HOMEWORK SOLUTIONS

Scattered Homework Solutions for Math 7550, Differential Geometry, Spring 2006. If students have solutions written in some form of TeX that they would like to submit to me for problems not posted, I'll check them and, if correct, post them in this file with proper credit given (and maybe even a little extra course credit).

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CHAPTER 2
MANIFOLDS

Exercises

Exercise 1*.** *Verify the calculations of Example 2.9a***. Show the two atlases given for S^n in Example 2.9a*** and Example 2.9b*** give the same differential structure and so may be merged.*

Solution. We calculate the formula for the stereographic projection from the south pole, the formula for the stereographic projection (from the north pole) given in Example 2.9a*** being derived in a completely analogous way. For $(x_1, \ldots, x_{n+1}) \in U_2 = S^n \setminus \{(0, \ldots, 0, -1)\}$ this projection is the point $(y_1, \ldots, y_n, 0)$ belonging to the parametrized line $t(x_1, \ldots, x_{n+1}) + (1-t)(0, \ldots, 0, -1)$. Solving in the last coordinate yields $0 = tx_{n+1} - (1-t)(-1)$, so that $t = 1/(x_{n+1} + 1)$. Thus $\phi_2 : U_2 \to \mathbf{R}^n$ is given by

$$\phi_2(x_1, x_2, \cdots, x_{n+1}) = \frac{1}{x_{n+1}+1}(x_1, \ldots, x_n) = (\frac{x_1}{1+x_{n+1}}, \cdots, \frac{x_n}{1+x_{n+1}}).$$

One verfies by direct computation that this is equal to the formula given for ϕ_2 given in Example 2.9a***, namely $\phi_2 = -\phi_1 \circ (-1)$. The inverse ϕ_2^{-1} is calculated by starting with $(y_1, \ldots, y_n) \in \mathbf{R}^n$ and calculating where the parametrized line $t(y_1, \ldots, y_n, 0) + (1-t)(0, \ldots, 0, -1)$ has norm 1 and is not the south pole. Thus $t^2(\sum_{i=1}^n y_i^2) + (1-t)^2 = 1$. Expanding the second term on the left, simplifying, and solving for nonzero t ($t = 0$ yields the south pole), we obtain $t = 2/(\sum_{i=1}^n y_i^2 + 1)$. Thus $\phi_2^{-1} : \mathbf{R}^n \to U_2$ is given by

$$\phi_2^{-1}(y_1, \ldots, y_n) = \left(\frac{2y_1}{\sum_{i=1}^n y_i^2 + 1}, \frac{2y_2}{\sum_{i=1}^n y_i^2 + 1}, \cdots, \frac{2y_n}{\sum_{i=1}^n y_i^2 + 1}, \ \frac{2}{\sum_{i=1}^n y_i^2 + 1} - 1\right).$$

The rest of the assertions of Example 2.9a*** are straightforward.

We show the the chart $\phi_{i,+} : U_{i,+} \to \mathbf{R}^n$ of Example 2.9b*** defined by

$$\phi_{i,+}(x_1, \cdots, x_{n+1}) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1})$$

is compatible with the stereographic projections. Using the formula for ϕ_2^{-1} from the previous paragraph, we easily see by direct computation that $\phi_{i,+} \circ \phi_2^{-1}$ is C^∞. We have

$$\phi_{i,+}^{-1}(y_1, \ldots, y_n) = (y_1, \ldots, y_{i-1}, 1 - \sum_{i=1}^n y_i^2, y_i, \ldots, y_n).$$

Again a straightforward computation establishes that

$$(\phi_2 \circ \phi_{i,+}^{-1})(y_1, \ldots, y_n) = \frac{1}{y_n + 1}(y_1, \ldots, y_{i-1}, 1 - \sum_{i=1}^n y_i^2, y_i, \ldots, y_{n-1}),$$

which is C^∞ in each coordinate on the interior of the unit ball in $\mathbf{R}^n$. The other compositions of hemispherical and stereographic projections follow similarly. $\square$

Exercise 2*.** *$S^1 \times S^1$ is a 2-manifold, $S^2 \times S^1$ is a 3-manifold,and $S^2 \times S^1 \times S^1$ is a 4-manifold.*

Of course these all follow from Proposition 2.13***. The reader should note, however, that there is an ambiguity in $S^2 \times S^1 \times S^1$, is it $(S^2 \times S^1) \times S^1$ or $S^2 \times (S^1 \times S^1)$? The reader should show that the atlases are compatible and so these are the same manifold.

