Unraveling the Mysteries of Infinity

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Thinking about the Infinite

There is a rich history of study and reflection on the concept and mystery of the infinite in a variety of contexts. This contemplation has often inspired feelings of awe, mystery, bafflement, skepticism (either about the reality of the infinite or our ability to make any sense of it), even fear.

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The French mathematician/philosopher/religious writer Blaise Pascal captures some of these feelings in the words: “When I consider the small span of my life absorbed in the eternity of all time, or the small part of space that I can touch or see engulfed by the infinite immensity of space, I am frightened and astonished.”

The Absolute Infinity

The term “Absolute Infinity” has been used to refer to beings who personify the infinite. Religions in general and theologians in particular, also philosophers, have contemplated beings who were infinite in various aspects of their beings. A most common aspect has been immortality, existing through time without beginning or end.

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The Hebrew Bible, for instance, gives vivid statement to this in Psalm 90:

> Before the mountains were born
>
> or you brought forth the earth and the world,
>
> from everlasting to everlasting you are God.

The Physically Infinite

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Is space infinite in extent? What about our universe? Are there an infinite number of heavenly bodies? Are there infinitely many (possible or actual) universes? Can matter be infinitely divided?

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- Galileo considered motion as a function of time over continuously varying instances of time;
- Newton and Leibnitz introduced the calculus with its calculations based on infinitesimally small, but non-zero, quantities.

George Cantor

The major mathematical breakthrough came in the late 19th century in groundbreaking work of George Cantor, who introduced the theory of sets as a foundation for mathematics and included a substantial mathematical theory of infinite sets. His highly original work was quite controversial in its day, but has made a major impact on modern mathematics.

One-to-One Correspondences

A basic insight of Cantor was that two sets should be compared by the existence or non-existence of one-to-one correspondences (members of one set could be paired up with members of the other so that everyone had a “dancing partner”), but not by whether one was a smaller set than the other.

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$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \ldots \\ 1 & 4 & 9 & 16 & 25 & 36 & \ldots \end{array}$$

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The set of perfect squares has the same number of elements as all the counting numbers though it is a much smaller subset.

The Infinite Number $\aleph$

Rather than trying to define the number 3, we need to learn how to tell whether a collection of objects has 3 members or some other number of members. We do this by counting and seeing whether we use precisely the numbers 1,2, and 3. We use a similar approach for $\aleph$ (aleph). We count the objects and see whether we use precisely all the numbers $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$. (Note the resemblance between $\aleph$ and N.)

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More precisely, we say a collection of objects has size or cardinality $\aleph$, or is countably infinite, if it can be put in one-to-one correspondence with the set $\mathbb{N}$. Intuitively this means that there is some method of pasting exactly one number on each of the objects.

A New Arithmetic

Cantor was able to show many surprising things about the arithmetic of infinite numbers. We will illustrate a few of these with a retelling of the story of Hercules cleaning the Augean stables, one of his twelve labors. Recall that Augeas was a man of vast herds of animals, and Hercules was able to cleanse the huge and filthy stables in one day by diverting a river through them.

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We modify the story so that the Augean herds are infinite in size and Hercules' tasks now have a significant mental component (the twelve intellectual labors of Hercules?). He has to assist him the greatest of the Greek mathematical minds, none other than Archimedes himself.

Task One

Augeas orders Hercules to place a newly acquired horse into his already full barn (with $\aleph$ stalls), one horse to a stall.

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Working at compound double speed, Hercules completes the task in eight minutes.

The Arithmetic of $\mathbb{N}$

If we add one horse to n horses, we obtain $n+1$ horses. In the same way adding one horse to $\aleph$ horses gives $\aleph+1$ horses.

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Applying the method used by Archimedes and Hercules, we observe that

$$\aleph + 1 = \aleph.$$

Task Two

In his full mare-with-colt barn, Augeas orders Hercules to separate the colts from their mothers and place each colt in its own stall.

