Calculating Submanifold Charts

Jimmie Lawson
Department of Mathematics
Louisiana State University

Spring, 2006

1 Introduction

The full-rank submanifold theorem states

Theorem 1.1 *The set $Q = f^{-1}(q)$ is a submanifold of M of dimension $m - n$ for $f : M \to N$, where f is a smooth map having full rank at points of Q, M is a manifold of dimension m, N is manifold of dimension n for $n < m$, and $q \in N$ is in the image of f.*

Proof. Let $p \in Q$, that is, $f(p) = q$. We seek to construct a slice chart for p. Let (V, ψ) be a centered chart for q so that $\psi(q) = 0$. Let (U, ϕ) be a centered chart for p. We may assume, by restricting the chart if necessary, that $f(U) \subseteq V$. Since $D(\psi \circ f \circ \phi^{-1})(0)$ has rank n by the full rank hypothesis, the image of $\{e_1, \ldots, e_m\}$ under $D(\psi \circ f \circ \phi^{-1})(0)$ is a spanning set for $\mathbb{R}^n$, where e_i, $1 \le i \le m$, is the unit vector in $\mathbb{R}^m$ with i^{th}-entry 1 and 0 elsewhere. Recall from linear algebra that a basis may always be extracted from a spanning set; we label such a basis

$$\{D(\psi \circ f \circ \phi^{-1})(0)(e_{\sigma(1)}), \ldots, D(\psi \circ f \circ \phi^{-1})(0)(e_\sigma(n))\}$$

where $\sigma : \{1, \ldots, n\} \to \{1, \ldots, m\}$ is a strictly increasing function ($i < j \Rightarrow \sigma(i) < \sigma(j)$). We label the remaining $m - n$ unit vectors in $\mathbb{R}^m$ by $\{e_{\tau(1)}, \ldots, e_{\tau(m-n)}\}$, where $\tau : \{1, \ldots, m - n\} \to \{1, \ldots, m\}$ is strictly increasing. We define the projection $\pi_\tau : \mathbb{R}^m \to \mathbb{R}^{m-n}$ by

$$\pi_\tau(x_1, \ldots, x_m) := (x_{\tau(1)}, \ldots, x_{\tau(m-n)}).$$

We define $F : \phi(U) \to \mathbb{R}^{m-n} \cong \mathbb{R}^n \oplus \mathbb{R}^n$ by $F(x) = (\pi_\tau(x), (\psi \circ f \circ \phi^{-1})(x))$; the first $m-n$ coordinates are given by $\pi_\tau(x)$ and the last n by $(\psi \circ f \circ \phi^{-1})(x)$. The derivative at 0 is computed coordinatewise to obtain

$$DF(0)(u) = (\pi_\tau(u), D(\psi \circ f \circ \phi^{-1})(0)(u),$$

where the derivative in the first coordinate follows from the fact that π_τ is linear. Suppose that $DF(0)(u) = 0$, where $u = (u_1, \ldots, u_m)$. Then $\pi_\tau(u) = 0$, so $u_{\tau(i)} = 0$, $i = 1, \ldots, m-n$, and thus $u = \sum_{i=1}^n u_{\sigma(i)} e_{\sigma(i)}$. It follows that

$$0 = D(\psi \circ f \circ \phi^{-1})(0)(u) = \sum_{i=1}^n u_{\sigma(i)} D(\psi \circ f \circ \phi^{-1})(0)(e_{\sigma(i)}),$$

and the fact that the $D(\psi \circ f \circ \phi^{-1})(0)(e_{\sigma(i)})$, $i = 1, \ldots, n$, form a basis for $\mathbb{R}^n$ implies that each $u_{\sigma(i)} = 0$. Hence $u = 0$, we conclude the kernel of $DF(0) : \mathbb{R}^m \to \mathbb{R}^m$ is trivial, and therefore $DF(0)$ is an isomorphism.

From the Inverse Function Theorem we conclude that F is a diffeomorphism on some neighborhood of $\phi(U_1)$ of 0, where U_1 is open, $p \in U_1 \subseteq U$. Hence $(U_1, F \circ \phi)$ is a centered chart at p since $F \circ \phi$ is a diffeomorphism and hence a chart and $F(0) = 0$. We observe for $x \in U_1$ that $F \circ \phi(x)$ is equal to 0 in the last n coordinates if and only if

$$0 = ((\psi \circ f \circ \phi^{-1}) \circ \phi)(x) = \psi(f(x))$$

if and only if $f(x) = q$, that is, $x \in Q$. Hence the chart $(U_1, F \circ \phi)$ is a slice-chart at p. Since p was an arbitrary point of Q, the proof is complete.

