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MULTIPLES OF WEIERSTRASS POINTS AS SPECIAL DIVISORS

R. F. LAX

Abstract. Complex spaces \( \mathbb{W}_n^* \) of Weierstrass points are isomorphic to the intersection, on the \( n \)th symmetric product of the universal curve over the Teichmüller space, of complex spaces \( \mathcal{O}_n^* \) of special divisors with the diagonal \( \Delta_n \), consisting of divisors which are multiples of a point. The tangent space at a point of this intersection is described and it is shown that \( \mathcal{O}_n^1 \) and \( \mathcal{O}_n^2 \) intersect transversally.

Let \( T = T_g \) denote the Teichmüller space for Teichmüller surfaces of genus \( g \geq 1 \) and let \( \tau : V \to T \) denote the universal curve of genus \( g \). Denote by \( V_T^{(n)} \) the \( n \)th symmetric product of \( V \) over \( T \). Let \( \mathcal{O}_n^* \) denote the closed complex subspace of \( V_T^{(n)} \) whose points are divisors of degree \( n \) and projective dimension at least \( r \) (see [3], [2]). We have proved

**Theorem 1 ([3]).** Suppose \( n \leq g \). Then \( \mathcal{O}_n^1 - \mathcal{O}_n^2 \) is smooth of pure dimension \( 2n + 2g - 4 \).

For \( 2 \leq n \leq g \), let \( \mathcal{W}_n^* \) denote the closed complex subspace of \( V \) consisting of those \( (t, P) \in V \) such that there are at least \( r \) gaps less than or equal to \( n \) in the Weierstrass gap sequence at \( P \) on \( V_t \). These spaces were introduced in [4] and, by employing methods similar to those used in the proof of Theorem 1, we proved

**Theorem 2 ([4]).** For \( 2 \leq n \leq g \), \( \mathcal{W}_n^1 - \mathcal{W}_n^2 \) is smooth of pure dimension \( n + 2g - 3 \).

In this note, we describe the relationship between the \( \mathcal{O}_n^* \) and the \( \mathcal{W}_n^* \) and show how Theorem 2 may be derived in a direct fashion from Theorem 1.

Let \( \Delta_n \) denote the image of \( \delta_n : V \to V_T^{(n)} \), the closed immersion which takes a point \( (t, P) \) to the point \( (t, nP) \). The following proposition follows easily from the definitions.

**Proposition 1.** For \( 2 \leq n \leq g \),

\[ \delta_n|_{\mathcal{W}_n^*} : \mathcal{W}_n^* \to \mathcal{O}_n^* \cap \Delta_n. \]

We now explicitly consider the intersection of \( \mathcal{O}_n^* \) and \( \Delta_n \). Suppose \( (t, nP) \in \mathcal{O}_n^* \cap \Delta_n \). Put \( X = V_t \). Let \( z \) be a local coordinate on \( X \) centered at \( P \) and let \( z_1, \ldots, z_n \) denote \( n \) copies of \( z \). Let \( \sigma_1, \ldots, \sigma_n \) denote the \( n \) elementary symmetric functions in \( z_1, \ldots, z_n \). Let \( c_1, \ldots, c_{3g-3} \) denote Patt's local coordinates on \( T \).

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centered at \( t \) (see [8]). Then \( c_1, \ldots, c_{3g-3}, \sigma_1, \ldots, \sigma_n \) are local coordinates on \( V_T^{(a)} \) centered at \((t, nP)\). Put

- \( Z = \) tangent space to \( V_T^{(a)} \) at \((t, nP)\);
- \( Z_1 = \) tangent space to \( \mathcal{O}_T^a \) at \((t, nP)\);
- \( Z_2 = \) tangent space to \( \Delta_n \) at \((t, nP)\).

