CHAPTER 7
VECTOR BUNDLES

We next begin addressing the question: how do we assemble the tangent spaces at various points of a manifold into a coherent whole? In order to guide the decision, consider the case of $U \subset \mathbf{R}^n$ an open subset. We reflect on two aspects.

The first aspect is that the total derivative of a C^∞ function should change in a C^∞ manner from point to point. Consider the C^∞ map $f : U \to \mathbf{R}^m$. Each point $x \in U$ gives a linear map $f_{*x} = Df(x) : \mathbf{R}^n \to \mathbf{R}^m$ the total derivative which is represented by an $m \times n$ matrix, the Jacobian matrix of $Df(x)$. The Jacobian matrix is a matrix of C^∞ functions in x. While for each $x \in U$ there is a linear map

$$Df(x) : TU_x = \mathbf{R}^n \to \mathbf{R}^m = T\mathbf{R}^m_{f(x)}$$

these fit together to give a C^∞ map on the product

$$\begin{aligned} U \times \mathbf{R}^n &\to \mathbf{R}^m \times \mathbf{R}^m \\ (x, v) &\mapsto (f(x), Df(x)(v)). \end{aligned}$$

The second aspect is that we wish to define vector fields. A vector field is a choice of tangent vector at each point. For an open subset U of $\mathbf{R}^n$, a vector field is just given by a function $g : U \to \mathbf{R}^n$ (as the reader probably learned in Advanced Calculus). In order to keep track of the tail, we write the vector field as

$$\begin{aligned} V : U &\to U \times \mathbf{R}^n \\ x &\mapsto (x, g(x)). \end{aligned}$$

Any C^∞ function g gives a vector field. The complication on a manifold M is that the vector with tail at $x \in M$ must be in the vector space TM_x and these vector spaces change with x. In this chapter, we study the required concepts to assemble the tangent spaces of a manifold into a coherent whole and construct the tangent bundle. The tangent bundle is an example of an object called a vector bundle.

Definition 7.1*.** *Suppose M^n is a manifold. A real vector bundle over M consists of a topological space E, a continuous map $\pi : E \to M$ and a real vector space V (called the fiber) such that for each $m \in M$, $\pi^{-1}(m)$ is a vector space isomorphic to V, and there exists an open neighborhood $\mathcal{U}$ of m, and a homeomorphism*

$$\mu_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times V$$

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such that $\mu_{\mathcal{U}}^{-1}(m,-) : \{m\} \times V \to \pi^{-1}(m)$ is a linear isomorphism.

The bundle is smooth if E is a smooth manifold, π is smooth, and $\mu_{\mathcal{U}}$ is a diffeomorphism.

In these notes, all vector bundles will be smooth. We may denote a vector bundle by $\pi : E \to M$ (and suppress the vector space) or as $\mathcal{E}$. If the dimension of the vector space is m then the bundle is often called an m-plane bundle. A 1-plane bundle is also called a line bundle. A bundle over a manifold is trivial if it is simply the Cartesian product of the manifold and a vector space. The neighborhoods $\mathcal{U}$ over which the vector bundle looks like a product are called trivializing neighborhoods.

Note that $\mu_{\mathcal{W}} \circ \mu_{\mathcal{U}}^{-1} : \{m\} \times V \to \{m\} \times V$ is a linear isomorphism. Denote this map $h_{\mathcal{W}\mathcal{U}}(m)$.

Definition 7.2*.** *If $\mu_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times V$ and $\mu_{\mathcal{W}} : \pi^{-1}(\mathcal{W}) \to \mathcal{W} \times V$ are trivial neighborhoods of a vector bundle then*

$$\begin{aligned} \mu_{\mathcal{W}} \circ \mu_{\mathcal{U}}^{-1} : (\mathcal{W} \cap \mathcal{U}) \times V &\to (\mathcal{W} \cap \mathcal{U}) \times V \\ (x, v) &\mapsto (x, h_{\mathcal{W}\mathcal{U}}(x)v) \end{aligned}$$

where $h_{\mathcal{W}\mathcal{U}} : \mathcal{W} \cap \mathcal{U} \to GL(V)$. The $h_{\mathcal{W}\mathcal{U}}$ are associated to each pair of trivial neighborhoods $(\mathcal{U}, \mu_{\mathcal{U}})$ and $(\mathcal{W}, \mu_{\mathcal{W}})$. They are called transition functions.

Theorem 7.3*.** *Every smooth vector bundle has smooth transition functions, i.e., $h_{\mathcal{W}\mathcal{U}} : \mathcal{W} \cap \mathcal{U} \to GL(V)$ is smooth.*

Proof. The map $\mu_{\mathcal{W}} \circ \mu_{\mathcal{U}}^{-1}$ defines $h_{\mathcal{W}\mathcal{U}}$ so the issue is to see that $h_{\mathcal{W}\mathcal{U}}$ is smooth. Let $h_{\mathcal{W}\mathcal{U}}(x)$ be the matrix $(h_{ij}(x))_{ij}$ in a fixed basis for V. Then, $\mu_{\mathcal{W}} \circ \mu_{\mathcal{U}}^{-1}(x, (r_1, \cdots, r_n)) = (x, (\sum_j h_{1j}(x) r_j, \cdots, \sum_j h_{nj}(x) r_j))$. To see that each $h_{ij}(x)$ is smooth let $\vec{r}$ vary over e_i for $i = 1, \cdots n$. Since $\mu_{\mathcal{W}} \circ \mu_{\mathcal{U}}^{-1}$ is smooth, so are its coordinate functions. $\square$

Example 7.4*.** *Line Bundles Over S^1.*

We take the circle to be $S^1 = \{e^{\theta i} \mid \theta \in \mathbf{R}\}$ the unit circle in the complex plane $\{(\cos\theta, \sin\theta) \mid \theta \in \mathbf{R}^2\}$.

One line bundle over the circle is $\epsilon^1_{S^1}$, the trivial bundle $\pi_\epsilon : S^1 \times \mathbf{R} \to S^1$ by $\pi_\epsilon((e^{\theta i}, r)) = e^{\theta i}$. For the trivialization neighborhoods, only one is needed: take $U = S^1$.

There is another, more interesting line bundle over S^1. Let $E = \{(e^{\theta i}, re^{\frac{\theta}{2} i}) \mid r, \theta \in \mathbf{R}\}$ and $\pi_\gamma : E \to S^1$ by $\pi_\gamma((e^{\theta i}, re^{\frac{\theta}{2} i})) = e^{\theta i}$. Denote this bundle $\gamma^1_{S^1}$ Notice that $\pi_\gamma^{-1}(e^{\theta i})$ is a real line in the complex plane. Two values of θ that differ by 2π determine the same point, so $\frac{\theta}{2}$ is not well-defined. Nevertheless, the line in the complex plane through $e^{\frac{\theta}{2} i}$ is well defined since $e^{\frac{2\pi}{2} i} = -1$.

We now construct the trivializing neighborhoods. Let $U = S^1 \setminus \{1\} = \{(e^{\theta i}, re^{\frac{\theta}{2} i}) \mid \theta \in (0, 2\pi)\}$ and $W = S^1 \setminus \{-1\} = \{(e^{\theta i}, re^{\frac{\theta}{2} i}) \mid \theta \in (\pi, 3\pi)\}$. Now,

$$\begin{aligned} \mu_U : \pi_\gamma^{-1}(U) &\to U \times \mathbf{R} \\ (e^{\theta i}, re^{\frac{\theta}{2} i}) &\mapsto (e^{\theta i}, r). \end{aligned} \tag{1}$$

This map is well defined since $\theta \in (0, 2\pi)$, a restricted domain which allows us to determine θ from $e^{\frac{\theta}{2}i}$. We similarly define

$$\begin{aligned} \mu_W : \pi_\gamma^{-1}(U) &\to U \times \mathbf{R} \\ (e^{\theta i}, re^{\frac{\theta}{2}i}) &\mapsto (e^{\theta i}, r) \end{aligned} \tag{2}$$

for $\theta \in (\pi, 3\pi)$.