The construction in this proof can frequently be adapted in an algorithmic fashion to calculate an atlas of charts for a submanifold.

2 The case $N = \mathbb{R}$

We begin with the fairly simple, but important, case that $N = \mathbb{R}$. Suppose we have $f : \mathbb{R}^n \to \mathbb{R}$, $q \in \mathbb{R}$, and $Q = f^{-1}(q)$. Then for $x \in \mathbb{R}^n$, the Jacobian Jf(x) is the transposed gradient vector $\nabla f(x)^T = [\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}]$ thought of as a linear operator $Jf(x) : \mathbb{R}^n \to \mathbb{R}$ given by multiplying the row vector $\nabla f(x)^T$ by any column vector $u \in \mathbb{R}^n$, or alternatively, by taking the dot product $\nabla f(x) \cdot u$.

Step 1. The rank condition in this setting is the condition that $Jf(x)$ has rank one on Q, which is equivalent to its not being the 0 or null vector for all $x \in Q = f^{-1}(q)$. We must check this condition to make sure that Q will indeed by a submanifold.

We fix $x \in Q$, and seek a slice-chart (U, F) where U is an open subset containing x and $F : U \to \mathbb{R}^n$ is a coordinate map satisfying for all $y \in U$,

$$F(y) \in \mathbb{R}^{n-1} \times \{0\} \Leftrightarrow y \in U \cap Q.$$

Step 2. We take the chart $\psi : \mathbb{R} \to \mathbb{R}$ centered at q and defined by $\psi(t) = t - q$. (We observe that this single chart is an atlas for the usual differentiable structure on $\mathbb{R}$.) The last coordinate of F is then given by $\psi \circ f$, i.e., $F(y) = (??, \psi(f(y)) = (??, f(y) - q)$.

We now turn to the problem of finding the formula for the first $n - 1$ coordinates. For a general M we would need to work in a chart around x. However, since $M = \mathbb{R}^n$, we simply take the chart to be the identity on $\mathbb{R}^n$ and henceforth can ignore the chart ϕ and work in $\mathbb{R}^n$, thought of both as the manifold and the chart image.

Step 3. We next identify a standard unit vector $e_i \in \mathbb{R}^n$ whose image under the Jacobian map $Jf(x) = \nabla f(x)^T$ spans. In this case it simply means the image is nonzero (since $\mathbb{R}$ is one-dimensional), i.e., $\nabla f(x) \cdot e_i \neq 0$, which occurs if and only if $\frac{\partial f}{\partial x_i}(x) \neq 0$. We then take the projection $\pi_{\neq i} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ that deletes the i^{th}-coordinate. The chart is then given by $F(y) = (\pi_{\neq i}(y), f(y) - q)$

Step 4. Restrict F to some open set U around x such that (i) F is one-to-one on U and (ii) $\frac{\partial f}{\partial x_i}(y) \neq 0$ for all $y \in U$. Repeat the procedure for other $x \in Q$ until enough such (U, F) are found to form an atlas for Q.

3 A specific example

Consider $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = x^2 + y^2 - z^2$, and let $q = 1 \in \mathbb{R}$. Then $Q = \{(x, y, z) : x^2 + y^2 - z^2 = 1\}$ is a hyperboloid of one sheet with axis of symmetry the z-axis.

Step 1. We have $\nabla f(x, y, z)^T = [2x, 2y, -2z]$, which is never the 0-vector for any point in Q; thus the full rank condition is satisfied.

Step 2. We define $F : \mathbb{R}^3 \to \mathbb{R}^3$ by $F(x, y, z) = (?, ?, x^2 + y^2 + z^2 - 1)$. Note that $F(x, y, z) = (?, ?, 0)$ if and only if $(x, y, z) \in Q$, one of the slice chart conditions for Q to be a submanifold.

Step 3. Let $(x, y, z) \in Q$ such that $y \neq 0$. Then $\nabla f(x, y, z) \cdot e_x = 2x(0) + 2y(1) - 2z(0) = 2y \neq 0$ on this open set. We choose for our projection the projection into the xz-plane. We then define $F(x, y, z) = (x, z, x^2 + y^2 - z^2 - 1)$.

