# MATHEMATICS AND ITS HISTORY

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# Chapter 1

# Mathematics of Ancient Egypt

## 1.1  History

Egyptian mathematics dates back at least almost 4000 years ago. The main sources about mathematics in ancient Egypt are the "Moscow Papyrus," dating back to around 1850 B.C., and "Rhind's Papyrus" in the British Museum, which was copied from the original in 1650 B.C. by someone named Ahmose. Egyptian mathematics apparently arose from various practical concerns such as surveying of fields after Nile flooding, construction projects such as the pyramids, the making of calendars, and various accounting activities and was practiced by scribes and perhaps priests.

The two papyri contain various specific elementary mathematical problems (for example: how many bricks does it take to build a ramp of certain dimensions?) and their solutions (often worked out). Although the problems often go well beyond practical concerns, we find no general methods of solutions described and nothing that could be considered theoretical, in particular theoretical justifications or proofs.

The Egyptians used a base 10 grouping system (with symbols representing various powers of 10 grouped together), but the scribes also developed a ciphered system (with symbols for all numbers of the form $k \cdot 10^n$ for $1 \leq k \leq 9$), which made writing of numbers much less cumbersome. They used fractions which they curiously insisted on representing as sums of integer reciprocals, i.e., fractions with numerator 1 (for example, Ahmose writes $\frac{2}{45} = \frac{1}{30} + \frac{1}{90}$). The methods used to rewrite fractions in this form are not clearly understood.

Adding and subtracting are easy in grouping systems. The Egyptians had a multiplication algorithm that reduced multiplication to repeated doubling of one muliplicand and a final addition. They used a modified version of this algorithm for division.

The Egyptian papyri present various word problems that involve solving linear equations and various methods for solution, including one similar to that we use today. Another technique was that of *false position*, where a convenient but incorrect answer is assumed and then appropriated adjusted to obtain the correct one. The solutions are much more challenging since no modern algebraic symbolism is used, only descriptions in words of what is being done, and since the Egyptian treatment of fractions was so cumbersome.

In geometry the papyri demonstrate a knowledge of basic area and volume formulas. One interesting formula is that used for the area of a circle, given the diameter: $A = [(8/9)d]^2$ (this may be viewed as an early form of "squaring the circle," since the area of the circle is approximately the square of 8/9 of the diameter).The most impressive formula obtained is that for the volume of a truncated pyramid with square base of side $a$ and square top of side $b$: $V = \frac{1}{3}h(a^2 + ab + b^2)$. It remains unknown how the Egyptians found this formula.

## 1.2 Problems

1. Rewrite the following numbers in hieroglyphics and add:
(i) $234 + 765$     (ii) $4,555 + 5,648$     (iii) $36,486 + 9,018$

2. The "doubling" method of Egyptian multiplication requires writing any whole number as a sum of powers of two. How do you know that this can be done for any number? Use the reverse remainder algorithm to write the following numbers as sums of powers of two:
(i) 73     (ii) 52     (iii) 98     (iv) 151

3. Multiply using the Egyptian method of doubling:
(i) $19 \times 29$     (ii) $25 \times 73$     (iii) $71 \times 211$

4. The validity of the Egyptian multiplication algorithm depends on the ability to write a number as a sum of powers of 2 and what basic arithmetic law?

5. The Egyptian multiplication algorithm can also be used with negative powers of 2. Use this idea to show the product of $1 + \frac{1}{2} + \frac{1}{4}$ by $\frac{1}{14}$ is equal to $\frac{1}{8}$ (Problem 12 of the Rhind Papyrus).