We next check the compatibility condition. The set $U \cap W$ is $S^1 \setminus \{1, -1\} = \{(e^{\theta i}, re^{\frac{\theta}{2}i}) \mid \theta \in (0, \pi) \cup (\pi, 2\pi)\}$. Suppose $\theta \in (0, \pi)$ then

$$\begin{aligned} \mu_W \mu_U^{-1}((e^{\theta i}, r)) &= \mu_W((e^{\theta i}, re^{\frac{\theta}{2}i})) \\ &= \mu_W((e^{(\theta+2\pi)i}, -re^{\frac{\theta+2\pi}{2}i})) \\ &= (e^{(\theta+2\pi)i}, -r) \\ &= (e^{\theta i}, -r) \end{aligned}$$

Notice that we had to change the expression for the second coordinate because formulas (1) and (2) require different domains. We have that $h_{UW}(e^{\theta i})(r) = -r$ for $\theta \in (0, \pi)$. Now, suppose $\theta \in (\pi, 2\pi)$, then

$$\begin{aligned} \mu_W \mu_U^{-1}((e^{\theta i}, r)) &= \mu_W((e^{\theta i}, re^{\frac{\theta}{2}i})) \\ &= (e^{\theta i}, r) \end{aligned}$$

We have that $h_{UW}(e^{\theta i})(r) = r$ for $\theta \in (\pi, 2\pi)$. Therefore the transition function $h_{UW} : U \cap W \to Gl(1, \mathbf{R})$ is

$$h_{UW}(x) = \begin{cases} -1 & \text{if } \mathrm{Im}(x) > 0 \\ 1 & \text{if } \mathrm{Im}(x) < 0 \end{cases}.$$

Example 7.5*.** *The Tautological Line Bundle Over* $\mathbf{RP}^n$

Define a $\mathbf{Z}_2$ action on $S^n \times \mathbf{R}$ by $(-1) \cdot (x, r) = (-x, -r)$. We show that this action satisfies the hypotheses of Theorem 2.23***. Suppose $(x, r) \in S^n \times \mathbf{R}$. Take U an open neighborhood of x in S^n that is entirely in one hemisphere. Then it follows that $U \cap -U = \emptyset$, and $U \times \mathbf{R}$ and $-U \times \mathbf{R}$ are disjoint neighborhoods of (x, r) and $(-1) \cdot (x, r)$. Let $E = S^n \times \mathbf{R}/\mathbf{Z}_2$. By 2.23***, E is a smooth manifold and the quotient map $\tilde{q} : S^n \times \mathbf{R} \to E$ is a local diffeomorphism. Let $\pi_E : E \to \mathbf{RP}^n$ by $\pi_E(x, r) = [(x, r)]$. The following diagram is a commutative diagram of smooth maps,

$$\begin{array}{ccc} S^n \times \mathbf{R} & \xrightarrow{\tilde{q}} & E \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_E} \\ S^n & \xrightarrow{q} & \mathbf{RP}^n \end{array}$$

where $\pi(x, r) = x$ and q is the quotient map from Example 2.25***, $\mathbf{RP}^n$. Let U be an open set in S^n that is entirely in one hemisphere so that $U \cap -U = \emptyset$. Then $\tilde{q}|_{U \times \mathbf{R}}$:

$U \times \mathbf{R} \to \pi_E^{-1}(q(U))$ is a diffeomorphism and linear on each fiber. If V is another such open subset of S^n then

$$(\tilde{q}|_{V\times\mathbf{R}})^{-1} \circ \tilde{q}|_{U\times\mathbf{R}} : U \cap (V \cup -V) \times \mathbf{R} \to U \cap (V \cup -V) \times \mathbf{R}$$
$$(x, r) \mapsto (x, h(x)r)$$

where

$$h(x) = \begin{cases} 1 \text{ if } x \in U \cap V \\ -1 \text{ if } x \in U \cap -V \end{cases}$$

Hence, $\pi_E : E \to \mathbf{RP}^n$ is a vector bundle.

Proposition 7.6*.** *Transition functions satisfy the following property:*

$$h_{\mathcal{W}\mathcal{U}}(x) = h_{\mathcal{W}\mathcal{O}}(x) \circ h_{\mathcal{O}\mathcal{U}}(x) \text{ for } x \in \mathcal{W} \cap \mathcal{O} \cap \mathcal{U}.$$

Proof. This property follows from the definition of transition functions and is the equation of the last coordinate of the equation below.
$(\mu_{\mathcal{W}} \circ \mu_{\mathcal{O}}^{-1}) \circ (\mu_{\mathcal{O}} \circ \mu_{\mathcal{U}}^{-1}) = \mu_{\mathcal{W}} \circ \mu_{\mathcal{U}}^{-1}$ □

The next theorem shows that a choice of transition functions consistent with the properties of the last proposition will determine the vector bundle. A bundle can be defined by the gluing (transition) functions. First a lemma.

Lemma 7.7*.** *Suppose X is a set and $\{U_i \mid i \in I\}$ is a collection of subsets. If $h_{ij} : U_i \cap U_j \to GL(n, \mathbf{R})$ for all $(i, j) \in I \times I$ satisfies $h_{ij}(x) = h_{ik}(x) \circ h_{kj}(x)$ for $x \in U_i \cap U_j \cap U_k$, then they also satisfy*

(1) $h_{ii}(x) = I_V$ *for* $x \in U_i$
(2) $h_{ij}(x) = (h_{ji}(x))^{-1}$ *for* $x \in U_i \cap U_j$

Proof. Since $h_{ii}(x) = h_{ii}(x) \circ h_{ii}(x)$ and $h_{ii}(x)$ has an inverse, multiply both sides by $h_{ii}(x)^{-1}$ to get (1). Since $h_{ij}(x)h_{ji}(x) = h_{ii}(x) = I_V$ by the hypothesis and (1), $h_{ij}(x) = h_{ji}(x)^{-1}$. □

Theorem 7.8*.** *Suppose M is a manifold and $\{(\mathcal{U}_i, \psi_i)|i \in I\}$ is a countable atlas for M. Suppose $V = \mathbf{R}^m$ is a vector space and for all $(i, j) \in I \times I$ there is a C^∞ function $f_{ij} : \mathcal{U}_i \cap \mathcal{U}_j \to GL(V)$ such that $f_{kj}(x) \circ f_{ji}(x) = f_{ki}(x)$ for all $x \in \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$.*

On $\{(u, v, i)|i \in I, u \in \mathcal{U}_i, v \in V\}$ let $(u, v, i) \backsim (u', v', i')$ if and only if $u = u'$, $f_{i'i}(u)(v) = v'$.

Then $\backsim$ is an equivalence relation. Furthermore, if $E = \dfrac{\{(u, v, i)|i \in I, u \in \mathcal{U}_i, v \in V\}}{\backsim}$ and $\pi([u, v, i]) = u$, then $\pi : E \to M$ is a smooth vector bundle with fiber V.