Step 4. If we restrict to the set $U^+ = \{(x, y, z) : y > 0\}$, then as we saw in step 3 that the image of e_2 under the Jacobian spans. Suppose that we choose $(x_1, y_1, z_1), (x_2, y_2, z_2) \in U^+$ with $F(x_1, y_1, z_1) = F(x_2, y_2, z_2)$. By looking at the first and second coordinates of the images, we conclude that $x_1 = x_2$ and $z_1 = z_2$. Since in the third coordinates $x_1^2 + y_1^2 - z_1^2 - 1 = x_2^2 + y_2^2 - z_2^2 - 1$, we conclude that

$$y_1^2 = 1 - x_1^2 + z_1^2 = 1 - x_2^2 + z_2^2 = y_2^2.$$

Since $y_1, y_2 > 0$, we conclude that $y_1 = y_2$. Therefore F is also one-to-one on U^+. Hence (U^+, F) is a chart for $\mathbb{R}^3$ that is also a submanifold chart for Q. If we consider the open set U^- for which $y < 0$, we can obtain a second chart by restricting F to this set. For the open sets V^+ with $x > 0$ and V^- with $x < 0$, we need to modify the definition of F to $G(x, y, z) = (y, z, x^2 + y^2 - z^2 - 1)$. Then G restricted to V^+ and V^- are also charts that satisfy the submanifold condition. We thus obtain four charts of $\mathbb{R}^3$ whose restrictions to Q form an atlas for Q (note that it is impossible for both x and y to be 0 at any point of Q).

We are done and our theory guarantees that (U^+, F), etc. are charts for $\mathbb{R}^3$, but we can verify this directly by noting that F is C^∞, one-to-one on U^+, and has invertible Jacobian:

$$JF(x, y, z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2x & 2y & 2z \end{pmatrix},$$

which has non-zero determinant if $y \neq 0$, and hence is invertible.

4 Exercises

Exercise 4.1 *Use the preceding method outlined for the example of the hyperboloid to find a slice chart for the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ for $a, b, c \neq 0$. How many such slice charts are needed in order to extract an atlas for the ellipsoid?*

Exercise 4.2 *The* 2×2 *real matrices of determinant* 1*, the so-called special linear group* $SL(2, \mathbb{R})$*, is a submanifold of the linear space of all* 2×2 *matrices* $Mat_{2\times 2}$*, which may be identified with* $\mathbb{R}^4$*:*

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \leftrightarrow (x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}).$$

(Indeed this correspondence defines a chart which is an atlas for $Mat_{2\times 2}$*.)*

(i) Show that the determinant mapping det: $\mathbb{R}^4 \to \mathbb{R}$ *defined by*

$$det(x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}) = x_{1,1}x_{2,2} - x_{1,2}x_{2,1}$$

has full rank 1 *at all points of* $SL(2, \mathbb{R})$*, and hence is a submanifold by the full-rank submanifold theorem.*

(ii) Compute the Jacobian matrix for $(x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}) \mapsto (x_{1,2}, x_{2,1}, x_{2,2}, x_{1,1}x_{2,2} - x_{1,2}x_{2,1} - 1)$ *and show that its determinant is non-zero if* $x_{2,2} \neq 0$ *(and hence the Jacobian is invertible). What is the projection map from* $\mathbb{R}^4$ *to* $\mathbb{R}^3$ *given by the first 3 coordinates?*

(iii) Show that $(x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}) \mapsto (x_{1,2}, x_{2,1}, x_{2,2}, x_{1,1}x_{2,2} - x_{1,2}x_{2,1} - 1)$ *is a slice chart for the open set* $x_{2,2} \neq 0$*.*

(iv) Find one other slice chart that suffices to construct an atlas for the submanifold $SL(2, \mathbb{R})$*.*

Exercise 4.3 *(i) Consider the map* $f : \mathbb{R}^4 \to \mathbb{R}^2$ *defined by*

$$f(x_1, x_2, x_3, x_4) = (x_1^2 + x_2, x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_2).$$

Show that f *restricted to* $Q = f^{-1}(0, 1)$ *has rank 2 at every point of* M*, and hence that* Q *is a submanifold. (Hint: Show that the two rows of the Jacobian matrix* $Jf(x)$ *are linearly independent for* $x \in Q$*.)*
(ii) For $x_3 \neq 0$*, show that the map* $F : \mathbb{R}^4 \to \mathbb{R}^4$ *defined by*

$$F(x_1, x_2, x_3, x_4) = (x_1, x_4, x_1^2 + x_2, x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_2 - 1)$$

has invertible Jacobian $JF(x_1, x_2, x_3, x_4)$ *and has value* 0 *in the last two coordinates precisely on* Q*.*
(iii) On the open set $x_3 > 0$*, show that* F *is injective. Conclude that* F *restricted to this open set a slice-chart. Note that* F *restricted to the open set* $x_3 < 0$ *is also a slice-chart.*
(iv) Repeat steps (ii) and (iii) for the cases $x_1 > 0$ *and* $x_4 > 0$*. Observe that since* $x_2 \neq 0$ *iff* $x_1 \neq 0$*, six slice-charts suffice to obtain an atlas for* Q*.*