1 The Tangent Bundle on $\mathbb{R}^n$

The tangent bundle gives a manifold structure to the set of tangent vectors on $\mathbb{R}^n$ or on any open subset $U$. Unlike the practice in typical lower level calculus courses, we distinguish between translates of vectors, that is, we may think of geometric vectors as directed line segments that have a specified and fixed initial and terminal point. If $x$ is a point in $\mathbb{R}^n$ and $v \in \mathbb{R}^n$ is a vector with initial point the origin, then we can think of a geometric vector stretching from $x$ to $x + v$. We give this picture mathematical substance by denoting the “arrow” from $x$ to $x + v$ as $(x, v)$, or occasionally as $v_x$.

We consider for $x \in \mathbb{R}^n$ the set $T_x \mathbb{R}^n := \{(x, v) : v \in \mathbb{R}^n\}$, called the tangent space at $x$. We note that $T_x \mathbb{R}^n$ carries a natural vector space structure with vector addition and scalar multiplication defined by

$$(x, v) + (x, w) := (x, v + w) \quad \text{and} \quad r(x, v) := (x, rv).$$

We define the tangent space of $\mathbb{R}^n$ to be the union of the tangent spaces of all the points:

$$T \mathbb{R}^n := \bigcup_{x \in \mathbb{R}^n} T_x \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n.$$

There is an obvious projection map $\pi : T \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by projection into the first coordinate: $\pi(x, v) = x$. The projection map allows us alternatively to denote a member of $T \mathbb{R}^n$ simply by a single letter, like $v$, since we can recover the first entry of a pair $v \in T \mathbb{R}^n$ as the projection $\pi(v)$, which gives the “initial point” of the vector.
The preceding considerations easily generalize to open subsets of euclidean space.

**Definition 1.1** For \( U \neq \emptyset \) open in \( \mathbb{R}^n \), we set

\[
TU = U \times \mathbb{R}^n = \{(x, v) : x \in U, v \in \mathbb{R}^n \},
\]

the tangent space of \( U \). For \( x \in U \), the set

\[
T_xU = \{(x, v) : v \in \mathbb{R}^n \} \subseteq TU
\]

is called the tangent space at \( x \). With respect to vector addition and scalar multiplication in the second coordinate, \( T_xU \) is an \( n \)-dimensional vector space. The projection map \( \pi_U : TU \to U \) is given by projection into the first coordinate.

**Remark 1.2** The tangent space \( TU \) has a natural manifold structure given by the chart from \( TU \) to \( \mathbb{R}^{2n} \) sending \( (x, v) \) to \((x_1, \ldots, x_n, v_1, \ldots, v_n)\), where \( x = (x_1, \ldots, x_n) \) and \( v = (v_1, \ldots, v_n) \), the chart that naturally identifies \( TU \) with an open subset of \( \mathbb{R}^{2n} \). With respect to this structure the projection map is smooth, since it is a projection map.

**Remark 1.3** Suppose that \( U \) is an open subset of \( \mathbb{R}^n \), \( V \) is an open subset of \( \mathbb{R}^m \), and \( f : U \to V \) is a \( C^\infty \)-map. We define \( f_* = Tf : TU \to TV \) by \( Tf(x, v) = (f(x), Df(x)(v)) \); the latter is easily verified to be \( C^\infty \) since \( f \) is. Furthermore, it follows immediately that we have a commuting diagram of smooth maps:

\[
\begin{array}{ccc}
TU & \xrightarrow{\pi_U} & TV \\
\pi \downarrow & & \downarrow \pi \\
U & \xrightarrow{f} & V
\end{array}
\]

**Exercise 1.1** Consider the map \( F : \mathbb{R}^4 \to \mathbb{R}^2 \) defined by

\[
F(x_1, x_2, x_3, x_4) = (x_1^2 + x_2, x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_2).
\]

Find the images of the four vectors \((1, -1, 2, 1), e_i\) under \( TF \), where \( e_i \) is the \( i \)th unit vector in \( \mathbb{R}^4 \) for \( i = 1, 2, 3, 4 \).
Exercise 1.2 Let $\alpha : I \rightarrow \mathbb{R}^n$ be a $C^\infty$-curve in $\mathbb{R}^n$, where $I$ is an open interval. For $t \in I$, let $\alpha'(t)$ denote the standardly defined tangent vector to the curve. Show that $\alpha'(t)_{\alpha(t)} = \alpha_s((t, 1))$.

