The Tangent Bundle

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Spring, 2006

1 The Tangent Bundle on $\mathbb{R}^n$

The tangent bundle gives a manifold structure to the set of tangent vectors on $\mathbb{R}^n$ or on any open subset U. Unlike the practice in typical lower level calculus courses, we distinguish between translates of vectors, that is, we may think of geometric vectors as directed line segments that have a specified and fixed initial and terminal point. If x is a point in $\mathbb{R}^n$ and $v \in \mathbb{R}^n$ is a vector with initial point the origin, then we can think of a geometric vector stretching from x to $x + v$. We give this picture mathematical substance by denoting the "arrow" from x to $x + v$ as (x, v), or occasionally as v_x.

We consider for $x \in \mathbb{R}^n$ the set $T_x\mathbb{R}^n := \{(x, v) : v \in \mathbb{R}^n\}$, called the *tangent space at* x. We note that $T_x\mathbb{R}^n$ carries a natural vector space structure with vector addition and scalar multiplication defined by

$$(x, v) + (x, w) := (x, v + w) \quad \text{and} \quad r(x, v) := (x, rv).$$

We define the *tangent space of* $\mathbb{R}^n$ to be the union of the tangent spaces of all the points:

$$T\mathbb{R}^n := \bigcup_{x\in\mathbb{R}^n} T_x\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n.$$

There is an obvious projection map $\pi : T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by projection into the first coordinate: $\pi(x, v) = x$. The projection map allows us alternatively to denote a member of $T\mathbb{R}^n$ simply by a single letter, like v, since we can recover the first entry of a pair $v \in T\mathbb{R}^n$ as the projection $\pi(v)$, which gi ves the "initial point" of the vector.

The preceding considerations easily generalize to open subsets of euclidean space.

Definition 1.1 For $U \neq \emptyset$ open in $\mathbb{R}^n$, we set

$$TU = U \times \mathbb{R}^n = \{(x, v) : x \in U,\ v \in \mathbb{R}^n\},$$

the *tangent space* of U. For $x \in U$, the set

$$T_x U = \{(x, v) : v \in \mathbb{R}^n\} \subseteq TU$$

is called the *tangent space at* x. With respect to vector addition and scalar multiplication in the second coordinate, $T_x U$ is an n-dimensional vector space. The *projection map* $\pi_U : TU \to U$ is given by projection into the first coordinate.

Remark 1.2 The tangent space TU has a natural manifold structure given by the chart from TU to $\mathbb{R}^{2n}$ sending (x, v) to $(x_1, \ldots, x_n, v_1, \ldots, v_n)$, where $x = (x_1, \ldots, x_n)$ and $v = (v_1, \ldots, v_n)$, the chart that naturally identifies TU with an open subset of $\mathbb{R}^{2n}$. With respect to this structure the projection map is smooth, since it is a projection map.

Remark 1.3 Suppose that U is an open subset of $\mathbb{R}^n$, V is an open subset of $\mathbb{R}^m$, and $f : U \to V$ is a C^∞-map. We define $f_* = Tf : TU \to TV$ by $Tf(x, v) = (f(x), Df(x)(v))$; the latter is easily verified to be C^∞ since f is. Furthermore, it follows immediately that we have a commuting diagram of smooth maps:

$$\begin{array}{ccc} TU & \xrightarrow{Tf} & TV \\ \pi \big\downarrow & & \big\downarrow \pi \\ U & \xrightarrow{f} & V \end{array}$$

Exercise 1.1 *Consider the map* $F : \mathbf{R}^4 \to \mathbf{R}^2$ *defined by*

$$F(x_1, x_2, x_3, x_4) = (x_1^2 + x_2, x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_2).$$

Find the images of the four vectors $\big((1, -1, 2, 1), e_i\big)$ *under* TF, *where* e_i *is the* i^{th} *unit vector in* $\mathbb{R}^4$ *for* $i = 1, 2, 3, 4$.