There is also a second approach that is sometimes used to define smooth functions. In this approach, one first defines a smooth function for $f : M \to \mathbf{R}$ only. The statement of the next exercise would be a defintion in some textbooks, e.g., Warner and Helgason, but for us, it is a proposition.

Exercise 3*.** *Show that a function $f : M^m \to N^n$ between manifolds is smooth if and only if for all open sets $\mathcal{U} \subset N$ and all smooth functions $g : \mathcal{U} \to \mathbf{R}$, $g \circ f$ is smooth on its domain.*

Solution. Suppose f is smooth and g is smooth then $\psi \circ f \circ \phi^{-1}$ and $g \circ \psi^{-1}$ are C^∞ on their domains for choices of charts and hence so is

$$g \circ f \circ \phi^{-1} = (g \circ \psi^{-1})(\psi \circ f \circ \phi^{-1}).$$

Therefore $g \circ f$ is smooth.

To prove the converse, take charts $(\mathcal{U}, \phi)$ and $(\mathcal{W}, \psi)$ of M and N respectively. Now, $\psi \circ f \circ \phi^{-1}$ is C^∞ if and only if $r_i \circ \psi \circ f \circ \phi^{-1}$ is C^∞ for $i = 1, \cdots, n$. This is true by hypothesis, since $r_i \circ \psi : W \to \mathbf{R}$ is smooth, i.e., let g run through $r_i \circ \psi$ for all charts and $i = 1, \cdots, n$. □

Exercise 4*.** *Consider $\mathbf{R}$ with the following three atlases:*

(1) $\mathcal{A}_1 = \{f_1\}$, *where* $f_1(x) = x$
(2) $\mathcal{A}_2 = \{f_2\}$ *where* $f_2(x) = x^3$
(3) $\mathcal{A}_3 = \{f_3\}$ *where* $f(x) = x^3 + x$

Which of these atlases determines the same differential structure. Which of the manifolds are diffeomorphic?

Solution. f_1 and f_2 do not determine the same differentiable structure since $(f_1 \circ f_2^{-1})(x) = x^{1/3}$, which is not differentiable at 0. f_1 and f_3 do determine the same differentiable structure since $(f_3 \circ f_1^{-1})(x) = x + x^3$, which is smooth, and $(f_1 \circ f_3^{-1})(x) = f_3^{-1}(x)$, which we now argue is C^∞. We note first that $f'(x) = 1 + x^2 > 0$ for all x, so f is strictly increasing, hence a bijection from $\mathbf{R}$ to $\mathbf{R}$. Its derivative everywhere has nonzero determinant (namely the value of the derivative), hence is invertible, hence by the Inverse Function Theorem locally a C^∞-map with C^∞-inverse. Since being C^∞ is a local property, we conclude that f_3^{-1} is C^∞.

The differential structures given by f_1 and f_2 do determine diffeomorphic manifolds. Indeed the mapping $g(x) = x^{1/3}$ is seen to be a diffeomorphism from (1) to (2) since $f_2 g f_1^{-1}$ and $f_1 g f_2^{-1}$ are both the identity mappings, hence C^∞. □

Exercise 5*.** *Let M, N, and Q be manifolds.*

(1) *Show that the projections $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ are smooth.*
(2) *Show that $f : Q \to M \times N$ is smooth iff $\pi_i f$ is smooth for $i = 1, 2$.*
(3) *Show for $b \in N$ that the inclusion $x \mapsto (x, b) : M \to M \times N$ is smooth.*

solution. (1) Let $(x, y) \in M \times N$, let (U, ϕ) be a chart into $\mathbf{R}^m$ for M at x, and let (V, ψ) be a chart into $\mathbf{R}^n$ for N at y. Then $(U \times V, \phi \times \psi)$ is a chart for $M \times N$ at (x, y), by the definition of the product differentiable structure. Now $\phi \circ \pi_1 \circ (\phi \times \psi)^{-1}$ is a projection from $\phi(U) \times \psi(V)$ to $\phi(U)$ and hence is the restriction of the first projection of $\mathbf{R}^{n+m} \cong \mathbf{R}^m \times \mathbf{R}^n$ to $\mathbf{R}^m$, which is C^∞ since it is linear. Thus the restriction to the open set $\phi(U) \times \psi(V)$ is C^∞, Hence by Proposition 2.18*** we conclude that $\pi_1 : M \times N \to M$ is smooth. In a similar way one shows that π_2 is smooth.

