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ON THE DIMENSION OF VARIETIES OF
SPECIAL DIVISORS

BY

R. F. LAX

ABSTRACT. Let $T_g$ denote the Teichmüller space and let $V$ denote the
universal family of Teichmüller surfaces of genus $g$. Let $V_{T_g}^{(n)}$ denote the $n$th
symmetric product of $V$ over $T_g$ and let $J$ denote the family of Jacobians
over $T_g$. Let $f: V_{T_g}^{(n)} 	o J$ be the natural relativization over $T_g$ of the classical
map defined by integrating holomorphic differentials. Let

$$u: f^*\Omega^1_{J/T_g} \to \Omega^1_{V_{T_g}^{(n)}/T_g}$$

be the map induced by $f$. We define $G^r_n$ to be the analytic subspace of $V_{T_g}^{(n)}$
defined by the vanishing of $\Lambda^{n-r+1}u$.

Put $\tau = (r + 1)(n - r) - rg$. We show that $G^1_n - G^2_n$, if nonempty, is
smooth of pure dimension $3g - 3 + \tau + 1$. From this result, we may conclude
that, for a generic curve $X$, the fiber of $G^1_n - G^2_n$ over the module point of $X$,
if nonempty, is smooth of pure dimension $\tau + 1$, a classical assertion.

Variational formulas due to Schiffer and Spencer and Rauch are employed
in the study of $G^r_n$.

0. Introduction. Let $X$ be a complete, nonsingular curve of genus $g$ over
an algebraically closed field $K$. Let $X^{(n)}$ denote the $n$th symmetric product of
$X$. Let $G^r_n(X)$ denote the subvariety of $X^{(n)}$ of all divisors $D$ of degree $n$
such that $\dim |D| \geq r$. (In the literature, e.g. [12], $G^r_n(X)$ is often used to denote
the subvariety of the Jacobian of $X$ consisting of all linear systems of degree $n$
and projective dimension at least $r$.)

Put $\tau$ equal to $(r + 1)(n - r) - rg$. Brill and Noether [2] asserted that if $\tau$
were nonnegative and $X$ were a generic curve, then $G^r_n(X)$ would have dimen-
sion $\tau + r$. The recent work of Kleiman and Laksov ([10], [11]) and Kempf
[8] shows that for $X$ any curve, if $\tau \geq 0$, then $G^r_n(X)$ has dimension at least
$\tau + r$. We will show, in the case $K = \mathbb{C}$, that if $X$ is a generic curve, then
$G^1_n(X) - G^2_n(X)$, if nonempty, has dimension $\tau + 1$. 

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We work in the category of analytic spaces over \( C \). We do this because we want to consider the Teichmüller space, an analytic, but not algebraic, variety [5]. We take the Séminaire Cartan, 1960–61, as our foundational reference. In particular, we allow the structure sheaf of an analytic space to contain nilpotents.

Let \( Y \) be an analytic space over \( C \) and let \( E \) and \( F \) be locally free \( \mathcal{O}_Y \)-modules of ranks \( g \) and \( n \) respectively. Suppose we are given a map \( u: E \to F \). In §1, we define the analytic space \( Z^r(u) \) to be given by the vanishing of the map \( \bigwedge^{n-r+1} u \). We then study the infinitesimal structure of \( Z^r(u) \).

Let \( S \) be an analytic space over \( C \) and let \( X \) be a family of nonsingular curves of genus \( g \) over \( S \). Let \( X^{(n)}_S \) denote the \( n \)th symmetric product of \( X \) over \( S \) and let \( J_S \) denote the family of Jacobians over \( S \) (cf. [7], [15]). Suppose we are given a map \( f: X^{(n)}_S \to J_S \). Let

\[
u: f^* \Omega^1_{S/S} \to \Omega^1_{X^{(n)}_S/S}
\]

be the map induced by \( f \). We study the analytic space \( Z^r(u) \subseteq X^{(n)}_S \) in the following situation: \( S = T_g \), the Teichmüller space, \( X \) is the universal family of Teichmüller surfaces of genus \( g \), and \( f \) is the natural relativization over \( T_g \) of the classical map from the \( n \)th symmetric product of a curve into its Jacobian defined by integrating a basis of homomorphic differentials (cf. §2). We let \( G_n^r \) denote \( Z^r(u) \) in this situation.