Exercise 1.2 *Let $\alpha : I \to \mathbb{R}^n$ be a C^∞-curve in $\mathbb{R}^n$, where I is an open interval. For $t \in I$, let $\alpha'(t)$ denote the standardly defined tangent vector to the curve. Show that $\alpha'(t)_{\alpha(t)} = \alpha_*\big((t,1)\big)$.*

Exercise 1.3 *(i) Show that $T1_U = 1_{TU}$, where 1_U is the identity map on U, and $T(gf) = Tg \circ Tf$ for composable C^∞-maps defined on open subsets of euclidean space. (We conclude that T is a functor on the category of open euclidean subspaces and C^∞-maps.)*
(ii) Deduce from (i) that if $f : U \to V$ is a diffeomorphism between open subsets of $\mathbb{R}^n$, then $Tf : TU \to TV$ is a diffeomorphism.

2 The Tangent Space of a Manifold

Some objects are much more visible by their traces than be direct observation. In the case of tangent vectors to manifolds, it is convenient to define them by their traces in the charts. A *chart vector at x* for a chart (U, ϕ) with $x \in U \subseteq M$, an n-dimensional smooth manifold, is a pair $(\phi(x), v) \in T\big(\phi(U)\big)$. The traces in different charts of a specific tangent vector should be appropriately related to one another. Thus we define two chart vectors at x for two charts (U, ϕ) and (V, ψ) to be *equivalent*, written $(\phi(x), u) \sim (\psi(x), v)$, if

$$\begin{aligned} T(\psi \circ \phi^{-1})\big((\phi(x), u)\big)v &:= \big(\psi(x), D(\psi \circ \phi^{-1})(\phi(x))(u)\big) = (\psi(x), v) \\ &\Leftrightarrow \;\; D(\psi \circ \phi^{-1})(\phi((x))(u) = v. \end{aligned}$$

Exercise 2.1 *Show that $\sim$ is an equivalence relation on the set of chart vectors at x. Show for any chart (U, ϕ) with $x \in U$, assigning to $u \in \mathbb{R}^n$ the equivalence class of $(\phi(x), u)$ defines a one-to-one correspondence between $\mathbb{R}^n$ and the set of equivalence classes of chart vectors at x.*

Exercise 2.2 *(i) Show that if $(\phi(x), u_1) \sim (\psi(x), v_1)$ and $(\phi(x), u_2) \sim (\psi(x), v_2)$, then $(\phi(x), u_1 + u_2) \sim (\psi(x), v_1 + v_2)$. Similarly show for $(\phi(x), u) \sim (\psi(x), v)$ and $r \in \mathbb{R}$ that $(\phi(x), ru) \sim (\psi(x), rv)$. Hence there is a well defined vector addition and scalar multiplication on the set of equivalence classes.*
(ii) Show for any chart (U, ϕ) with $x \in U$, the map that sends $(\phi(x), u) \in T_{\phi(x)}(\phi(U))$ to its equivalence class is a vector space isomorphism from $T_{\phi(x)}(\phi(U)$ to the space of equivalence classes equipped with vector addition and scalar multiplication defined in (i).

Given a smooth n-dimensional manifold M and $x \in M$, we see that a tangent vector in the tangent space at x should give rise to a $\sim$-equivalence class of chart vectors. We can reverse the procedure and *define* a tangent vector at x to be a $\sim$-class of chart vectors.

Definition 2.1 Let M be an n-dimensional manifold, and let $x \in M$. The *tangent space at* x, denoted T_xM consists of the set of equivalence classes of chart vectors at x endowed with the induced vector space structure. The *tangent space of* M, denoted TM, is given by $TM = \bigcup_{x \in M} T_xM$, the disjoint union of the tangent spaces. The *projection* $\pi : TM \to M$ is defined by $\pi(v) = x$ if $v \in T_xM$.