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Archimedes advises Hercules to move the mare in stall n to stall $2n$ and put her colt just before her in stall $2n - 1$. For example, the mare in stall 50 is moved to stall 100 and her colt to stall 99.

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Working at compound double speed, Hercules completes the task in one hour. At the end the mares occupy the even numbered stalls and the colts the odd numbered.

More Arithmetic

If we have n colts and n mares, then we have in total $n+n$ horses. In the same way $\aleph$ colts plus $\aleph$ mares gives $\aleph+\aleph$ horses.

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Again using the rearrangement method applied by Archimedes and Hercules, we observe that

$$\aleph + \aleph = \aleph.$$

Task 3

On the back part of his property Augeas has built a brand new stable. He wants Hercules to move all the horses from all of his stables ($\aleph$ stables with $\aleph$ horses in each) into the one new stable, with one horse per stable, a seemingly impossible task.

Many Barns

Below is the scheme. Each row represents an old barn, with entry $(4, 3)$, for instance, representing the 3rd horse in the 4th barn.

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$$
\begin{array}{ccccc}
(1,1) & (1,2) & (1,3) & (1,4) & \ldots \\
(2,1) & (2,2) & (2,3) & (2,4) & \ldots \\
(3,1) & (3,2) & (3,3) & (3,4) & \ldots \\
(4,1) & (4,2) & (4,3) & (4,4) & \ldots \\
(5,1) & (5,2) & \ldots & &
\end{array}
$$

Stall 1 is Filled

After consider thought, Archimedes suggests a scheme. The first horse in Barn 1 goes to Stall 1 in the new barn.

$$
\begin{array}{ccccc}
(1,1)^{\mathbf{1}} \;\; \color{red}{\longrightarrow} & (1,2) & (1,3) & (1,4) & \ldots \\
(2,1) & (2,2) & (2,3) & (2,4) & \ldots \\
(3,1) & (3,2) & (3,3) & (3,4) & \ldots \\
(4,1) & (4,2) & (4,3) & (4,4) & \ldots \\
(5,1) & (5,2) & \ldots & &
\end{array}
$$

Stall 2 is Filled

The second horse in Barn 1 goes to Stall 2 in the new barn.

$$
\begin{array}{ccccc}
(1,1)^{\mathbf{1}} & \rightarrow \quad (1,2)^{\mathbf{2}} & (1,3) & (1,4) & \ldots \\
 & \swarrow & & & \\
(2,1) & (2,2) & (2,3) & (2,4) & \ldots \\
(3,1) & (3,2) & (3,3) & (3,4) & \ldots \\
(4,1) & (4,2) & (4,3) & (4,4)\ldots & \\
(5,1) & (5,2) \quad \ldots & & &
\end{array}
$$

Stall 3 is Filled

The first horse in Barn 2 goes to Stall 3 in the new barn.

$$
\begin{array}{ccccccccc}
(1,1)^{\mathbf{1}} & \rightarrow & (1,2)^{\mathbf{2}} & & (1,3) & & (1,4) & & \dots \\
 & \swarrow & & & & & & & \\
(2,1)^{\mathbf{3}} & & (2,2) & & (2,3) & & (2,4) & & \dots \\
\color{red}{\downarrow} & & & & & & & & \\
(3,1) & & (3,2) & & (3,3) & & (3,4) & & \dots \\
 & & & & & & & & \\
(4,1) & & (4,2) & & (4,3) & & (4,4) \dots & & \\
 & & & & & & & & \\
(5,1) & & (5,2) & \dots & & & & &
\end{array}
$$

Stalls 4 through 6

We proceed along the next diagonal to place the next 3 horses.