6. Perform the following divisions using the Egyptian method:
(i) $96 \div 8$     (ii) $805 \div 35$     (iii) $84 \div 5$

7. The Egyptians wrote their fractions as sums of *distinct* unit fractions. Historians are unsure about how the Egyptians reduced fractions to this form. One method for doing this that has been referred to as the *splitting method* depends on the identity

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}.$$

For example,
$$\frac{2}{19} = \frac{1}{19} + \frac{1}{19} = \frac{1}{19} + \frac{1}{20} + \frac{1}{380}.$$
For fractions with larger numerators, the splitting may need to be repeated.
(i) Verify the algebraic identity behind the splitting method.
(ii) Show that for any positive integers $p, q$,

$$\frac{1}{pq} = \frac{1}{p(p+q)} + \frac{1}{q(p+q)}.$$

(ii) Represent $\frac{3}{7}$ and $\frac{4}{15}$ as sums of distinct unit fractions by using (a) the splitting method and (b) Fibonacci's method.

8. The *Method of False Position* is a guess method for solving *linear* equations. For example, a number plus its double plus its third equals 20. What is the number? We make some initial guess, typically one that is easy to evaluate. If we guess 12, then we obtain $12+2(12)+(12/3) = 12+24+4 = 40$. We then form the fraction (desired answer)/(answer from guess)$= 20/40 = 1/2$ and multiply it by our first guess $(1/2) \times 12 = 6$ to get the desired answer.
(i) Use the method of false position to solve

$$3x - \frac{x}{4} + \frac{3}{8}x = 10$$

(ii) Problem 32 of the Rhind Papyrus states that a quantity, its third, and its fourth added together yield 2. Use the method of false position to find the quantity. Express your answer in Egyptian fashion.

9. The Moscow and Rhind Papyri give what are probably the earliest remaining records of area and volume formulas. Problem 41 of the Rhind Papyrus asks for the volume of grain that can be stored in a cylindrical granary of diameter 9 cubits and height 10 cubits. Solve the problem using the Egyptian value $\pi = 4(8/9)^2$.

10. Problem 58 of the Rhind Papyrus asks for the "seked" of a pyramid (with square base) if it is $93\frac{1}{3}$ cubits high and the side of its base is 140 cubits. (Explanation: The seked of an isosceles triangle is given by $s \div 2h$ where $s$ is the length of the base and $h$ is the height or altitude on the base. For the pyramid in question the seked would be the seked of a isosceles slice of the pyramid with base connecting the midpoints of opposing sides.)
(i) Verify that the seked of an isosceles triangle equals the cotangent of one of the base angles.
(ii) Solve Problem 58. How does this compare to the slope of a lateral face of the pyramid? (The seked in this problem has been associated with the slope of the lateral faces of the Second Pyramid at Gizeh.)

11. The Moscow Papyrus (1850 B.C.) shows that the Egyptians were familiar with the correct formula for the volume of a truncated pyramid:

$$V = \frac{h}{3}(a^2 + ab + b^2),$$

where $h$ is the altitude, $a$ is the length of the square base, and $b$ is the length of the square top.
(i) Solve Problem 14, the volume of a truncated pyramid of vertical height

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6, a side of length 4 for the base, and a side of length 2 on the top.
(ii) Show that the Egyptian formula is equivalent to one known to the Babylonians:

$$V = h\left[\left(\frac{a+b}{2}\right)^2 + \frac{1}{3}\left(\frac{a-b}{2}\right)^2\right].$$

# Chapter 2

# Mathematics of Ancient Babylon

## 2.1 History

The Babylonians lived in Mesopotamia (currently southern Iraq) from about 2000 B.C. until around 500 B.C, having displaced the earlier Sumerians and Akkadians. They exhibited developed systems of agriculture, government, laws, religion, achooling, and city communities, among others. They also possessed a substantial body of mathematical knowledge, well before most other civilizations. Their writings have been passed down to us on hundreds of clay tablets, on which they wrote in cuneiform (wedge-shaped) symbols. Most of the Babylonian mathematics that has come down to us on these tablets is from the period 1800-1600 B.C.

Some important characteristics of Old Babylonian mathematics stand out. Almost always, the goal of a problem was the computation of a number. Students were not asked to produce a figure, nor give what we would call a proof. Because of this emphasis on numeric computation based on procedures, Babylonian mathematics might be called algorithmic, as opposed to the geometric focus of the Greeks and their deductive, axiomatic approach. But the Babylonians never stated a general procedure; instead they gave worked examples. The students would work through lots of examples until they could do the problem with any set of initial values.