In order to understand explicitly the above map \( f \), we must use certain variational formulas which are similar to those derived by Schiffer and Spencer [19], but much closer in form to those appearing in Rauch [18]. We also need a theorem due to Patt [17] concerning local coordinates at a point of \( T_g \).

Our main result is:

**Theorem.** Suppose \( y \in G^1_n - G^2_n \). Then the dimension of the tangent space to \( G^1_n \) at \( y \) is \( 3g - 3 + \tau + 1 \).

From this result, we can conclude that if \( X \) is a generic compact Riemann surface, then \( G^1_n(X) - G^2_n(X) \), if nonempty, is smooth of pure dimension \( \tau + 1 \).

As an application, we show that the subvariety of \( T_g \), for \( g \geq 4 \), of curves with nonempty \( G^1_3 \) (so-called "trigonal" curves), is of dimension \( 2g + 1 \), a result which was known to Severi [22] and B. Segre [20].

In a sequel to this paper, we will show that if \( \tau \geq 0 \) then \( G^2_n \) (resp. \( G^3_n \)) has a component of dimension \( 3g - 3 + \tau + 2 \) (resp. \( 3g - 3 + \tau + 3 \)). The proof involves computations using the examples of Riemann surfaces given by Meis [16].

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1. $Z'(u)$ and its infinitesimal structure. Let $S$ be an analytic space over $\mathbb{C}$. Denote by $((An/S))$ the category of analytic spaces over $S$. Let $Y$ be an analytic space over $S$ and let $E$ and $F$ be locally free $\mathcal{O}_Y$-modules of ranks $g$ and $n$ respectively. Suppose we are given a map $u: E \rightarrow F$. Define the functor $Z'(u): ((An/S))^0 \rightarrow ((\text{Sets}))$ by

$$Z'(u)(T) = \left\{ g \in \text{Hom} (T, Y) : \bigwedge^{n-r+1} g^* u = 0 \right\}.$$ 

We wish to show that this functor is represented by an analytic subspace of $Y$.

**Definition 1 [5].** Let $S$ be an analytic space and let

$$G: ((An/S))^0 \rightarrow ((\text{Sets}))$$

be a functor. We say that $G$ is of a local nature if for every $T$ the presheaf $U \mapsto G(U)$, where $U$ runs through the open sets of $T$, is a sheaf.

**Remark.** This is the analog to the notion of a Zariski sheaf in the category of contravariant functors from $((\text{Schemes}))$ to $((\text{Sets})).$

**Lemma 1.** Let $(S_i)$ be a covering of an analytic space $S$ by open sets. Let $G: ((An/S))^0 \rightarrow ((\text{Sets}))$ be a functor. Then $G$ is representable iff $G$ is of a local nature and for every $i$, the functor $G/S_i: ((An/S_i))^0 \rightarrow ((\text{Sets}))$ is representable.

**Proof.** [5, Corollary 5.7 of Exposé 7].

Our functor $Z'(u)$ is clearly of a local nature. Hence, by the lemma, its representability is a local question.

Let $y$ be a point of $Y$. Since $E$ and $F$ are locally free of ranks $g$ and $n$ respectively, the map $u$ is given locally at $y$ by an $n \times g$ matrix $[f_{ik}]$ of functions regular at $y$. The functor $Z'(u)$ is then locally represented by the closed analytic subspace defined by the vanishing of the minors of order $n - r + 1$ of the matrix $[f_{ik}]$. Thus we have

**Proposition 1.** $Z'(u)$ is represented by a closed analytic subspace of $Y$.

We will also use $Z'(u)$ to denote this analytic subspace.

Put $\rho = \text{rank}(u \otimes \kappa(y))$. Locally at $y$, both $E$ and $F$ split off a direct summand of rank $\rho$, and $u$ maps one summand isomorphically onto the other. The map that $u$ induces on the other two summands is given by an $(n - \rho) \times (g - \rho)$ matrix $[e_{jk}]$ of functions regular at $y$. The analytic space $Z'(u)$ is also defined locally at $y$ by the vanishing of the minors of order $(n - r + 1 - \rho)$ of the matrix $[e_{jk}]$ (cf. [10]).

**Proposition 2.** Assume $r > 0$. Then the points of $Z^{r+1}(u)$ are singular points of $Z'(u)$.
PROOF. Suppose \( y \in Z^{r+1}(u) \). Then we have \( \rho < n - r \). By construction, the \( e_{jk} \) above vanish at \( y \), hence are in the maximal ideal \( m \) of \( \mathcal{O}_{Y,Y} \). The analytic space \( Z'(u) \) is defined locally at \( y \) by the vanishing of the minors of order \( (n - r + 1 - \rho) \) of the matrix \( [e_{jk}] \) and, since \( \rho < n - r \), all these minors are of order at least 2, hence are in \( m^2 \). Thus \( y \) cannot be a smooth point of \( Z'(u) \).