(2) If f is smooth then $\pi_i \circ f$ is smooth for $i = 1, 2$ since by part (1) π_i is smooth and the composition of smooth maps is smooth. The converse is argued by passing to charts in a manner analogous to the proof of part (1) and using the fact the for open subsets of Euclidean space a function into a product is C^∞ if and only if each coordinate function is.

(3) Each of the two coordinate functions is smooth, one being a constant function and one being the idenity, and hence by (2) the given inclusion if smooth.

The following is a difficult exercise.

Exercise 6*.** *Prove that the set of all $n \times n$ matrices of rank k (where $k < n$) is a smooth manifold. What is its dimension?*

If this is too hard, then prove that the set of all $n \times n$ matrices of rank 1 is a smooth manifold of dimension $2n - 1$.

Warmup Exercises, Chapter 2

Basic Facts: In the following exercises these basic facts witll be needed about C^∞-functions defined on open subspaces of Euclidean spaces:

(1) The composition of C^∞-functions is C^∞.
(2) The restriction of a C^∞ to an open subset of its domain is C^∞.
(3) A function is C^∞ if every point in the domain has an open neighborhood on which the function is C^∞.

Exercise 1*. *Suppose that $\{(U_i, \phi_i) : i \in I\}$ is an atlas for M. Argue that a chart (V, ψ) is compatible with the atlas if for each $x \in V$, there exists an open set W, $x \in W \subseteq V$ and an $i_x \in I$ such that $x \in U_{i_x}$ and $\phi_{i_x} \circ (\psi|_W)^{-1}$ and $\psi|_W \circ \phi_{i_x}^{-1}$ are C^∞. (Hint: See the proof of Theorem 2.5***.)*

Exercise 2*. *Suppose that $\{(U_i, \phi_i) : i \in I\}$ is an atlas for M, $J \subseteq I$ and $\bigcup_{i\in J} U_i = M$. Argue that $\{(U_i, \phi) : i \in I\}$ is an atlas that generates the same differentiable structure on M.*

Exercise 3*. *Prove the following corollary of Theorem 2.5*** and its proof.*

Corollary. *Let $\mathcal{A}$ be an atlas for a smooth manifold M. A chart (U, ϕ) belongs to the smooth differentiable structure generated by the atlas if and only if it is compatible with all charts in $\mathcal{A}$. In particular if $\mathcal{A}$ is a maximal atlas (i.e., a differentiable structure), the chart belongs to $\mathcal{A}$ if it is compatible with all charts in $\mathcal{A}$.*

Exercise 4*. *(i) Let $\phi : U \to \mathbf{R}^n$ be a chart for a smooth manifold M and let V be a nonempty open subset of U. Argue that $\phi|_V : V \to \mathbf{R}^n$ is also a chart in the differentiable structure of M.*
(ii) Let $\{(U_j, \phi_j) : j \in J\}$ be a family of charts belonging to a differentiable structure on M such that $\phi_i|_{U_i\cap U_j} = \phi_j|_{U_i\cap U_j}$ for all $i, j \in J$. Argue that $(\bigcup_{j\in J} U_j, \phi)$ is in the differentiable structure, where $\phi|_{U_j} = \phi_j|_{U_j}$ for each $j \in J$.

Exercise 5*. *Let U be a nonempty open subset of a manifold M. Show that the charts of M with domain contained in U form a differentiable structure on U. Show that the restriction of any chart on M to U belongs to this differentiable structure. Show that the restrictions to U of any atlas of charts for M yields an atlas for U.*

Exercise 6*. *The restriction of a smooth map $f : M \to N$ to a nonempty open subset U of M is again smooth. In particular, if $f : M \to N$ is a diffeomorphism, then $f|_U : U \to f(U)$ is a diffeomorphism.*

Exercise 7*. *Show that a function from an open subset U of $\mathbf{R}^n$ to an open subset V of $\mathbf{R}^m$ is smooth (with respect to the manifolds U and V) if and only if it is C^∞.*