The Babylonians were strong believers in word problems. Apart from a few procedure texts for finding things like square roots, most Old Baby-

lonian problems were couched in a language of measurement of everyday objects and activities. Students had to find lengths of canals dug, weights of stones, lengths of broken reeds, areas of fields, numbers of bricks used in a construction, and so on. Despite this focus, Babylonian mathematics problems were often artificial and had little to do with the real world. However, many of the problems used realistic coefficients and settings, especially those concerned with defense construction and economic activities.

Many of the unearthed mathematical tablets of the Babylonians consist of arithmetical tables of squares, square roots, cube roots, reciprocals, and some powers. These tables, which are easier to translate than more complicated texts, were an important tool in their attainment of an advanced level of calculation. The tables make it clear that the Babylonians used a base-60 or sexagesimal system, not the usual base-10 system. Part of that system has been preserved even to the present in the way we measure time (and latitude and longitude): sixty minutes to each hour, sixty seconds to each minute, 24 hours in the day. For their base-60 system, the Babylonians needed units from 1 to 59, which they made by grouping the two cuneiform strokes $\nabla = 1$ and $\triangleleft = 10$. The number system and associated methods of calculation were complicated enough they were generally reserved for the priesthood. On the other hand, 60 has a distinct advantage over 10, because it has many more divisors, which led the Babylonians generally to avoid fractions and work with sexagesimal decimal representation.

The Babylonians used the formulas

$$ab = [(a + b)^2 - a^2 - b^2]/2 \text{ and } ab = [(a + b)^2 - (a - b)^2]/4$$

to make multiplication easier. These show that a table of squares is all that is necessary to multiply numbers, simply taking the difference of the two or three squares that were looked up in the table then taking a half or a half twice of the answer.

The Babylonians did not have an algorithm for long division. Instead they based their method on the fact that

$$a/b = a \times (1/b)$$

so all that was necessary was a table of reciprocals. We still have their reciprocal tables going up to the reciprocals of numbers up to several billion.

The Babylonians were adept at linear and quadratic equation. The Babylonian method for solving quadratics was essentially based on completing the

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square. The method(s) are not as clean as the modern quadratic formula, because equations were set in a form for which there was a positive solution. Negative solutions (indeed negative numbers) would not be allowed until the 16th century CE. The method of writing a problem was verbal and did not include mathematical symbols. As such problems have a rather intuitive feel. Anyone could understand the problem, but without the proper tools, the solution would be impossibly difficult. Perhaps this rendered a sense of the mystic to the mathematician. Consider this example:

> I added twice the side to the square; the result is 2,51,60. What is the side?

In modern terms we have the simple quadratic $x^2 + 2x = 10300$. The student would then follow his "procedure" for quadratics to find the solution.

For geometric objects, problems typically involved lengths of sides or diagonals, or determination of area or volume. The Babylonians had no measurement of angle, which to us is such a basic part of geometry. Angles were a later development. Some tablets do have figures drawn on them, and there are surviving problems treating all the standard shapes such as squares, rectangles, triangles, trapezoids, circles, and so on. Some of the problems are sophisticated. Solid geometry is dominated by bricks and ramps, but cylinders and truncated cones and pyramids do make an appearance. They estimated $\pi$ and were aware of the Pythagorean theorem, as is shown by the famous Plimpton 322 tablet, which appears to be a table of Pythagorean triples in a variant form, and by the following tablet inscription:

> 4 is the length and 5 the diagonal. What is the breadth? Its size is not known. 4 times 4 is 16. 5 times 5 is 25. You take 16 from 25 and there remains 9. What times what shall I take in order to get 9? 3 times 3 is 9. 3 is the breadth.