Proof. We first show that $\backsim$ is an equivalence relation, The functions f_{ij} also satisfy the hypotheses of Lemma 7.7***. The relation $\backsim$ is an equivalence relation as reflexivity, symmetry and transitivity are guaranteed by conditions 1 and 2 of Lemma 7.7***, and condition 3 is the hypothesis on the f_{ij}'s.

We must verify the various requirements of the definition of a vector bundle for $\pi : E \to M$. The longest part is done first, showing that E is a manifold.

We first introduce the map μ. Let $\mu_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times V$ by $\mu_i([u, v, i]) = (u, v)$. This map is a bijection: if $u \in \mathcal{U}_i$ then the class $[u, v, i]$ has (u, v, i) as the unique representative with the last coordinate i by the definition of $\backsim$ as $f_{ii}(u) = I_V$.

We verify that E is a smooth manifold by using Theorem 2.15*** Let $\phi_i = (\psi_i \times I_V) \circ \mu_i$ so $\phi_i : \pi^{-1}(\mathcal{U}_i) \to \psi_i(\mathcal{U}_i) \times V \subset \mathbf{R}^n \times V = \mathbf{R}^{n+m}$. We claim that E is a smooth manifold with atlas $\mathcal{A} = \{(\pi^{-1}(\mathcal{U}_i), \phi_i) | i \in I\}$. We verify the conditions of Theorem 2.15*** with $X = E$ and $\mathcal{A}$. The first two conditions are true, $\pi^{-1}(\mathcal{U}_i) \subset E$ and $\bigcup_{i \in I} \pi^{-1}(\mathcal{U}_i) = E$ by construction. Next, the map ϕ_i is a bijection since both $\mu_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times V$ and $\psi_i \times I_V : \mathcal{U}_i \times V \to \psi_i(\mathcal{U}_i) \times V$ are bijections. The fourth condition is also immediate: $\phi_i : \pi^{-1}(\mathcal{U}_i) \to \psi_i(\mathcal{U}_i) \times V \subset \mathbf{R}^n \times V$ and $\phi_i : \pi^{-1}(\mathcal{U}_i) \cap \pi^{-1}(\mathcal{U}_j) \to \psi_i(\mathcal{U}_i \cap \mathcal{U}_j) \times V \subset \mathbf{R}^n \times V$ are both open. The last condition is that $\phi_j \phi_i^{-1} : \phi_i(\pi^{-1}(\mathcal{U}_i) \cap \pi^{-1}(\mathcal{U}_j)) \to \phi_j(\pi^{-1}(\mathcal{U}_i) \cap \pi^{-1}(\mathcal{U}_j))$ is smooth. We now show that the map is smooth. Suppose $x \in \psi_i(\mathcal{U}_i \cap \mathcal{U}_j)$ and $v \in V$. Then,

$$\begin{aligned}
\phi_j \phi_i^{-1}(x, v) &= (\psi_j \times I_V) \circ \mu_j \circ \mu_i^{-1} \circ (\psi_i^{-1} \times I_V)(x, v) \\
&= (\psi_j \times I_V) \circ \mu_j \circ \mu_i^{-1}(\psi_i^{-1}(x), v) \\
&= (\psi_j \times I_V) \circ \mu_j([\psi_i^{-1}(x), v, i]) \\
&= \psi_j \times I_V \circ \mu_j([\psi_i^{-1}(x), f_{ji}(\psi_i^{-1}(x))(v), j]) \qquad \text{by the equivalence relation} \\
&= \psi_j \times I_V(\psi_i^{-1}(x), f_{ji}(\psi_i^{-1}(x))(v)) \\
&= (\psi_j \psi_i^{-1}(x), f_{ji}(\psi_i^{-1}(x))(v))
\end{aligned}$$

We have shown that $\phi_j \phi_i^{-1}(x, v) = (\psi_j \psi_i^{-1}(x), f_{ji}(\psi_i^{-1}(x))(v))$. The first coordinate is smooth since M is a manifold. The second coordinate map is smooth since it is a composition of the following smooth maps:

$$\psi(\mathcal{U}_i \cap \mathcal{U}_j) \times \mathbf{R}^m \xrightarrow{f_1} (\mathcal{U}_i \cap \mathcal{U}_j) \times \mathbf{R}^m \xrightarrow{f_2} \mathrm{GL}(n, \mathbf{R}) \times \mathbf{R}^m \xrightarrow{f_3} \mathbf{R}^m$$

where $f_1(y, v) = (\psi^{-1}(y), v)$ is smooth since ψ^{-1} is smooth, $f_2(x, v) = (f_{ji}(x), v)$ is smooth by Theorem 7.3***, and f_3 is smooth since it is evaluation of a linear function at a vector, i.e., matrix multiplication. The hypotheses of Theorem 2.15*** are verified. We show in the next two paragraphs that E is second countable and Hausdorff, and hence a manifold.

Since I is countable, $\{\pi^{-1}(\mathcal{U}_i) | i \in I\}$ is countable and so E is a second countable space.

Suppose $[u, v, i], [u', v', j] \in E$. If $u \neq u'$, then there are open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ that separate u and u' in M. Hence $\pi^{-1}(\mathcal{O}_1)$ and $\pi^{-1}(\mathcal{O}_1)$ separate $[u, v, i]$ and $[u', v', j]$ in E. If $u = u'$, then $[u, v, i], [u', v', j] \in \pi^{-1}(\mathcal{U}_i)$ which is Hausdorff. Therefore E is Hausdorff and E is a manifold.

We now show that $\pi : E \to M$ is a vector bundle. The vector space structure on $\pi^{-1}(m)$ is defined by the structure on V: $a[m, v, i] + b[m, v', i] = [m, av + bv', i]$. This is well defined since if $[m, v, i] = [m, w, j]$ and $[m, v', i] = [m, w', j]$ then $f_{ij}(m)(w) = v$ and

$f_{ij}(m)(w') = v'$ so

$$\begin{aligned} a[m,w,j] + b[m,w',j] &= a[m, f_{ij}(m)(w), i] + b[m, f_{ij}(m)(w'), i] \\ &= [m, af_{ij}(m)(w) + bf_{ij}(m)(w'), i] \\ &= [m, f_{ij}(m)(aw + bw'), i] \\ &= [m, av + bv', i] \end{aligned}$$

The map $\mu_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times V$ is a diffeomorphism. This was shown above as these maps are part of the manifold structure for E (Theorem 2.21***).

$\mu_i^{-1}(m,-) : V \to \pi^{-1}(m)$ is an isomorphism as $\mu_i^{-1}(m,-)(v) = \mu_i^{-1}(m,v) = [m,v,i]$.

π is smooth since if m is in the coordinate neighborhood given by $(\mathcal{U}_i, \psi_i)$, then $[m,v,i]$ is in the coordinate neighborhood given by $(\pi^{-1}(\mathcal{U}_i), \phi_i)$ and $\psi_i \circ \pi \circ \phi_i^{-1}(x,v) = x$ for $(x,v) \in \phi_i(\pi^{-1}(\mathcal{U}_i)) \subset \mathbf{R}^n \times V$. $\square$

Definition 7.9*.** *A bundle map between two vector bundles $E_1 \xrightarrow{\pi_1} M_1$ and $E_2 \xrightarrow{\pi_2} M_2$ is a pair of smooth maps $f : E_1 \to E_2$ and $g : M_1 \to M_2$ such that $\pi_2 \circ f = g \circ \pi_1$ and $f\big|_x : \pi_1^{-1}(x) \to \pi_2^{-1}(g(x))$ is linear.*

Definition 7.10*.** *Suppose that $E_1 \xrightarrow{\pi_1} M$ and $E_2 \xrightarrow{\pi_2} M$ are bundles over a manifold M. A bundle equivalence between these bundles is a a bundle map (f, I_M) over the identity is an isomorphism on each fiber.*

Please note the result Exercise 2***: if $\mathcal{E}_1$ an $\mathcal{E}_2$ are vector bundles over the manifold M and if $\mathcal{E}_1$ is bundle equivalent to $\mathcal{E}_2$ by a bundle equivalence f, then $\mathcal{E}_2$ is bundle equivalent to $\mathcal{E}_1$ by f^{-1} which is also a bundle equivalence.