Notation. For $x \in M$, an n-dimensional smooth manifold, and a chart (U, ϕ) with $x \in U$ and $(\phi(x), v) \in T_{\phi(x)}(\phi(U))$, we write $[\phi(x), v]$ for the $\sim$-equivalence class in T_xM containing $(\phi(x), v)$. For U open in M, we denote $\pi^{-1}(U)$ by TU. We define $\phi_* = T\phi : TU \to T(\phi(U)) = U \times \mathbb{R}^n$ as follows: for $v \in TU$, set $x = \pi(v)$. By Exercise 2.2(ii) there exists a unique $\hat{v} \in \mathbb{R}^n$ such that $[\phi(x), \hat{v}] = v$. We set $\phi_*(v) = (\phi(x), \hat{v})$. This is the inverse of the map given in Exercise 2.2(ii), so is a vector space isomorphism when restricted to each tangent space T_xM for $x \in U$.

Definition 2.2 We define an atlas on TM consisting of all

$$\{(TU, \phi_*) : (U, \phi) \text{ is a chart for } M\}$$

and endow TM with the resulting differentiable structure. The triple (TM, π_M, M) is called the *tangent bundle* of the smooth manifold M.

Exercise 2.3 *(i) Show for any atlas $\{((U_i, \phi_i) : i \in I\}$ for a smooth manifold M, the collection $\{(TU_i, \phi_{i*}) : i \in I\}$ is an atlas for TM. (Hint: Show that the transition map between charts is given by $T(\psi \circ \phi^{-1})|_{\phi(U \cap V)}$ and use Remark 1.3.) Why do any two atlases for the given differentiable structure on M give the same differentiable structure on TM?*
(ii) Show the the projection map $\pi : TM \to M$ is smooth.

Definition 2.3 Let $f : M \to N$ be a smooth map. We define $Tf : TM \to TN$ as follows: for $v \in TM$ and $x = \pi_M(v)$, let (U, ϕ) be a chart at x and let (V, ψ) be a chart at $f(x)$. We set $Tf(v) = \psi_*^{-1}\big(T(\psi \circ f \circ \phi^{-1})(\phi_*(v))\big)$.

Exercise 2.4 *(i)Show that the definition of Tf given in Definition 2.3 is independent of the charts ϕ and ψ.*
(ii) Show that $Tf : TM \to TN$ is smooth.
(iii) Show that the following diagram commutes:

$$\begin{array}{ccc} TM & \xrightarrow{Tf} & TN \\ {\scriptstyle \pi_M}\downarrow & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{f} & N \end{array}$$

Exercise 2.5 *(i) Show that $T1_M = 1_{TM}$, where 1_M is the identity map on a smooth manifold M, and $T(gf) = Tg \circ Tf$ (the chain rule for smooth manifolds) for smooth maps $f : M_1 \to M_2$ and $g : M_2 \to M_3$. (We conclude that T is a functor on the category of smooth manifolds and smooth maps.)*
(ii) Deduce from (i) that if $f : M \to N$ is a diffeomorphism between two smooth manifolds, then $Tf : TM \to TN$ is a diffeomorphism.

Remark 2.4 If U is a nonempty open subset of $\mathbb{R}^n$ and $i : U \to \mathbb{R}^n$ is the inclusion map, then $i : U \to i(U) = U$ is a chart that forms an atlas for U. We then have $Ti : TU \to U \times \mathbb{R}^n$ is a diffeomorphsim, and in this way identify the tangent bundle of U, considered as a manifold, with its tangent bundle, considered as an open subset of $\mathbb{R}^n$. This applies in particular to $U = \mathbb{R}^n$.

The following exercise indicates how we can give geometric interpretation to tangent spaces of embedded manifolds.

Exercise 2.6 *(i) The circle $S^1 = \{(x, y) : x^2 + y^2 = 1\}$ is an embedded submanifold of $\mathbb{R}^2$. Let $j : S^1 \to \mathbb{R}^2$ be the inclusion map. For a point $(x, y) \in S^1$, identify the set $Tj(T_{(x,y)}S^1)$ as a subset of $T\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$.*
(ii) Repeat the exercise for the inclusion of S^2 in $\mathbb{R}^3$.