We describe coordinates for \( Z \). Suppose \( \xi \in Z \). We may view \( \xi \) as a \( \mathbb{C} \)-homomorphism of local rings

\[
\xi: \Theta_{\mathcal{O}^{(a)},(t,nP)} \to \mathbb{C}[\varepsilon]/(\varepsilon^2)
\]

(cf. [6, p. 332]). Then \( \xi \) is determined by its values on a set of local parameters of \( \Theta_{\mathcal{O}^{(a)},(t,nP)} \). So, if \( \xi(c_m) = b_m\varepsilon, m = 1, \ldots, 3g-3 \), and \( \xi(\sigma_i) = u_i\varepsilon, i = 1, \ldots, n \), then \((u_1, \ldots, u_n, b_1, \ldots, b_{3g-3})\) serve as coordinates for \( Z \).

**Proposition 2.** \( \xi = (u_1, \ldots, u_n, b_1, \ldots, b_{3g-3}) \) is in \( Z_2 \) if and only if \( u_2 = u_3 = \ldots = u_n = 0 \).

**Proof.** Suppose \((t_1, Q_1 + \ldots + Q_n) \in V_T^{(a)}\) is a point near \((t, nP)\). Then \((t_1, Q_1 + \ldots + Q_n) \in \Delta_n \Leftrightarrow z(Q_1) = \ldots = z(Q_n) = z_0 \Leftrightarrow z_0 \) is an \( n \)-fold root of

\[
F(Y) = \prod_{i=1}^{n} (Y - z(Q_i)) = Y^n - \sigma_1(z(Q_1), \ldots, z(Q_n)) Y^{n-1} + \ldots + (-1)^n \sigma_n(z(Q_1), \ldots, z(Q_n))
\]

\( \Leftrightarrow F(z_0) = F'(z_0) = \ldots = F^{(n-1)}(z_0) = 0 \)

\( \Leftrightarrow \sigma_k(z(Q_1), \ldots, z(Q_n)) = \binom{n}{k} [\sigma_1(z(Q_1), \ldots, z(Q_n))]^{k/n} \)

for \( k = 2, 3, \ldots, n \), and \( \sigma_1(z(Q_1), \ldots, z(Q_n)) = nz_0 \).

So, near \((t, nP), \Delta_n \) is defined by the equations \( \{ \sigma_k = \binom{n}{k} \sigma_k^{1/n} \}, k = 2, \ldots, n \). Thus \( \xi \) is tangent to \( \Delta_n \) at \((t, nP)\) if and only if \( \xi(\sigma_k) = 0 \) for \( k = 2, 3, \ldots, n \).

We next recall the description of \( Z_1 \), which was given in [3]. Let \( \gamma_2, \ldots, \gamma_g \) denote the Weierstrass gaps at \( P \in X \). Choose a basis of holomorphic 1-forms \( dz_1, \ldots, dz_g \) on \( X \) such that \( \text{ord}_P dz_j = \gamma_j - 1 \). Write

\[
dz_j = \sum_{i=0}^{\infty} a_{i,j} z^i dz.
\]

For details concerning the following result, we refer the reader to [3].

**Proposition 3.** Suppose \( n \leq g \) and \( \xi \in Z \). Then \( \xi \in Z_1 \) if and only if all minors of order \( n - r + 1 \) of the matrix

\[
\begin{bmatrix}
(-1)^i a_{i,j} & \varepsilon \left[ \sum_{i=1}^{n} (-1)^{i+j-1} a_{i,j} u_i + \sum_{m=1}^{3g-3} \tau_{P,j}(Q_m) z_j(Q_m) b_m \right] \\
\end{bmatrix}
\]

\( i = 0, \ldots, n-1; \quad j = 1, \ldots, n - r \)

vanish, where \( \tau_{P,k} \) is an elementary integral of the second kind on \( X \) with pole of order \( k + 1 \) at \( P \) and where \((Q_1, \ldots, Q_{3g-3})\) is any point chosen from an open subset of \( X^{3g-3} \).
Now, suppose \((t, nP) \in \mathcal{R}^r_n - \mathcal{R}^{r+1}_n, n < g\). Then \(\mathcal{R}\) will have a nonzero minor of order \(n - r\), call it \(\mu\), and in order that all minors of order \(n - r + 1\) of \(\mathcal{R}\) vanish, it is sufficient that those minors of order \(n - r + 1\) which contain \(\mu\) should vanish. This gives rise to \(r(g - n + r)\) linear equations \(\{E_k\}\) in \(u_1, \ldots, u_n, b_1, \ldots, b_{3g-3}\). These equations are of the form

\[
E_k: \sum_{i=1}^{n} e_{k,i}u_i + \sum_{m=1}^{3g-3} \alpha_k(Q_m)b_m = 0,
\]

where the \(\alpha_k\) are (not necessarily finite) quadratic differentials on \(X\). (The \(\alpha_k\) arise from the products \(dr_{P,i}d\xi_j\) which appear in \(\mathcal{R}\)–see [3].)