Definition 7.11*.** *A section of a vector bundle, $E \xrightarrow{\pi} M$ is a smooth map $s : M \to E$ such that $\pi \circ s = I_M$.*

Notation 7.12*.** *Suppose $\mathcal{E}$ is $E \xrightarrow{\pi} M$ a vector bundle with fiber V. Let $\Gamma(\mathcal{E})$ or $\Gamma(E)$ denote the sections of the vector bundle $\mathcal{E}$.*

Example 7.13*.** *The sections of the trivial bundle.*

The sections of ϵ_M^n are smooth maps $s : M \to M \times \mathbf{R}^n$ with $\pi(s(x)) = x$ for all $x \in M$. Hence, the first coordinate of $s(x)$ is x and $s(x) = (x, f(x))$ for any smooth function $f : M \to \mathbf{R}^n$. Therefore, $C^\infty(M, \mathbf{R}^n) = \Gamma(\epsilon_M^n)$ by the correspondence: if $f \in C^\infty(M, \mathbf{R}^n)$ then $s \in \Gamma(\epsilon_M^n)$ with $s(x) = (x, f(x))$.

Example 7.14*.** *The sections of line bundles over S^1*

We first note that the function $p : \mathbf{R} \to S^1$ by $p(\theta) = (\cos\theta, \sin\theta) \in \mathbf{R}^2$ or $e^{\theta i} \in \mathbf{C}$ is a local diffeomorphism. An inverse of $p : (\theta_0 - \frac{\pi}{2}, \theta_0 + \frac{\pi}{2}) \to p(\theta_0 - \frac{\pi}{2}, \theta_0 + \frac{\pi}{2})$ is $p^{-1}(x,y) = \arcsin(y\cos\theta_0 - x\sin\theta_0) + \theta_0$. Of course, p^{-1} isn't globally well defined as $p^{-1}(\cos\theta, \sin\theta) = \{\theta + 2k\pi \mid k \in \mathbf{Z}\}$

In Example 7.13*** it was shown that the sections of $\epsilon_{S^1}^1$ were given by the smooth functions $f : S^1 \to \mathbf{R}$. We also note that these sections can be given by the functions $\{g : \mathbf{R} \to \mathbf{R} \mid g(\theta + 2\pi) = g(\theta)\}$, i.e., periodic functions of the angle rather than of a point on the circle. Given a section $s \in \Gamma(\epsilon^1_{S^1})$ with $s(\cos\theta, \sin\theta) = ((\cos\theta, \sin\theta), f(\cos\theta, \sin\theta)$, the function g is $g(\theta) = f(\cos\theta, \sin\theta)$. Conversely, given a periodic function g let the section s be $s(\cos\theta, \sin\theta) = ((\cos\theta, \sin\theta), g(\theta))$. The section is well-defined since g is periodic and it is smooth since p has a smooth local inverse.

We now describe the sections of $\gamma^1_{S^1}$. The bundle was described in Example 7.4***. We show that $\Gamma(\gamma^1_{S^1})$ correspond to the functions $\{g : \mathbf{R} \to \mathbf{R} \mid g(\theta + 2\pi) = -g(\theta)\}$. Given a function g take s to be the section $s(e^{\theta i}) = (e^{\theta i}, g(\theta)e^{\frac{\theta}{2}i})$. Conversely, given a section s, it can be written as $s(e^{\theta i}) = (e^{\theta i}, h(e^{\theta i}))$, where by the definition of $\gamma^1_{S^1}$, $h(e^{\theta i})$ is a real multiple of $e^{\frac{\theta}{2}i}$. Then let $g(\theta) = h(e^{\theta i})e^{-\frac{\theta}{2}i}$.

Remark 7.15*.** *A section of a vector bundle is a way of choosing an element in each fiber that varies in a smooth manner. One speaks of an "element of a vector space" and the appropriate generalization to a vector bundle usually is a "section of a vector bundle."*

Example 7.16*.** *The zero section*

Every vector space has a distinguished element: zero, the additive identity. If $E \xrightarrow{\pi} M$ is a vector bundle, let $0_x \in \pi^{-1}(x)$ be the zero. Let $z : M \to E$ be defined by $z(x) = 0_x$ for all $x \in M$. The section z is called the zero section.

We check that this map is a smooth section. If $x \in M$ take an open neighborhood that is both part of a coordinate neighborhood (U, ϕ) and a trivializing neighborhood for E, $\mu : U \to U \times \mathbf{R}^m$. The map z is smooth on U if and only if $Z = (\phi \times I_V) \circ \mu \circ z \circ \phi^{-1} : \phi(U) \to \phi(U) \times \mathbf{R}^m$ is smooth. The map Z is $Z(x) = (x, 0)$ which is a smooth map of an open subset of $\mathbf{R}^n$ to $\mathbf{R}^n \times \mathbf{R}^m$. Therefore, z is a smooth section.

Proposition 7.17*.** *A bundle $\pi : E \to M$ with an n-dimensional fiber is a trivial bundle if and only if it has n sections $\{s_1, \cdots, s_n\}$ such that $s_1(x), \cdots, s_n(x)$ are a basis for $\pi^{-1}(x)$ for each $x \in M$.*

Proof. If $f : M \times \mathbf{R}^n \to E$ is a bundle equivalence over M, then let s_i be defined by $s_i(x) = f(x, e_i)$. The sections $\{s_1, \cdots, s_n\}$ are the required set of sections. Note that $s_i(x) = (x, e_i)$ are the required sections for the trivial bundle.

If $\pi : E \to M$ is a bundle with sections $\{s_1, \cdots, s_n\}$ such that $s_1(x), \cdots, s_n(x)$ are a basis for $\pi^{-1}(x)$ for each $x \in M$, then

$$f : M \times \mathbf{R}^n \to E$$

defined by $f(x, (a_1, \cdots, a_n)) = \sum_{i=1}^n a_i s_i(x)$ is a smooth map and isomorphism on each fiber. Therefore f is a bundle equivalence. $\square$

Theorem 7.18*.** *If N is an n-manifold, then there is a vector bundle $\pi_N : TN \to N$ such that $\pi_N^{-1}(x) = TM_x$. Furthermore, if M is an m-manifold, and $f : N \to M$ is a smooth map, then there is a bundle map $f_* : TN \to TM$ define by $f_*|_{TN_x} = f_{*x}$ for each $x \in N$, i.e., the bundle map on TN_x is the derivative map.*

The first part of the theorem asserts the existence of a vector bundle. The substance, the justification to call the bundle the tangent bundle is in the second paragraph. The derivative map at each point combines to form a smooth map. It indicates that the construction of the tangent bundle is the philosophically correct manner to combine the tangent spaces into a coherent whole.