This document shows that although the Babylonians had a good understanding of how to apply the theorem, they tended to think algorithmically; that is, in terms of a sequence of steps that could always be followed to arrive at the answer to a problem. They did not attempt any formal proof of the statement; that would not come until much later, when Pythagoras offered a proof in the 6th century B.C.

The level of mathematical skill displayed in Old Babylonian mathematics goes considerably beyond that required for day-to-day usage, even as a scribe. Indeed there are some examples of virtuoso work.

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## 2.2 Problems

1. (a) Express 10,000 in Babylonian cuneiform notation.
(b) We normally write Babylonian numbers in sexagesimal notation with the individual blocks written as Hindu-Arabic numbers: $2, 15, 45; 12 = 2 \times 60^2 + 15 \times 60 + 45 + 12 \times 60^{-1} = 8145\frac{12}{60}$. Rewrite ◁▽ ◁ ◁▽▽ ◁ ◁ ◁ ▽ in the usual sexagesimal way of expressing Babylonian numbers and in our base 10 system.

2. (a) Express the fractions $\frac{1}{6}$, $\frac{5}{12}$, $\frac{1}{9}$ in sexagesimal notation.
(b) Express the reciprocals of 40 and 64 in sexagesimal notation.

3. What is the condition on an integer for it to be a regular sexagesimal, that is, its reciprocal is a finite sexagesimal decimal?

4. Convert the Babylonian approximation $1; 24, 51, 10$ for $\sqrt{2}$ to decimals and determine the number of decimal places that it is correct.

5. Given $11^2 = 121$, $12^2 = 144$, $13^2 = 169$, $14^2 = 196$, $15^2 = 225$, $16^2 = 256$, $17^2 = 289$, $18^2 = 324$, $19^2 = 361$, $20^2 = 40$, $30^2 = 900$, $40^2 = 1600$, $50^2 = 2500$, use the Babylonian method of squares to find $6 \times 13$, $13 \times 27$, and $34 \times 16$.

6. Multiply $12, 3; 45, 6$ by 60. What is a simple rule for multiplying any sexagesimal number by 60?

7. In their astronomical calculations, the Babylonians were interested in chord calculations, such as the following one taken from a Babylonian tablet. Given that the circumference of a circle is 60 units and the length of the radius through the center of a chord that lies between the chord and the circle is 2 units, what is the length of the chord? Assume $\pi = 3$. (Hint: Use the Pythagorean theorem in the top triangle.)
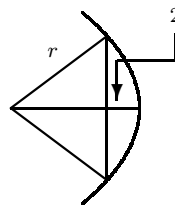


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8. The most famous of the Babylonian tablets is Plimpton 322, which appears to be a table of Pythagorean triples of integers, integers such that $x^2+y^2 = d^2$. Two of the columns in the table give values for $x$ and $d$, and the third gives $(x/y)^2$, which values may have been used in astronomical calculations. The Pythagorean triples found are quite large, so the Babylonians undoubtedly had some method for generating them.

(a) Given the values $x = 319$ and $d = 481$, find $y$ and show that $(x/y)^2$ agrees with the entry 0.7851929 in the Plimpton table.

(b) For $n < m$, verify algebraically that $(m - n)^2 + 4mn = (m + n)^2$. Hence if $4mn$ is a perfect square, a Pythagorean triple is generated. Fine the triple for $m = 81$ and $n = 16$. The $x$ and $d$ values appear in Plimpton 322.

9. The Babylonian solution $x = \sqrt{c + b^2/4} - b/2$ for the quadratic equation $x^2 + bx = c$, where $b, c > 0$, suggests that it arose from the ancient geometric method of "completing the square." The method of solution consisted of first forming the rectangle of sides $x$ and $x + b$ so that the area was $x(x + b) = x^2 + bx = c$.