We want now to study the infinitesimal structure of \( Z'(u) \). Let \( \xi \) denote a tangent vector to \( Y \) at \( y \). We will also use \( \xi \) to denote the comorphism, which is a \( \mathcal{O}_Y \)-homomorphism of local rings \( \xi: \mathcal{O}_{Y,Y} \to \mathcal{O}((\varepsilon))/((\varepsilon^2)) \).

We are interested in seeing when \( \xi \) is a tangent vector to \( Z'(u) \) at \( y \). By definition, this will be true if \( \wedge^{n-r+1} \xi^* u = 0 \).

PROPOSITION 3. \( \xi \) is a tangent vector to \( Z'(u) \) at \( y \) iff the minors of order \( n-r+1 \) of the matrix \( [\xi(f_{jk})] \) are all zero.

PROOF. It is easy to see that the map \( \xi^* u \) is given by the matrix \( [\xi(f_{jk})] \). Thus we have \( \wedge^{n-r+1} \xi^* u = 0 \) iff the minors of order \( n-r+1 \) of \( [\xi(f_{jk})] \) all vanish.

We now assume that \( Y \) is smooth of dimension \( m \) over \( \mathbb{C} \). Let \( y \in Y \) and let \( \sigma_1, \ldots, \sigma_m \) be local parameters on \( Y \) at \( y \). Let \( s_i \) in \( \mathbb{C} \) be given by

\[
\xi(\sigma_l) = s_i \varepsilon, \quad l = 1, 2, \ldots, m.
\]

Then, by Taylor's Theorem, we have

\[
\xi(f_{jk}) = f_{jk}(y) + \varepsilon \sum_{i=1}^{m} s_i \frac{\partial f_{jk}}{\partial \sigma_l}(y).
\]

The vanishing of the minors of order \( n-r+1 \) of the matrix \( [\xi(f_{jk})] \) gives rise to linear equations in the \( s_i \). These equations must be satisfied for \( \xi \) to be a tangent vector to \( Z'(u) \) at \( y \). If we view \( s_1, \ldots, s_m \) as being unknowns, then the dimension of the solution space of this system of equations is the dimension of the tangent space to \( Z'(u) \) at \( y \).

If \( y \in Z'(u) - Z^{r+1}(u) \), we will want to use the following lemma.

LEMMA 2. Let \( A \) be a commutative ring (with unit). Let \( M = [a_{jk}] \) be an \( m \times n \) matrix over \( A \). Suppose that a minor \( \mu \) of order \( r \) is a unit, and that every minor of order \( r+1 \) containing \( \mu \) vanishes. Then every minor of order \( r+1 \) vanishes.

PROOF. Without loss of generality, we may assume that \( \mu \) is the leading (i.e., upper left) minor of order \( r \). Since \( \mu \) is a unit, we may perform column operations using the first \( r \) columns to change \( M \) to the matrix
\[
M' = \begin{bmatrix}
\mu & \cdots & 0 \\
\cdots & \cdots & \cdots \\
a_{r+1,1} & \cdots & a_{r+1,r} \\
a_{m,1} & \cdots & a_{m,r} \\
N \\
\end{bmatrix}
\]

where \( N \) is an \((m - r) \times (n - r)\) matrix.

Then by row operations, using the first \( r \) rows, we may change \( M' \) to the
matrix
\[
M'' = \begin{bmatrix}
\mu & 0 \\
\cdots & \cdots \\
0 & \cdots \\
N \\
\end{bmatrix}
\]
where \( N \) is the same matrix as before.

Now, no minor containing \( \mu \) is affected by performing these row and column
operations. Hence, the minors of order \( r + 1 \) of \( M'' \) which contain \( \mu \) are all
zero. Thus \( N \) is the zero matrix.