*Proof of Theorem 7.18***.* For U an open set in $\mathbf{R}^n$, the tangent bundle is a simple cross product $U \times \mathbf{R}^n$. A basis in the fiber over a point p is $\displaystyle\sum_{i=1}^{n} a_i \left.\frac{\partial}{\partial x_i}\right|_p$ where $\left.\frac{\partial}{\partial x_i}\right|_p$ represents the equivalence class of the path $t \mapsto te_i + p$. Observe that $TU = U \times \mathbf{R}^n$ agrees with the usual notions from Calculus as discussed in the beginning of this chapter.

We now construct the tangent bundle for a smooth n-manifold M. Let $\{(U_i, \phi_i) | i \in I\}$ be a countable atlas, and let $f_{ji} : U_i \cap U_j \to GL(\mathbf{R}^n)$ be defined by

$$f_{ji}(u) = (\phi_j \circ \phi_i^{-1})_{*\phi_i(u)} = D(\phi_j \circ \phi_i^{-1})(\phi_i(u)).$$

First note that since $\phi_j \circ \phi_i^{-1}$ is a C^∞ function on an open set of $\mathbf{R}^n$ into $\mathbf{R}^n$, each matrix entry of the Jacobian of $\phi_j \circ \phi_i^{-1}$ is a C^∞ function. Therefore, $D(\phi_j \circ \phi_i^{-1})(\phi_i(u)) = f_{ji}(u)$ is a smooth function.

We next show that these maps satisfy the conditions of Theorem 7.8*** and so define a vector bundle, $TM \to M$. The above functions $f_{ji}(u) = (\phi_j \circ \phi_i^{-1})_{*\phi_i(u)}$ satisfy the condition for transition functions. Suppose $u \in U_i \cap U_j \cap U_k$, then $(\phi_j \circ \phi_k^{-1})_{*\phi_k(u)}(\phi_k \circ \phi_i^{-1})_{*\phi_i(u)} = (\phi_j \circ \phi_i^{-1})_{*\phi_i(u)}$ by the chain rule. Hence $f_{jk}(u) \circ f_{ki}(u) = f_{ji}(u)$. The hyposthesis of Theorem 7.8*** are satisfied and so give a well-defined bundle, the tangent bundle of M.

The next task is to show that if $f : N \to M$ is a smooth map, then there is a bundle map $f_* : TN \to TM$ define by $f_*|_{TN_x} = f_{*x}$ for each $x \in N$.

The map f_* is well defined and linear on each tangent space TN_x by the given formula. It must be shown that it is a smooth map from TN to TM. Let $\pi_N : TN \to N$ and $\pi_M : TM \to M$ be the projections for the tangent bundles. Take any point in TN. It is in the fiber over some point in N, say $x \in N$.

Let (W, ψ) be a chart on M with $f(x) \in W$. By construction of the tangent bundle, $\pi_M^{-1}(W)$ is open in TM and

$$\begin{aligned}\Psi : \pi_M^{-1}(W) &\to \psi(W) \times \mathbf{R}^m \\ v &\mapsto (\psi(\pi_M(v)), \psi_{*\pi_M(v)}(v))\end{aligned}$$

is a coordinate chart.

Let (U, ϕ) be a chart on N with $x \in U$ and $f(U) \subset W$. Again, by construction of the tangent bundle, $\pi_N^{-1}(U)$ is open in TN and

$$\begin{aligned}\Phi : \pi_N^{-1}(U) &\to \phi(U) \times \mathbf{R}^n \\ v &\mapsto (\phi(\pi_N(v)), \phi_{*\pi_N(v)}(v))\end{aligned}$$

is a coordinate chart. We check smoothness by using Proposition 2.18***. Suppose $(x, v) \in \Phi(\pi^{-1}(U) \subset \mathbf{R}^n \times \mathbf{R}^n$. Then compute

$$\begin{aligned}\Psi \circ f_* \circ \Phi^{-1}((x, v)) &= (\psi \circ f \circ \phi^{-1}(x), \psi_{*f(\phi^{-1}(x)} \circ f_{*\phi^{-1}(x)} \circ \phi_{*x}^{-1}(v)) \\ &= (\psi \circ f \circ \phi^{-1}(x), (\psi \circ f \circ \phi^{-1})_{*x}(v)).\end{aligned}$$

The last line follows by the chain rule, Theorem 5.3*** part 2. This map is C^∞ in (x, v) since it the derivative map for $\psi \circ f \circ \phi^{-1}$ a C^∞ function between open subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$. $\square$

Definition 7.19*.** *A section of the tangent bundle of a manifold M is called a vector field on M.*

Example 7.20*.** *The tangent bundle to the 2-sphere TS^2.*

Let $S^2 = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Recall from Calculus that the tangent plane to S^2 translated to $(0, 0, 0)$ is $P_{(x,y,z)} = \{(v_1, v_2, v_3) \in \mathbf{R}^3 \mid (x, y, z) \cdot (v_1, v_2, v_3) = 0\}$. Let $E = \{(\vec{x}, \vec{v}) \in \mathbf{R}^3 \times \mathbf{R}^3 \mid |\vec{x}| = 1 \text{ and } \vec{v} \in P_{\vec{x}}\}$ and $\pi : E \to S^2$ by $\pi((\vec{x}, \vec{v})) = \vec{x}$ is a vector bundle which is bundle equivalent to the tangent bundle.

We first examine the tangent bundle of S^2 using the atlas constructed in Example 2.9A*** for S^2. Let $U_1 = S^2 \setminus \{(0, 0, 1)\}$ and $U_2 = S^2 \setminus \{(0, 0, -1)\}$. Let $\phi_i : U_i \to \mathbf{R}^2$ for $i = 1$ and 2 be $\phi_1(x, y, z) = (\frac{2x}{1-z}, \frac{2y}{1-z})$ and $\phi_2(x, y, z) = (\frac{2x}{1+z}, \frac{2y}{1+z})$. Then $\phi_1^{-1}(x, y) = (\frac{4x}{x^2+y^2+4}, \frac{4y}{x^2+y^2+4}, \frac{x^2+y^2-4}{x^2+y^2+4})$ and $\phi_2 \circ \phi_1^{-1}(x, y) = (\frac{4x}{x^2+y^2}, \frac{4y}{x^2+y^2})$. The transition function is then $g_{21} : U_1 \cap U_2 \to Gl(2, \mathbf{R})$ defined by

$$g_{21}(\vec{x}) = D(\phi_2 \circ \phi_1^{-1})(\phi_1(\vec{x})) \tag{Eq 1***}$$

and

$$D(\phi_2 \circ \phi_1^{-1})((x, y)) = \begin{pmatrix} \frac{4(y^2-x^2)}{(x^2+y^2)^2} & \frac{-8xy}{(x^2+y^2)^2} \\ \frac{-8xy}{(x^2+y^2)^2} & \frac{4(x^2-y^2)}{(x^2+y^2)^2} \end{pmatrix}$$

It is C^∞ since ϕ_1 and $D(\phi_2 \circ \phi_1^{-1})$ are C^∞.

We now turn to $\pi : E \to S^2$. For each $\vec{x} \in S^2$, we note that $\pi^{-1}(\vec{x}) = P_{\vec{x}}$ is a vector space, a two dimensional subspace of $\mathbf{R}^3$. We define homeomorphisms $h_i : U_i \times \mathbf{R}^2 \to \pi^{-1}(U_i)$ for $i = 1$ and 2. These maps will be used to show E is a 4-manifold. They will commute with the projections and be linear on each fiber. Their inverses will be the trivialization maps for the bundle. The compatibility condition for the charts is the smoothness of the transition functions.