Exercise 1.3 (i) Show that $T1_U = 1_{TU}$, where $1_U$ is the identity map on $U$, and $T(gf) = Tf \circ Tg$ for composable $C^\infty$-maps defined on open subsets of euclidean space. (We conclude that $T$ is a functor on the category of open euclidean subspaces and $C^\infty$-maps.)

(ii) Deduce from (i) that if $f : U \rightarrow V$ is a diffeomorphism between open subsets of $\mathbb{R}^n$, then $Tf : TU \rightarrow TV$ is a diffeomorphism.

2 The Tangent Space of a Manifold

Some objects are much more visible by their traces than be direct observation. In the case of tangent vectors to manifolds, it is convenient to define them by their traces in the charts. A chart vector at $x$ for a chart $(U, \phi)$ with $x \in U \subseteq M$, an $n$-dimensional smooth manifold, is a pair $(\phi(x), v) \in T(\phi(U))$. The traces in different charts of a specific tangent vector should be appropriately related to one another. Thus we define two chart vectors at $x$ for two charts $(U, \phi)$ and $(V, \psi)$ to be equivalent, written $(\phi(x), u) \sim (\psi(x), v)$, if

$$T(\psi \circ \phi^{-1})((\phi(x), u))v := (\psi(x), D(\psi \circ \phi^{-1})(\phi(x))(u)) = (\psi(x), v)$$

$$\Leftrightarrow D(\psi \circ \phi^{-1})(\phi(x))(u) = v.$$ 

Exercise 2.1 Show that $\sim$ is an equivalence relation on the set of chart vectors at $x$. Show for any chart $(U, \phi)$ with $x \in U$, assigning to $u \in \mathbb{R}^n$ the equivalence class of $(\phi(x), u)$ defines a one-to-one correspondence between $\mathbb{R}^n$ and the set of equivalence classes of chart vectors at $x$.

Exercise 2.2 (i) Show that if $(\phi(x), u_1) \sim (\psi(x), v_1)$ and $(\phi(x), u_2) \sim (\psi(x), v_2)$, then $(\phi(x), u_1 + u_2) \sim (\psi(x), v_1 + v_2)$. Similarly show for $(\phi(x), u) \sim (\psi(x), v)$ and $r \in \mathbb{R}$ that $(\phi(x), ru) \sim (\psi(x), rv)$. Hence there is a well defined vector addition and scalar multiplication on the set of equivalence classes.

(ii) Show for any chart $(U, \phi)$ with $x \in U$, the map that sends $(\phi(x), u) \in T_{\phi(x)}(\phi(U))$ to its equivalence class is a vector space isomorphism from $T_{\phi(x)}(\phi(U))$ to the space of equivalence classes equipped with vector addition and scalar multiplication defined in (i).
Given a smooth $n$-dimensional manifold $M$ and $x \in M$, we see that a tangent vector in the tangent space at $x$ should give rise to a $\sim$-equivalence class of chart vectors. We can reverse the procedure and define a tangent vector at $x$ to be a $\sim$-class of chart vectors.

**Definition 2.1** Let $M$ be an $n$-dimensional manifold, and let $x \in M$. The *tangent space at $x$*, denoted $T_xM$ consists of the set of equivalence classes of chart vectors at $x$ endowed with the induced vector space structure. The *tangent space of $M$*, denoted $TM$, is given by $TM = \bigcup_{x \in M} T_xM$, the disjoint union of the tangent spaces. The projection $\pi : TM \to M$ is defined by $\pi(v) = x$ if $v \in T_xM$.

