inherit its structure from $M$.

We first review the situation for functions $f : \mathbf{R}^n \to \mathbf{R}^m$ for $n \le m$. The infinitesimal condition of a function to be one-to-one is that the derivative is one-to-one. That the derivative is one-to-one is not required for the function to be one-to-one, but it is sufficient to guarantee the function is one-to-one in some neighborhood (by the Inverse Function Theorem). On the other hand, if $f(y_0) = f(z_0)$, then there is a point $x_0$ on the segment between $y_0$ and $z_0$ where $Df(x_0)$ is not one-to-one. This last statement is a consequence of Rolle's Theorem. This discussion, perhaps, serves as some motivation to study functions whose derivative is injective. A second justification is that if $f$ is to be a diffeomorphism to its image, then the derivative must be invertible as a linear map.

While the phrase "$f(N)$ inherits manifold structure form $M$" is vague, it certainly includes that "$f(N)$ inherits its topology from $M$" which is precise.

**Definition 6.1***.** *Suppose $f : N \to M$ is a smooth map between manifolds. The map $f$ is called an immersion if $f_{*x} : T_xN \to T_{f(x)}M$ is injective for all $x \in N$.*

The derivative is injective at each point is not enough to guarantee that the function is one-to-one, as very simple example illustrate. Take $f : \mathbf{R} \to \mathbf{R}^2$ by $f(x) = (\sin(2\pi x), \cos(2\pi x))$. This function is infinite-to-one as $f(x+1) = f(x)$, but $Df(x)$ is injective for all $x \in \mathbf{R}$. Hence it is clear that we will need some other condition to obtain an inclusion. An obvious first guess, that turns out to be inadequate, is that $f$ is also one-to-one.

**Example 6.2***.** *A one-to-one immersion $f : N \to M$ in which $f(N)$ is not a topological manifold.*

Let $N = (\frac{-\pi}{4}, \frac{3\pi}{4})$, $M = \mathbf{R}^2$, and $f(x) = (\cos(x)\cos(2x), \sin(x)\cos(2x))$. The image $f(\frac{-\pi}{4}, \frac{3\pi}{4})$ is two petals of a four leafed rose. The map is one-to-one: only $x = \frac{\pi}{4}$ maps to $(0,0)$. Note that if $\frac{-\pi}{4}$ or $\frac{3\pi}{4}$ were in the domain, then they would also map to $(0,0)$. $Df(x)$ is rank one, so $f$ is a one-to-one immersion. However, $f(N)$ is not a topological manifold. Suppose $U \subset B_{1/2}((0,0))$, then $U \cap f(N)$ cannot be homeomorphic to an open interval. An interval with one point removed has two components, by $U \cap f(N) \setminus (0,0)$ has at least four components. Hence no neighborhood of $(0,0) \in f(N)$ is homeomorphic to an open set in $\mathbf{R}$.

**Definition 6.3***.** *Suppose $f : N \to M$ is a smooth map between manifolds. The map $f$ is called an embedding if $f$ is an immersion which is a homeomorphism to its image.*

This extra topological condition is enough to guarantee that $f(N)$ is a submanifold in the strong sense of Definition 3.2***.

**Theorem 6.4***.** *Suppose $N^n$ and $M^m$ are manifolds and $f : N \to M$ is a smooth map of rank $n$. If $f$ is a homeomorphism to its image, then $f(N)$ is a submanifold of $M$ and $f$ is a diffeomorphism to its image.*

*Proof.* To show that $f(N)$ is a submanifold of $M$, we suppose $x_0 \in N$ and we must produce a slice neighborhood of $f(x_0) \in f(N) \subset M$. We produce this neighborhood in three steps. The first step is to clean up the local picture by producing coordinate neighborhoods of $x_0$ and $f(x_0)$ that properly align. The second step is to produce a coordinate neighborhood of $f(x_0)$ in $M$ in which $f(N)$ looks like the graph of a function. The graph of a function was already seen to be a submanifold, and we have virtually completed the construction. The third step is to construct the slice neighborhood.