Let

$$\begin{aligned} h_1 : U_1 \times \mathbf{R}^n &\to \pi^{-1}(U_1) \\ (\vec{x}, \vec{v}) &\mapsto (\vec{x}, D\phi_1^{-1}(\phi_1(\vec{x}))\vec{v}) \end{aligned} \quad \text{and} \quad \begin{aligned} h_2 : U_2 \times \mathbf{R}^n &\to \pi^{-1}(U_2) \\ (\vec{x}, \vec{v}) &\mapsto (\vec{x}, D\phi_2^{-1}(\phi_2(\vec{x}))\vec{v}) \end{aligned}$$

The inverses are

$$\begin{aligned} \mu_1 : \pi^{-1}(U_1) &\to U_1 \times \mathbf{R}^n \\ (\vec{x}, \vec{v}) &\mapsto (\vec{x}, D\phi_1(\vec{x})\vec{v}) \end{aligned} \quad \text{and} \quad \begin{aligned} \mu_2 : \pi^{-1}(U_2) &\to U_2 \times \mathbf{R}^n \\ (\vec{x}, \vec{v}) &\mapsto (\vec{x}, D\phi_2(\vec{x})\vec{v}) \end{aligned}$$

These maps are homeomorphisms as

$$D\phi_i^{-1}(\phi_i(\vec{x}))D\phi_i(\vec{x}) = I_{P_{\vec{x}}} \text{ and } D\phi_i(\vec{x})D\phi_i^{-1}(\phi_i(\vec{x})) = I_{\mathbf{R}^2}$$

by the chain rule. Note that the transition function for the bundle E is

$$g_{21}(\vec{x}) = (D\phi_2(\vec{x})D\phi_1^{-1}(\phi_1(\vec{x})) = D(\phi_2 \circ \phi_1^{-1})(\phi_1(\vec{x}))$$

which is the same as we computed for the tangent bundle.

We further pursue Example 7.20*** and prove that TS^2 is not trivial. We actually show more. We prove that S^2 doesn't have any nowhere zero section.

The proof of this theorem requires that the reader knows some of the basics of homotopy. We will require

(1) The definition of a homotopy class.
(2) The fact that the map $f : S^1 \to \mathbf{R}^2 \setminus \{(0,0)\}$ defined by $f(e^{\theta i}) = e^{2\theta i}$ or $f((\cos\theta, \sin\theta)) = (\cos 2\theta, \sin 2\theta)$ is not homotopic to a constant map.

We state the specific elements we will use in the following lemma.

Lemma 7.21*.** *Suppose $g : S^1 \to GL(2, \mathbf{R})$ is a continuous function. For any continuous function $f : S^1 \to \mathbf{R}^2 \setminus \{(0,0)\}$, let $G(f) : S^1 \to \mathbf{R}^2 \setminus \{(0,0)\}$ by $G(f)(p) = g(p)(f(p))$. We then have*

(1) *If $f : S^1 \to \mathbf{R}^2 \setminus \{(0,0)\}$ be any continuous map, then the map f is homotopic to $-f$.*
(2) *If $f, h : S^1 \to \mathbf{R}^2 \setminus \{(0,0)\}$ and f is homotopic to h, then $G(f)$ is homotopic to $G(h)$.*
(3) *Let $S^1 \subset S^2$ be the equator and g the transition function that from TS^2 defined in* (Eq I***) *of Example 7.20***. Furthermore, let $\iota : S^1 \to \mathbf{R}^2 \setminus \{(0,0)\}$ be the constant map $\iota(p) = (1,0)$. Then the map $G(\iota)$ is not homotopic to a constant map.*

Proof. To show item 1 we give a homotopy from f to $-f$. Let $H : S^1 \times [0,1] \to \mathbf{R}^2 \setminus \{(0,0)\}$ by

$$H(p,t) = \begin{pmatrix} \cos \pi t & \sin \pi t \\ -\sin \pi t & \cos \pi t \end{pmatrix} f(p).$$

Item 2 is also easy to see by the appropriate homotopy. Suppose that $H : S^1 \times [0,1] \to \mathbf{R}^2 \setminus \{(0,0)$ is the homotopy between f and h, then $(p,t) \mapsto g(p)(H(p,t))$ is the homotopy between $G(f)$ and $G(h)$.

To show item 3, we first write $p = (x, y, 0)$ for p on the equator of S^2 and recall that $\phi_1(p) = (2x, 2y)$ for the map ϕ, stereographic projection (see Example 7.20***). Since the transition function $g(p)$ is the matrix given in (Eq I***) of Example 7.20***,

$$\begin{aligned} G(\iota)(p) &= D(\phi_2 \circ \phi_1^{-1})(\phi_1(p))(\iota(p)) \\ &= \begin{pmatrix} \frac{4((2y)^2-(2x)^2)}{((2x)^2+(2y)^2)^2} & \frac{-8(2x2y)}{((2x)^2+(2y)^2)^2} \\ \frac{-8(2x2y)}{((2x)^2+(2y)^2)^2} & \frac{4((2x)^2-(2y)^2)}{((2x)^2+(2y)^2)^2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} y^2 - x^2 \\ -2xy \end{pmatrix} \end{aligned}$$

The last line follows since $x^2 + y^2 = 1$. Writing $x = \cos\theta$ and $y = \sin\theta$, $G(\iota)(\cos\theta, \sin\theta) = -(\cos 2\theta, \sin 2\theta)$ using the double angle identities. By item 1, $G(\iota)$ is homotopic to the map $(\cos\theta, \sin\theta) \mapsto (\cos 2\theta, \sin 2\theta)$, which is not homotopic to a constant map. $\square$

Theorem 7.22*.** *There is no nonzero vector field on S^2.*

As a consequence of this theorem and Proposition 7.17***, TS^2 is not equivalent to a trivial vector bundle.

Proof. We use proof by contradition. Suppose V is a nowhere zero section of TS^2, i.e., V is a nowhere zero vector field. We examine the two maps that characterize the vector field in the trivializing neighborhoods for TS^2 as given in Example 7.20***. We use the maps and notation from Example 7.20***. For each $p \in S^2 \setminus \{(0,0,1)\}$, $\phi_{1*}(V_p) = (\phi_1(p), f(p)) \in \mathbf{R}^2 \times \mathbf{R}^2 \setminus \{(0,0)\}$ and for each $p \in S^2 \setminus \{(0,0,-1)\}$, $\phi_{2*}(V_p) = (\phi_2(p), h(p)) \in \mathbf{R}^2 \times \mathbf{R}^2 \setminus \{(0,0)\}$. The images of the maps f and g avoid $(0,0)$ precisely because the vector field is nowhere zero. It is easy to see that f and g are homotopic to constant maps. The null homotopy for f is $H : S^1 \times [0,1] \to \mathbf{R}^2 \setminus \{(0,0)\}$ by $H(p,t) = f(\phi_1^{-1}(t\phi_1(p)))$ and for h is $K : S^1 \times [0,1] \to \mathbf{R}^2 \setminus \{(0,0)\}$ by $K(p,t) = h(\phi_1^{-1}(t\phi_1(p)))$. The maps f and h are also related by the transition function on the sphere by $h(p) = g_{21}(p)f(p)$. In other words, if f is null homotopic then $G(f)$ is also null homotopic. This conclusion contradicts item 3 of the lemma. Therefore TS^2 doesn't have a nonzero section. $\square$

Duality and Dual Bundles.

We review the notion of duality from linear algebra. Suppose V is a vector space. Let V^* be the dual vector space $\mathrm{Hom}(V, \mathbf{R})$, the vector space of linear functions from V to $\mathbf{R}$. The relationship between V and V^* is usually expressed in the evalutation pairing:

$$V^* \times V \to \mathbf{R}$$
$$(f, v) \mapsto f(v)$$

We give the basic properties of V^* in the next two theorems. The first theorem is usually covered in a linear algebra course. The second isn't difficult, but has a flavor different from a first course in linear algebra.