**Notation.** For $x \in M$, an $n$-dimensional smooth manifold, and a chart $(U, \phi)$ with $x \in U$ and $(\phi(x), v) \in T_{\phi(x)}(\phi(U))$, we write $[\phi(x), v]$ for the $\sim$-equivalence class in $T_xM$ containing $(\phi(x), v)$. For $U$ open in $M$, we denote $\pi^{-1}(U)$ by $TU$. We define $\phi_* = T\phi : TU \to T(\phi(U)) = U \times \mathbb{R}^n$ as follows: for $v \in TU$, set $x = \pi(v)$. By Exercise 2.2(ii) there exists a unique $\hat{v} \in \mathbb{R}^n$ such that $[\phi(x), \hat{v}] = v$. We set $\phi_*(v) = (\phi(x), \hat{v})$. This is the inverse of the map given in Exercise 2.2(ii), so is a vector space isomorphism when restricted to each tangent space $T_xM$ for $x \in U$.

**Definition 2.2** We define an atlas on $TM$ consisting of all
\[ \{(TU, \phi_i) : (U, \phi) \text{ is a chart for } M\} \]
and endow $TM$ with the resulting differentiable structure. The triple $(TM, \pi_M, M)$ is called the *tangent bundle* of the smooth manifold $M$.

**Exercise 2.3** (i) Show for any atlas $\{((U_i, \phi_i) : i \in I\}$ for a smooth manifold $M$, the collection $\{(TU_i, \phi_{iu}) : i \in I\}$ is an atlas for $TM$. (Hint: Show that the transition map between charts is given by $T(\psi \circ \phi^{-1})|_{\psi(U \cap V)}$ and use Remark 1.3.) Why do any two atlases for the given differentiable structure on $M$ give the same differentiable structure on $TM$?

(ii) Show the the projection map $\pi : TM \to M$ is smooth.

**Definition 2.3** Let $f : M \to N$ be a smooth map. We define $Tf : TM \to TN$ as follows: for $v \in TM$ and $x = \pi_M(v)$, let $(U, \phi)$ be a chart at $x$ and let $(V, \psi)$ be a chart at $f(x)$. We set $Tf(v) = \psi_*^{-1}(T(\psi \circ f \circ \phi^{-1})(\phi_*(v)))$. 

4
Exercise 2.4  (i) Show that the definition of $Tf$ given in Definition 2.3 is independent of the charts $\phi$ and $\psi$.

(ii) Show that $Tf : TM \to TN$ is smooth.

(iii) Show that the following diagram commutes:

$$
\begin{array}{ccc}
TM & \xrightarrow{Tf} & TN \\
\downarrow{\pi_M} & & \downarrow{\pi_N} \\
M & \xrightarrow{f} & N
\end{array}
$$

Exercise 2.5  (i) Show that $T1_M = 1_{TM}$, where $1_M$ is the identity map on a smooth manifold $M$, and $T(gf) = Tg \circ Tf$ (the chain rule for smooth manifolds) for smooth maps $f : M_1 \to M_2$ and $g : M_2 \to M_3$. (We conclude that $T$ is a functor on the category of smooth manifolds and smooth maps.)

(ii) Deduce from (i) that if $f : M \to N$ is a diffeomorphism between two smooth manifolds, then $Tf : TM \to TN$ is a diffeomorphism.

Remark 2.4 If $U$ is a nonempty open subset of $\mathbb{R}^n$ and $i : U \to \mathbb{R}^n$ is the inclusion map, then $i : U \to i(U) = U$ is a chart that forms an atlas for $U$. We then have $Ti : TU \to U \times \mathbb{R}^n$ is a diffeomorphism, and in this way identify the tangent bundle of $U$, considered as a manifold, with its tangent bundle, considered as an open subset of $\mathbb{R}^n$. This applies in particular to $U = \mathbb{R}^n$.

The following exercise indicates how we can give geometric interpretation to tangent spaces of embedded manifolds.

Exercise 2.6  (i) The circle $S^1 = \{(x,y) : x^2 + y^2 = 1\}$ is an embedded submanifold of $\mathbb{R}^2$. Let $j : S^1 \to \mathbb{R}^2$ be the inclusion map. For a point $(x,y) \in S^1$, identify the set $T_j(T_{(x,y)}S^1)$ as a subset of $T\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$.

(ii) Repeat the exercise for the inclusion of $S^2$ in $\mathbb{R}^3$. 