As a first step, we produce coordinate neighborhoods:

(1) $(O_2, \psi)$ a coordinate neighborhood in $N$ centered at $x_0$
(2) $(U_2, \tau)$ a coordinate neighborhood in $M$ centered at $f(x_0)$
with $f^{-1}(U_2 \cap N) = O_2$ and $(\tau \circ f)_{*x_0}(TN_{x_0}) = \mathbf{R}^n \times \{0\}$

Take $O_1 \subset N$ a coordinate neighborhood in $N$ centered at $x_0$, and $U_1 \subset M$ such that $f^{-1}(U_1 \cap N) = O_1$. Such a $U_1$ exists since $f$ is a homeomorphism to its image, and $f(N)$ has the subspace topology. Take $(U_2, \phi)$ a coordinate neighborhood of $M$ centered at $f(x_0)$ with $U_2 \subset U_1$. Let $O_2 \subset f^{-1}(U_2)$, $x_0 \in O_2$. Then $(O_2, \psi)$ is a coordinate neighborhood of $N$ centered at $x_0$. Let $v_1, \cdots, v_n$ span $(\phi \circ f)_{*x_0}(TN_{x_0}) \subset \mathbf{R}^m$, and let $v_1, \cdots, v_m$ be a basis of $\mathbf{R}^m$. Let $H : \mathbf{R}^m \to \mathbf{R}^m$ be the isomorphism $H(\sum_{i=1}^m a_i v_i) = (a_1, \cdots, a_m)$. Then $(U_2, H \circ \phi)$ is a coordinate neighborhood in $M$ centered at $f(x_0)$ and $(H \circ \phi \circ f)_{*x_0}(TN_{x_0}) = \mathbf{R}^n \times \{0\}$. Let $\tau = H \circ \phi$ and the coordinate neighborhoods are constructed.

The second step is to cut down the neighborhood of $f(x_0)$ so that $f(N)$ looks like the graph of a function. This step requires the inverse function theorem. We produce coordinate neighborhoods:

(1) $(O_3, \psi)$ a coordinate neighborhood in $N$ centered at $x_0$
(2) $(U_3, \tau)$ a coordinate neighborhood in $M$ centered at $f(x_0)$, $\tau : U_3 \to W_3 \times W_2 \subset \mathbf{R}^n \times \mathbf{R}^{m-n}$
(3) a $C^\infty$ function $g : W_3 \to W_2$
such that $\tau(f(N) \cap U_4)$ is the graph of $g$.

Let $W_2 \subset \mathbf{R}^n$ and $W_4 \subset \mathbf{R}^{m-n}$ be open sets such that $W_4 \times W_2$ is a neighborhood of $0 \in \tau(U_2) \in \mathbf{R}^m$. Now define $U_4 = \tau^{-1}(W_4 \times W_2)$ and $O_4 = f^{-1}(U_4)$. Then $O_4 \subset O_2$, $(O_4, \psi)$ is a chart centered at $x_0$, and $U_4 \subset U_2$, $(U_4, \tau)$ is a chart centered at $f(x_0)$. Let $p_1 : \mathbf{R}^m \to \mathbf{R}^n$ be the projection onto the first $n$ coordinates and $p_2 : \mathbf{R}^m \to \mathbf{R}^{m-n}$ be the projection onto the last $m - n$ coordinates. The function $p_1 \circ \tau \circ f \circ \psi^{-1}$ maps the open set $\psi(O_4)$ to $W_4$. Since $(\tau \circ f)_{*x_0}(TN_{x_0}) = \mathbf{R}^n \times \{0\}$, $D(p_1 \circ \tau \circ f \circ \psi^{-1})(0)$ has rank $n$, i.e., it is an isomorphism. By the Inverse Function Theorem, there is a neighborhood $V$ of $0 \in \mathbf{R}^n$, $V \subset \psi(O_4)$ and a neighborhood $W_3$ of $0 \in \mathbf{R}^n$, $W_3 \subset W_4$ such that

$$p_1 \circ \tau \circ f \circ \psi^{-1} : V \to W_3$$

is a diffeomorphism. Let $O_3 = \psi^{-1}(V)$ and $U_3 = \tau^{-1}(W_3 \times W_2)$. Then $(U_3, \tau)$ is a coordinate chart centered at $f(x_0)$, $\tau : U_3 \to W_3 \times W_2$, and $(O_3, \psi)$ is a coordinate chart centered at $x_0$. Let $g$ be the composition

$$W_3 \xrightarrow{(p_1 \circ \tau \circ f \circ \psi^{-1})^{-1}} \psi^{-1}(O_3) \xrightarrow{(p_2 \circ \tau \circ f \circ \psi^{-1})} W_2$$

The function $g$ is the composition of two $C^\infty$ functions. We now observe that the graph of $g$ is $\tau(f(N) \cap U_3)$. The points in $\tau(f(N) \cap U_3)$ are $\tau \circ f \circ \psi^{-1}(\psi(O_3))$. If $x \in \psi(O_3)$, then its coordinates in $W_3 \times W_2$ is $(p_1 \circ \tau \circ f \circ \psi^{-1}(x), p_2 \circ \tau \circ f \circ \psi^{-1})(x))$ which agrees with the graph of $g$. The second step is established.