Theorem 7.23. *For any finite dimensional vector space V, let V^* be the dual space of linear functionals.*

1. *V^* is a vector space of the same dimension as V.*
2. *Suppose that $e_1, \cdots, e_n$ is a basis for V. Let $e_i^*(\sum_{j=1}^n a_j e_j) = a_i$. Then $e_1^*, \cdots, e_n^*$ is a basis for V^* called the dual basis.*
3. *Let V and W be vector spaces and $f : V \to W$ be a linear map. Let $f^* : W^* \to V^*$ by $f^*(g) = g \circ f$.*

Proof. We can add linear functionals and multiply be a constant. If fact $V^* = \mathrm{Hom}(V, \mathbf{R})$ is isomorphic to the vector space of n by 1 matrices. The isomorphism is given by a choice of basis. If the choice of basis is $e_1, \cdots, e_n$, then e_i^* is the matrix with a 1 in the i^{th} row and zeros elsewhere. This shows items 1 and 2.

Item 3 is a short computation. Suppose $g_1, g_2 \in V^*$. Then,

$$\begin{aligned} f^*(ag_1 + bg_2) &= (ag_1 + bg_2) \circ f \\ &= ag_1 \circ f + bg_2 \circ f \\ &= f^*(ag_1) \\ &= af^*(g_1) + bf^*(ag_2) \end{aligned}$$

$\square$

Theorem 7.24*.** *The following are properties of $*$. Suppose V, W and X are finite dimensional vector spaces.*

(1) *If $h : V \to W$ and $f : W \to X$ are linear, then $(f \circ h)^* = h^* \circ f^*$*
(2) *$I_V^* = I_{V^*}$*
(3) *If $f : W \to V$ is linear and one-to-one, then f^* is onto.*
(4) *If $f : W \to V$ is linear and onto, then f^* is one-to-one.*
(5) *If $f : W \to V$ is an isomorphism, then f^* is an isomorphism and $(f^{-1})^* = (f^*)^{-1}$.*
(6) *The map $F : GL(V) \to GL(V^*)$ by $F(f) = f^*$ is a C^∞ map.*

Proof. We check each item.

Proof of item 1:

$$\begin{aligned}(f \circ h)^*(\alpha) &= \alpha \circ f \circ h \\ &= h^*(\alpha \circ f) \\ &= f^* \circ h^*(\alpha)\end{aligned}$$

Proof of item 2: $I_V^*(\alpha) = \alpha \circ I_V = \alpha = I_{V^*}$

Proof of item 3: The function f is one-to-one if and only if there is a function h such that $h \circ f = I_V$. Hence $(h \circ f)^* = I_V^*$. By 1 and 2, $f^* \circ h^* = I_{V^*}$. Since f^* has a right inverse, f^* is onto.

Proof of item 4: The function f is onto if and only if there is a function h such that $f \circ h = I_V$. Hence $(f \circ h)^* = I_V^*$. By 1 and 2, $h^* \circ f^* = I_{V^*}$. Since f^* has a left inverse, f^* is one-to-one.

Proof of item 5: Since f is one-to-one and onto, so is f^* by 3 and 4. Now, $f \circ f^{-1} = I_V$. Items 1 and 2 imply, $(f^{-1})^* \circ f^* = I_{V^*}$, therefore $(f^{-1})^* = (f^*)^{-1}$

Proof of item 6: We pick charts on $GL(V)$ and $GL(V^*)$ by picking bases for V and V^*. Suppose $\{e_i | i = 1 \cdots n\}$ is a basis for V. Let $e_i^* \in V^*$, where $e_i^*(\sum_{j=1}^n b_j e_j) = b_i$. Take $\{e_i^* | i = 1 \cdots n\}$ as a basis for V^*. If f is represented by the matrix M in the e basis then f^* is represented by M^T in the e^* basis. We check this fact.

$$\begin{aligned}f^*(\sum_{i=1}^n a_i e_i^*)(\sum_{j=1}^n b_j e_j) &= (\sum_{i=1}^n a_i e_i^*) \sum_{j=1}^n \sum_{k=1}^n m_{kj} b_j e_k \\ &= \sum_{i,j=1}^n a_i m_{ij} b_j \\ &= (\sum_{i,j=1}^n m_{ij} a_i e_j^*)(\sum_{k=1}^n b_k e_k)\end{aligned}$$

So, $f^*(\sum_{i=1}^n a_i e_i^*) = \sum_{i,j=1}^n m_{ij} a_i e_j^*$ and M^T represents f^* in the e^* basis. If we consider $GL(V)$ and $GL(V^*)$ as open sets in $\mathbf{R}^{n^2}$ via matrix coordinates, then F is $M \mapsto M^T$. Since transposing is just a permutation of the entries, the map is C^∞ and in fact, linear. $\square$

If $\pi : E \to M$ is a vector bundle $\mathcal{E}$ with fiber the vector space V, then we can then construct $\mathcal{E}^*$, a dual bundle $\rho : E^* \to M$. The fiber should be V^*, but how should the fibers be assembled? Consider the following: each point in V^* gives a map from V to $\mathbf{R}$, so each point in $\rho^{-1}(x)$ should give a map from $\pi^{-1}(x)$ to $\mathbf{R}$. A section of $\rho : E^* \to M$ picks out a point in each fiber and varies in a smooth manner. Therefore, we want a section of $\rho : E^* \to M$ to give a map to $\mathbf{R}$ which is linear on each fiber. Another way to think of the above description is that a section of the dual bundle should give a bundle map of $\mathcal{E}$ to ϵ^1_M, the trivial line bundle over M.

Before proceeding to the construction of the cotangent bundle, we first undertake a discussion of the meaning of a dual of a bundle, i.e., the co in cotangent. For a vector space the meaning is expressed in the evaluation pairing,

$$\begin{aligned} V^* \times V &\to \mathbf{R} \\ (f, v) &\mapsto f(v). \end{aligned}$$

For vector bundles, there should be a pairing for each $x \in M$ and this pairing should vary smoothly in x. Therefore there should be an evaluation pairing,

$$\Gamma(\mathcal{E}^*) \times \Gamma(\mathcal{E}) \to \Gamma(\epsilon^1_M) \tag{2}$$

with $(f, s)(x) = (x, f(x)(s(x)))$. Note that this condition is guided by Remark 7.15***.

Definition 7.25*.** *Suppose $\mathcal{E}$ is a vector bundle $\pi : E \to M$. The vector bundle $\rho : E^* \to M$ is the dual bundle to $\mathcal{E}$ if $\rho^{-1}(x) = (\pi^{-1}(x))^*$ for all $x \in M$ and if $\mathcal{E}^*$ satisfies the following property: There is an evaluation pairing*

$$F : \Gamma(\mathcal{E}^*) \times \Gamma(\mathcal{E}) \to \Gamma(\epsilon^1_M) \tag{2}$$

defined by $F(f, s) = f(x)(s(x))$ for all $x \in M$.

The pairing is the usual pairing on each fiber, but the fibers fit together smoothly so the evaluation varies smoothly over M.

The reader should notice that the dual bundle to the trivial bundle is again a trivial bundle since the dual to the bundle $\pi : M \times V \to M$ is the bundle $\pi : M \times V^* \to M$.