Exercise 8*. *Verify the note after Definition 2.17***.*

Exercise 9*. *Prove Proposition 2.21***.*

CHAPTER 5 ADDENDUM
THE TANGENT BUNDLE

A *chart vector at x* for a chart (U, ϕ) with $x \in U \subseteq M$, an n-dimensional smooth manifold, is a pair $(\phi(x), v) \in T\big(\phi(U)\big)$. We define two chart vectors at x for two charts (U, ϕ) and (V, ψ) to be *equivalent*, written $(\phi(x), u) \sim (\psi(x), v)$, if

$$\begin{aligned} T(\psi \circ \phi^{-1})\big((\phi(x), u)\big)v := \big(\psi(x), D(\psi \circ \phi^{-1})(\phi(x))(u)\big) = (\psi(x), v) \\ \Leftrightarrow D(\psi \circ \phi^{-1})(\phi((x))(u) = v. \end{aligned}$$

Exercise 2.1. *Show that $\sim$ is an equivalence relation on the set of chart vectors at x. Show for any chart (U, ϕ) with $x \in U$, assigning to $u \in \mathbf{R}^n$ the equivalence class of $(\phi(x), u)$ defines a one-to-one correspondence between $\mathbf{R}^n$ and the set of equivalence classes of chart vectors at x.*

Solution. We use the last equivalence in the definition of $\sim$. Let (U, ϕ) be a chart at x, $u \in \mathbf{R}^n$. Then $u = D(\phi \circ \phi^{-1})(\phi(x))(u)$ since the derivative of the identity map is the identity map, and hence $\sim$ is reflexive.

For symmetry one can apply the Inverse Function Theorem, or argue directly as follows. Assume $D(\psi \circ \phi^{-1})(\phi((x))(u) = v$. Then

$$\begin{aligned} D(\phi \circ \psi^{-1})(\psi(x))(v) &= D(\phi \circ \psi^{-1})(\psi(x))(D(\psi \circ \phi^{-1})(\phi((x))(u)) \\ &= D(\phi \circ \psi^{-1} \circ \psi \circ \phi^{-1})(\phi(x))(u) \quad \text{(Chain Rule)} \\ &= D(\mathrm{Id})(\phi(x))(u) = u. \end{aligned}$$

We conclude that $\sim$ is symmetric.

For transitivity, assume that (U, ϕ), (V, ψ) and (W, θ) are charts at x, and that $D(\psi \circ \phi^{-1})(\phi((x))(u) = v$ and $D(\theta \circ \psi^{-1})(\psi((x))(v) = w$. Then

$$\begin{aligned} D(\theta \circ \phi^{-1})(\phi((x))(u) &= D(\theta \circ \psi^{-1} \circ \psi \circ \phi^{-1})(\phi((x))(u) \\ &= [D(\theta \circ \psi^{-1})(\psi \circ \phi^{-1})(\phi(x)) \circ D(\psi \circ \phi^{-1})(\phi(x))](u) \\ &= D(\theta \circ \psi^{-1})(\psi(x))(v) = w, \end{aligned}$$

and thus $\sim$ is transitive.

If for some chart (U, ϕ) at x, we have $(\phi(x), u) \sim (\phi(x), v)$, then $D(\phi \circ \phi^{-1})\phi(x)(u) = v$. Thus $u = v$ since the derivative on the left is the identity. Hence the map $\gamma : \mathbf{R} \to T_x M$ that sends u to the set of equivalence classe of $(\phi(x), u)$ is injective. Let $(\psi(x), v)$ be any chart vector at x. Then for $u := D(\phi \circ \psi^{-1})(\psi(x))(v)$, we have $(\psi(x), v) \sim (\phi(x), u)$. Hence $\gamma(u)$ is the equivalence class of $(psi(x), v)$, and γ is surjective.

Exercise 2.2. *(i) Show that if* $(\phi(x), u_1) \sim (\psi(x), v_1)$ *and* $(\phi(x), u_2) \sim (\psi(x), v_2)$*, then* $(\phi(x), u_1 + u_2) \sim (\psi(x), v_1 + v_2)$*. Similarly show for* $(\phi(x), u) \sim (\psi(x), v)$ *and* $r \in \mathbf{R}$ *that* $(\phi(x), ru) \sim (\psi(x), rv)$*. Hence there is a well defined vector addition and scalar multiplication on the set of equivalence classes.*
(ii) Show for any chart (U, ϕ) *with* $x \in U$*, the map that sends* $(\phi(x), u) \in T_{\phi(x)}(\phi(U))$ *to its equivalence class is a vector space isomorphism from* $T_{\phi(x)}(\phi(U)$ *to the space of equivalence classes equipped with vector addition and scalar multiplication defined in (i).*