But this implies that every column of \( M \) is a linear combination of the first
\( r \) columns of \( M \). Hence, every minor of order \( r + 1 \) of \( M \) is zero. \( \square \)

Suppose now that \( y \in Z'(u) - Z'^{+1}(u) \). Then the matrix \([f_j]\) has rank
\( n - r \). We may thus assume that the leading minor of order \( n - r \) of \([f_j]\), call it \( \mu_0 \), is nonzero. Let \( \mu' \) denote the leading minor of order \( n - r \) of \([\xi(f_{jk})]\).
Then \( \mu' = \mu + ce \), for some \( c \in \mathbb{C} \). Since \( \mu \) is nonzero, \( \mu' \) does not lie in
the maximal ideal of \( \mathbb{C}[e]/(e^2) \), hence is a unit. We then have, by Proposition 3
and Lemma 2, that \( \xi \) is a tangent vector to \( Z'(u) \) at \( y \) iff the minors of order
\( n - r + 1 \) of \([\xi(f_{jk})]\) which contain \( \mu' \) all vanish. Obviously, there are
\( r(g - n + r) \) such minors. If the equations in the \( s_i \) given by the vanishing of
these minors are linearly independent (over \( \mathbb{C} \)), then the dimension of the tangent
space to \( Z'(u) \) at \( y \) is \( m - r(g - n + r) \). We could then conclude that \( y \) is a
smooth point of \( Z'(u) \) by virtue of the following proposition.

**Proposition 4.** Either \( Z'(u) \) is empty, or each component has codimension at most \( r(g - n + r) \) \( \) in \( Y \).

**Proof.** This is proved in [9] for \( Y \) a scheme. With the obvious modifications, the proof is valid for \( Y \) an analytic space.

2. The universal family of Teichmüller surfaces. In [5], Grothendieck
proved the following

**Theorem 1.** There exist an analytic space \( T_g \) and a family \( V \) of
Teichmüller surfaces of genus \( g \) over \( T_g \) which is universal in the following
sense: for every family \( X \) of Teichmüller surfaces of genus \( g \) over an analytic
space \( S \), there exists a unique map \( \Phi: S \rightarrow T_g \) such that \( X \) is isomorphic
(as a family of Teichmüller surfaces) to the pullback via \( \Phi \) of \( V/T_g \).
$T_g$ is called the Teichmüller space (for Teichmüller surfaces of genus $g$). The Teichmüller space is a smooth, irreducible, and simply connected analytic space [5].

Let $h : V \to T_g$ denote the structural morphism. By well-known topological facts, since $T_g$ is simply connected, the fiber bundle $R^1h_*Z$ is trivial. Thus, there are sections of this bundle which give rise to cycles $\gamma_i(s), \delta_i(s), i = 1, \cdots, g,$ which form a canonical homology basis for $H_1(V_s, \mathbb{Z}), s \in T_g$ [15].

Consider the sheaf $\Omega^1_{V/T_g}$. For all $s \in T_g$, we have

$$\dim H^0(V_s, \Omega^1_{V/T_g} \otimes \kappa(s)) = \dim H^0(V_s, \Omega^1_V) = g.$$ 

Hence, $h_\ast \Omega^1_{V/T_g}$ is a vector bundle of rank $g$ over $T_g$ and we have

$$h_\ast \Omega^1_{V/T_g} \otimes \kappa(s) \cong H^0(V_s, \Omega^1_V)$$

by [4].

Choose holomorphic sections $d\xi_i^s, i = 1, \cdots, g,$ of $h_\ast \Omega^1_{V/T_g}$ such that $\{d\xi_i^s(s)\}_{i=1}^g$ is a basis for $H^0(V_s, \Omega^1_V), s \in T_g$ (cf. [15]). Put

$$a_{ij}(s) = \int_{\gamma_i(s)} d\xi_j^s(s), \quad b_{ij}(s) = \int_{\delta_j(s)} d\xi_j^s(s), \quad i, j = 1, \cdots, g.$$ 

For each $s \in T_g$, the matrix $[a_{ij}(s), b_{ij}(s)]$ is the period matrix of $V_s$. Recall that the columns of this matrix generate a maximal lattice subgroup of $\mathbb{C}^g$. Let $J$ be the quotient of $T_g \times \mathbb{C}^g$ by this family of lattices. The induced projection $J \to T_g$ gives a complex analytic family of complex tori, the fiber $J_s$ being the Jacobian variety of the Teichmüller surface $V_s$ [15].

Since our concern will only be local, we assume that there exist sections of $V \to T_g$. Let $P_0^s(s)$ be such a section. As in [15], define a map $\psi : V \to J$ by

$$\psi(s, P) = \left( s, \int_{P_0^s(s)} d\xi_1^s(s), \cdots, \int_{P_0^s(s)} d\xi_g^s(s) \right) \mod \text{periods}$$

for $P \in V_s$.