The third step is to produce the slice neighborhood. Take $W_1$ an open set with compact closure and $\bar{W_1} \subset W_3$. Let $\epsilon$ be such that $0 < \epsilon < \max\{|g(x) - y \mid x \in \bar{W_1}, y \in \mathbf{R}^n \setminus W_2\}$. Let $V_1 \subset W_1 \times W_2$ be the open set $\{(x, y) \in W_1 \times W_2 \mid |g(x) - y| < \epsilon\}$. Let $\gamma : V_1 \to W_1 \times B_\epsilon(0)$ by $\gamma(x, y) = (x, y - g(x))$. The map $\gamma$ is a diffeomorphims with inverse $(x, y) \mapsto (x, y + g(x))$. The image of the graph of $g$ under $\gamma$ is $W_1 \times \{0\}$. Let $U = \tau^{-1}(V_1)$, then $(U, \gamma \circ \tau)$ is the slice neighborhood: $y = g(x)$ if and only if $\gamma(x, y) = (x, 0)$.

It remains to show that $f$ is a diffeomorphism. Since $f$ is a homeomorphism to its image, $f$ has a continuous inverse. We need to see that $f$ is smooth as is its inverse. We use Proposition 2.18***. Given $x \in N$, there is a chart of $f(N)$ about $x$ that arises from a slice chart about $x$ in $M$, Proposition 3.3***. Let $(U, \phi)$, $\phi : U \to W_1 \times W_2$ be the slice chart and $(U \cap f(N), p_1 \circ \phi)$ the chart for $f(N)$. The map $f$ is a diffeomorphism if $p_1 \circ \phi \circ f \circ \psi^{-1}$ and its inverse are $C^\infty$ in a neighborhood of $\psi(x)$ and $\phi(f(x))$, respectively. Now, since $(U, \phi)$ is a slice neighborhood,

$$p_1 \circ \phi \circ f \circ \psi^{-1} = \phi \circ f \circ \psi^{-1}.$$

The derivative $D(\phi \circ f \circ \psi^{-1})(x)$ has rank $n$ since $\phi$ and $\psi$ are diffeomorphisms, and $f$ has rank $n$. By the Inverse Function Theorem, $p_1 \circ \phi \circ f \circ \psi^{-1}$ is $C^\infty$ on a neighborhood of $\phi(f(x))$. By Proposition 2.18***, $f$ and $f^{-1}$ are smooth functions. □

Some authors use the terminology that the image of a manifold under an immersion is a submanifold, but this usage is less common. Furthermore it requires the immersion in the definition. We will use the term immersed submanifold.

**Definition.** *Suppose $N$ and $M$ are manifolds and $f : N \to M$ is an immersion. Then $(N, f)$ is an immersed submanifold.*

This terminology is suggested by Exercise 1.***

**Proposition 6.5***.** *Suppose $N$ and $M$ are manifolds and $f : N \to M$ is a one-to-one immersion. If $N$ is compact, then $f$ is an embedding.*

*Proof.* We just need to show that $f$ is a homeomorphism to its image. It is a one-to-one continuous map from a compact space to a Hausdorff space. By a standard result in general topology, $f$ is a homeomorphism. □

## Exercises

**Exercise 1***.** *Suppose that* $f : N \to M$ *is a one-to-one immersion. Show that for every* $x \in N$ *there is a neighborhood* $U$ *of* $x$ *such that* $f\,|_U : U \to M$ *is an embedding. Show that the result holds even if* $f$ *is not one-to-one.*

The next exercise is a difficult and interesting exercise.

**Exercise 2***.** *Every compact* $n$*-manifold embeds in* $\mathbf{R}^N$ *for some* $N$*.*

This result is true without the hypothesis of compactness.

The dimension $N$ can be taken to be $2n$. That every $n$-manifold embedds in $\mathbf{R}^{2n}$ is a result by H. Whitney. It is also interesting to note that every compact $n$-manifold immerses in $\mathbf{R}^{2n-\alpha(n)}$ where $\alpha(n)$ is the number of ones in the dyadic expansion of $n$. This result was proven by Ralph Cohen. The connection to the dyadic expansion and that this result is the best possible arose in work by William S. Massey.

**Exercise 3***.** *Let* $f : \mathbf{RP}^2 \to \mathbf{R}^3$ *by* $f([x, y, z]) = (xy, xz, yz)$*. Show that* $f$ *is a well-defined smooth function. Is* $f$ *one-to-one? Is* $f$ *an immersion?*

*Let* $g : \mathbf{RP}^2 \to \mathbf{R}^4$ *by* $g([x, y, z]) = (xy, xz, yz, x^4)$*. Is* $g$ *an embedding or an immersion?*