**Theorem 3.** Suppose \(n < g\), \(r(g - n + r) < 3g - 3\), and \((t, nP) \in \mathcal{R}^r_n - \mathcal{R}^{r+1}_n\). If the above \(\alpha_k\), \(k = 1, \ldots, r(g - n + r)\), are linearly independent quadratic differentials, then:

1) \(\dim Z_1 = 3g - 3 + (r + 1)(n - r) - rg + r\) and \(\mathcal{R}^r_n\) is smooth at \((t, nP)\).

2) \(\dim Z_1 \cap Z_2 = 3g - 2 - r(g - n + r)\) and \(\mathcal{R}^r_n\) and \(\Delta_n\) intersect transversally at \((t, nP)\).

**Proof.** One may show, as in [3], that if the \(\alpha_k\) are linearly independent, then since \(\{Q_1, \ldots, Q_{3g-3}\}\) is any point from an open subset of \(X^{3g-3}\), the matrix \([\alpha_k(Q_m)], k = 1, \ldots, r(g - n + r)\) and \(m = 1, \ldots, 3g - 3\), will have maximum rank. It then follows that the systems of equations which define \(Z_1\) and \(Z_1 \cap Z_2\) will have maximum rank, establishing the theorem.

We showed in [3] that at least \(g - n + r\) of the \(\alpha_k\) are linearly independent. In particular, if \(r = 1\), then all the \(\alpha_k\) are linearly independent. As a consequence we have

**Theorem 4.** Suppose \((t, nP) \in \mathcal{R}^r_n - \mathcal{R}^{r+1}_n, n < g\). Then

1) \(\dim Z_1 < 2g + 2n - r - 3\); in particular, if \(r = 1\), then \(\dim Z_1 = 2g + 2n - 4\) and \(\mathcal{R}^r_n\) is smooth at \((t, nP)\).

2) \(\dim Z_1 \cap Z_2 < 2g + n - r - 2\); in particular, if \(r = 1\), then \(\dim Z_1 \cap Z_2 = 2g + n - 3\) and \(\mathcal{R}^r_n\) and \(\Delta_n\) intersect transversally at \((t, nP)\).

**Corollary.** For \(n < g\),

1) \(\dim \mathcal{W}^r_n < 2g + n - r - 2\);

2) \(\mathcal{W}^r_n - \mathcal{W}^3_n\) is smooth of pure dimension \(2g + n - 3\).

**Remarks.** (1) The smoothness of \(\mathcal{W}^r_n - \mathcal{W}^3_n\) has also recently been demonstrated by Arbarello-Cornalba [1] and Namba [7].

(2) Arbarello-Cornalba [1] have shown that \(\mathcal{W}^3_n - \mathcal{W}^3_n\) is smooth, but it does not necessarily follow that the \(\alpha_k\) are then linearly independent or that this space intersects \(\Delta_n\) transversally.

(3) In [5], we defined \(\mathcal{W}^r_n\) for \(n > g\). The points of this space are those \((t, P) \in V\) such that there are at least \(r\) gaps greater than \(n\) in the gap sequence at \(P \in V_r\). We showed that for \(n > g\), \(\mathcal{W}^r_n - \mathcal{W}^3_n\) is smooth of pure dimension \(4g - n - 3\). This result can also be obtained as above by considering the intersection of \(\mathcal{R}^r_n\) and \(\Delta_n\) for \(n > g\), but we note that, by our definition of \(\mathcal{W}^r_n\), for \(n > g\), \(\mathcal{W}^r_n = \delta^{-1}(\mathcal{R}^r_n - \mathcal{W}^3_n)\).
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