Theorem 7.26*.** *Suppose $\mathcal{E}$ is a vector bundle $\pi : E \to M$ with fiber V, then there is a dual vector bundle $\mathcal{E}^*$, $\pi : E^* \to M$ with fiber V^*.*

Furthermore, if $\mathcal{U} = \{(U_i, \varphi_i) \mid i \in I\}$ is a countable atlas for M with each U_i also a trivializing neighborhood for $\mathcal{E}$ and $g_{ij} : U_i \cap U_j \to GL(V)$ are the transition functions for $\mathcal{E}$, then $f_{ij} : U_i \cap U_j \to GL(V^*)$ defined by $f_{ij}(x) = (g_{ji}(x))^*$ for each $x \in U_i \cap U_j$ forms a set of transition functions for $\mathcal{E}^*$.

Proof. The cover $\mathcal{U}$, the vector space V^* and the functions f_{ij} satisfy Theorem 7.8*** and so define a vector bundle $\mathcal{E}^*$. To see this fact, we must check the condition the f_{ij} must satisfy.

$$\begin{aligned} f_{ik}(x) f_{kj}(x) &= (g_{ki}(x))^* (g_{jk}(x))^* \\ &= (g_{jk}(x) g_{ki}(x))^* \text{ by properties of } {*}, \text{ Theorem 7.24***} \\ &= (g_{ji}(x))^* \text{ by properties of transition functions, Proposition 7.6***} \\ &= f_{ij}(x) \end{aligned}$$

which is the required property. It remains to check the property of the pairing (2). This condition is a local condition and so we check it in the trivializations. Suppose that $\mu : \pi^{-1}(U_i) \to U_i \times V$ by $\mu(e) = (\pi(e), h_i(e))$ and $\gamma : \pi^{*-1}(U_i) \to U_i \times V^*$ by $\gamma(e) = (\pi(e), k_i(e))$. If s_1 and s_2 are sections then for $x \in U_i$ the pairing (2) is

$$(s_2(x), s_1(x)) \mapsto (x, k_i(s_2(x))(h_i(s_1(x)))).$$

Since s_1, s_2, h_i, and k_i are smooth, the pairing is also smooth, but we must check that the pairing is well-defined. If $x \in U_j$, then $f_{ij}(x)(k_j(s_2(x))) = g^*_{ji}(x)(k_j(s_2(x))) = k_i(s_2(x))$ and $g_{ij}(x)(h_j(s_1(x))) = h_i(s_1(x))$. Hence,

$$\begin{aligned} k_i(s_2(x))(h_i(s_1(x)) &= f_{ij}(x)(k_j(s_2(x)))(g_{ij}(x)(h_j(s_1(x)))) \\ &= g^*_{ji}(x)(k_j(s_2(x)))(g_{ij}(x)(h_j(s_1(x)))) \\ &= (k_j(s_2(x)))(g_{ji}(x)g_{ij}(x)(h_j(s_1(x)))) \\ &= k_j(s_2(x))(h_j(s_1(x))) \end{aligned}$$

So the pairing is well-defined. □

Example 7.27*.** *The Cotangent Bundle*

Suppose M is a smooth n-manifold. The dual bundle to the tangent bundle is called the cotangent bundle. The cotangent space at a point x is $(TM_x)^*$ which we denote T^*M_x. The bundle dual to the tangent bundle is $(TM)^*$ in the duality notation, but is usually denoted T^*M.

Let $f_{ij}(m) = (\varphi_j \varphi_i^{-1})^*_{\varphi_j(m)}$. Note that if g_{ij} are the transition functions from which we constructed the tangent bundle above, then $f_{ij}(x) = (g_{ji}(x))^*$. This is the construction specified by Theorem 7.25*** to construct the bundle dual to the tangent bundle. Call it the cotangent bundle and denote it T^*M.

If $\phi : N \to M$ is a smooth map between smooth manifolds, then there is an induced map $\phi^*_{\phi(x)} : T^*M_{\phi(x)} \to T^*N_x$ define by $\phi^*_{\phi(x)}(\gamma) = \gamma \circ \phi_{*x}$.

Example 7.28*.** *A section of the cotangent bundle, df.*

Suppose $f : M \to \mathbf{R}$, then let $df_x = \pi \circ f_*$ where $\pi : T\mathbf{R} = \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is projection in the fiber direction, i.e., $\pi(a \left.\frac{\partial}{\partial x_i}\right|_p) = a$. Suppose that $\phi : M \to N$ is a smooth map and an onto map. Then,

$$\phi^*(df_x) = df_x \circ \phi_* = \pi \circ f_* \circ \phi_* = d(f \circ \phi).$$

Suppose $f : M \to \mathbf{R}$ then we will show that df defines a section of T^*M. If $x \in U_i$, then $x \mapsto d(f \circ \varphi_i^{-1})_{\varphi_i(x)}$ is a smooth map since $x \mapsto (f \circ \varphi_i^{-1})_{*\varphi_i(x)}$ is smooth. In the notation of Theorem 7.8***, df_x is represented by $[x, d(f \circ \varphi_i^{-1})_{\varphi_i(x)}, i]$. We have that if $x \in U_i \cap U_j$, then

$$[x, d(f \circ \varphi_i^{-1})_{\varphi_i(x)}, i] = [x, d(f \circ \varphi_j^{-1})_{\varphi_j(x)}, j]$$

as $f_{ij}(x)((d(f \circ \varphi_j^{-1})_{\varphi_j(x)}) = (\varphi_j \circ \varphi_i^{-1})^* d(f \circ \varphi_j^{-1})_{\varphi_j(x)} = d(f \circ \varphi_j^{-1} \circ \varphi_j \circ \varphi_i^{-1})_{\varphi_i(x)} = d(f \circ \varphi_i^{-1})_{\varphi_i(x)}$ using the equivalence relation of Theorem 7.8***.

Remark 7.29. *Sections of the tangent and cotangent bundle are both used to generalize types of integration.*

Sections of the tangent bundle are used to integrate, solve a differential equation. Sections of the cotangent bundle are used to compute definite integrals.

Exercises

Exercise 1*.** *Let $\pi : E \to M$ be a vector bundle. Show that it has sections that are not identically zero.*

This next exercise justifies the terminology bundle equivalence.

Exercise 2*.** *If $\mathcal{E}_1$ an $\mathcal{E}_2$ are vector bundles over the manifold M and if $\mathcal{E}_1$ is bundle equivalent to $\mathcal{E}_2$ by a bundle equivalence f, then $\mathcal{E}_2$ is bundle equivalent to $\mathcal{E}_1$ by f^{-1} which is also a bundle equivalence.*

Show that bundle equivalence is an equivalence relation.

Exercise 3*.** *Show that every vector bundle over an interval is trivial.*

Exercise 4*.** *Show that up to bundle equivalence, there are exactly two distinct line bundles over the circle.*

This next exercise introduces the notion of a Reimannian metric.

Exercise 5*.** *Suppose that $\mathcal{E}$ is a vector bundle $\pi : E \to M$. A Riemannian metric on $\mathcal{E}$ is a choice of inner product $<,>_x$ for each fiber $\pi^{-1}(x)$ such that there is an induced map on sections*

$$<,>: \Gamma(\mathcal{E}) \times \Gamma(\mathcal{E}) \to \Gamma(\epsilon^1_M)$$

defined by $< s_1, s_2 > (x) =< s_1(x), s_2(x) >_x$.
a. Show that every vector bundle has a Riemannian metric. This argument will require a partition of unity.
b. Show that $\mathcal{E}$ is bundle equivalent to $\mathcal{E}^*$.