$$A = x(x + b) \qquad A = (x + b/2)^2 - (b/2)^2$$
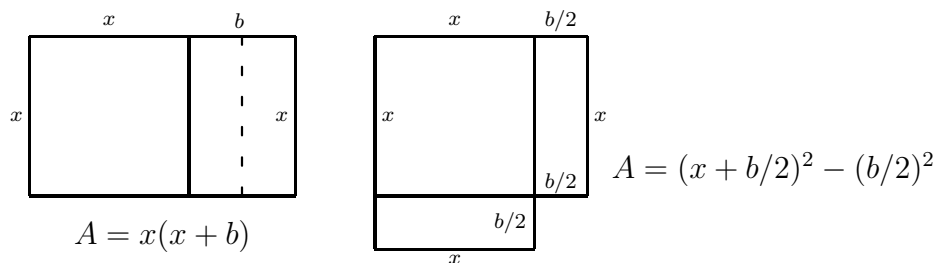
One then split the right rectangle (with sides $x$ and $b$) and pasted the right most half to the bottom. One then obtained a square of side $x + b/2$, except that the bottom right corner, a square of side $b/2$ was missing. Such figures are known as *gnomons* (squares with square corners missing). By taking the area of the small square from the larger, we obtain an area of $(x + b/2)^2 - (b/2)^2$. Since the gnomon is a rearrangement of the original rectangle, it must also have area $c = (x + b/2)^2 - (b/2)^2$. Thus $(x + b/2)^2 = c + (b/2)^2$. Taking square roots of both sides and moving $b/2$ to the right gives the Babylonian formula.

(a) Solve the equation $x^2 + bx = c$ algebraically by algebraically completing the square of the left-hand side, and then solving for $x$.

(b) Modify the Babylonian process to find the Babylonia formula for the solution of $x^2 - bx = c$.

# Chapter 3

# Euclid's *Elements*

## 3.1 Problems

A very useful web reference for Euclid's *Elements* is the web version prepared by D. E. Joyce at the Clark University site.

1. Euclid's notion of "equal" is equality of some geometric measurement (such as length or area). It corresponds in modern math to what we now call an *equivalence relation.* (What we would call "equality" Euclid calls "coinciding.")

   Suppose that we use $\cong$ for the relation "equal" in Euclid's Common Notions. Define what it means for $\cong$ to be an equivalence relation and show that Common Notions 1 and 4 may be interpreted to show that $\cong$ is an equivalence relation. In his 1899 reformulation of Euclid's geometry, Hilbert defined equality of length by postulating an equivalence relation on line segments.

2. Give Pappus' proof that the base angles of an isosceles triangle are equal (Proposition I.5) (Hint: show that flipping the triangle gives a congruence of the triangle with itself.)

3. The assertion that if two lines cut one another, then they make vertical angles that are equal (Proposition I.15) has been attributed to Thales. Prove it by appealing to Proposition I.13, which says that if a ray is drawn from a point on a line, then the supplementary angles that are formed equal two right triangles (180°).

4. Show that the angles of a triangle sum to two right angles (180°) by constructing a line through a vertex parallel to the opposite side.

5. Prove that if the opposite sides of a quadrilateral are equal and parallel, then the quadrilateral is a parallelogram (Proposition I.33). (Hint: Draw a diagonal and prove the triangles are congruent.)

6. Illustrate with a square of side having length $a + b$ that $(a + b)^2 = a^2 + 2ab + b^2$ (see Proposition II-4).

7. Describe how to carry out the following constructions:

   (i) the perpendicular bisector of a line segment;
   (ii) the tangent to a circle.

   How does Euclid define a tangent line to a circle? How does that differ from the way we define it in calculus? Show that your construction in part (ii) satisfies Euclid's definition.

8. Given a line segment of length $a$, describe the construction for obtaining $x$ such that $a : x = x : (a - x)$ (Proposition II-11). Show that $a : x = \sqrt{5} + 1 : 2$, the **golden ratio**.

9. Show that Proposition II-13 is equivalent to the law of cosines for an acute-angled triangle: In acute-angled triangles the square on the side opposite the acute angle is less than the sum of the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acute angle.

10. Describe the construction for inscribing a regular hexagon in a circle. Explain why it works.

11. Explain using the sum of angles at a vertex why there can be at most 5 regular polyhedra.