$$\begin{array}{ccccccccc}
(1,1)^{\mathbf{1}} & \rightarrow & (1,2)^{\mathbf{2}} & & (1,3)^{\mathbf{6}} & {\color{red}\rightarrow} & (1,4) & & \\
& \swarrow & & \nearrow & & & & & \\
(2,1)^{\mathbf{3}} & & (2,2)^{\mathbf{5}} & & (2,3) & & (2,4) & & \\
\downarrow & \nearrow & & & & & & & \\
(3,1)^{\mathbf{4}} & & (3,2) & & (3,3) & & (3,4) & & \ldots \\
& & & & & & & & \\
(4,1) & & (4,2) & & (4,3) & & (4,4)\ldots & & \\
& & & & & & & & \\
(5,1) & & (5,2) & \ldots & & & & &
\end{array}$$

Stalls 7 through 10

We proceed along the next diagonal to place the next 4 horses.

$$
\begin{array}{cccccccc}
(1,1)^{\mathbf{1}} & \longrightarrow & (1,2)^{\mathbf{2}} & & (1,3)^{\mathbf{6}} & \longrightarrow & (1,4)^{\mathbf{7}} & \\
 & \swarrow & & \nearrow & & \swarrow & & \\
(2,1)^{\mathbf{3}} & & (2,2)^{\mathbf{5}} & & (2,3)^{\mathbf{8}} & & (2,4) & \\
\downarrow & \nearrow & & \swarrow & & & & \\
(3,1)^{\mathbf{4}} & & (3,2)^{\mathbf{9}} & & (3,3) & & (3,4) & \ldots \\
 & \swarrow & & & & & & \\
(4,1)^{\mathbf{10}} & & (4,2) & & (4,3) & & (4,4)\ldots & \\
\color{red}{\downarrow} & & & & & & & \\
(5,1) & & (5,2) & \ldots & & & &
\end{array}
$$

And So Forth

We continue traversing the diagonals by this pattern, eventually reaching and assigning all horses to stalls.

$$
\begin{array}{cccccccc}
(1,1)^{\mathbf{1}} & \rightarrow & (1,2)^{\mathbf{2}} & & (1,3)^{\mathbf{6}} & \rightarrow & (1,4)^{\mathbf{7}} & \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow \\
(2,1)^{\mathbf{3}} & & (2,2)^{\mathbf{5}} & & (2,3)^{\mathbf{8}} & & (2,4)^{\mathbf{14}} & \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & \swarrow \\
(3,1)^{\mathbf{4}} & & (3,2)^{\mathbf{9}} & & (3,3)^{\mathbf{13}} & & (3,4) & \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow \\
(4,1)^{\mathbf{10}} & & (4,2)^{\mathbf{12}} & & (4,3) & & (4,4)\ldots & \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & \swarrow \\
(5,1)^{\mathbf{11}} & & (5,2) & & \ldots & & &
\end{array}
$$

Multiplication by Infinity

If we had eight barns with 7 horses each, then we would have a total of $8 \times 7 = 56$ horses.

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Applying the same method for multiplying infinite quantities, we have just derived the following multiplication fact:

$$\aleph \times \aleph = \aleph.$$

Larger Infinities?

As we leave our heroes, we are left with a question. Are all infinite sets of size $\aleph$? Cantor showed that the answer is no, which surprised many, who thought all infinities, were after all, just infinity.

The sets of size $\aleph$ are the smallest of the infinite sets. For this reason, $\aleph$ is denoted $\aleph_0$, the first of the infinite cardinal numbers.

More Cowherds Than Cows

Imagine a cow barn with $\aleph_0$ cows, one in each stall $1, 2, 3, \ldots$. Now imagine all possible different herds that could be formed by taking some of the cows into the pasture and leaving others behind, and suppose for each possible herd there was a different cowherd to attend that specific herd. Then the number of cowherds needed is a larger infinity than $\aleph_0$.

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In math jargon, the set of all possible subsets of the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ is strictly larger in size than $\aleph_0$. Cantor gave a beautiful and surprising proof of this result called "Cantor's diagonal argument." You can google to find it at several sites on the web.

Conclusion

The study of large infinite cardinal numbers remains an active area of mathematical research to the current day.

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General References

Rudy Rucker, Infinity and the Mind, Princeton University Press, 1982.

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