Denote by $V_{T_g}^{(n)}$ the $n$th symmetric product of $V$ over $T_g$ (cf. [7]). Extend $\psi$ to a map $f : V_{T_g}^{(n)} \to J$ as follows. If $s \in T_g$ and $D \in (V_{T_g}^{(n)})_s$ is the divisor $\Sigma_{i=1}^nP_i$ on $V_s$, then

$$f(s, D) = \left( s, \sum_{i=1}^n \int_{P_0^s(s)} d\xi_i^s(s), \cdots, \sum_{i=1}^n \int_{P_0^s(s)} d\xi_g^s(s) \right) \mod \text{periods}.$$ 

Let
be the map induced by \( f \). Since \( J \) and \( V_{T_g}^{(n)} \) are smooth over \( T_g \) of relative dimensions \( g \) and \( n \) respectively, the sheaves

\[
f^* \Omega^1_{J/T_g} \quad \text{and} \quad \Omega^1_{V_{T_g}^{(n)}/T_g}
\]

are locally free of ranks \( g \) and \( n \) respectively. Thus, we may consider the analytic subspace \( Z'(u) \subseteq V_{T_g}^{(n)} \) of \( \S 1 \). We will denote by \( \mathcal{G}_n^r \) the analytic space \( Z'(u) \) which arises in this situation. We will see in \( \S 4 \) that \( (\mathcal{G}_n^r)_s \) is what was denoted by \( \mathcal{G}_n^r(V_s) \) in \( \S 0 \).

We wish to study the infinitesimal structure of \( \mathcal{G}_n^r \). To do this, we need explicit knowledge of the above map \( f \). And to obtain this knowledge, we need certain variational formulas which are contained in the next section.

3. The variational formula. For a detailed treatment of the material in this section, the reader is referred to Rauch [18] or Patt [17].

Let \( X \) be a compact Riemann surface of genus \( g > 0 \). Let \( \Gamma = (\gamma_1, \cdots, \gamma_g) \) and \( \Delta = (\delta_1, \cdots, \delta_g) \) be a canonical homotopy basis and let \( \Pi \) be the simply connected surface obtained by the canonical dissection of \( X \) determined by \( \Gamma \) and \( \Delta \) (cf. [23]).

Let \( w \) be a point in the interior of \( \Pi \) and let \( \tau_{w, \nu}(z) \) denote the (normalized) elementary integral of the second kind with pole of order \( \nu + 1 \) at \( w \) and zero \( \Gamma \)-periods.

Let \( \xi \) be an Abelian integral of the first kind. Let \( a_i, \ i = 1, \cdots, g \), denote the \( \Gamma \)-periods of \( d\xi \); that is,

\[
a_i = \int_{\gamma_i} d\xi, \quad i = 1, \cdots, g.
\]

The value of the derivatives of a determination of \( \xi \) at \( w \) and the periods of the differentials \( d\tau_{w, \nu} \) are related by

\[
\xi^{(\nu+1)}(w) = \frac{\nu!}{2\pi i} \sum_{j=1}^{g} a_j \int_{\delta_j} d\tau_{w,\nu}(z)
\]

which follows from the bilinear relation for differentials of the first and second kinds [23, p. 260].

Let \( Q_1, \cdots, Q_n \) be distinct points in the interior of \( \Pi \) such that all the \( Q_j \) are different from \( w \) and none of the \( Q_j \) is a zero of \( d\xi \). Let \( t_j, \ j = 1, \cdots, n, \) be a local parameter at \( Q_j \). Let \( D_1, \cdots, D_n \) be disjoint disks about \( Q_1, \cdots, Q_n \) respectively, such that \( D_j \) lies in the domain of \( t_j \), is completely
contained in the interior of $\Pi$ and such that no $D_j$ contains either $w$ or any zero of $d\xi$.

Inside $D_j$, we can vary the local parameter $t_j$ to a new parameter $t_j^*$ given by

$$t_j^* = t_j + c_j/t_j, \quad j = 1, \cdots, n,$$

where $c_j$ is sufficiently small. This defines a new Riemann surface $X^*$, having the same canonical homotopy basis as $X$ has (since all variations take place in the interior of $\Pi$).

Let $\xi^*$ be the Abelian integral of first kind on $X^*$ with the same $\Gamma$-periods as $\xi$. We wish to compute

$$\Delta\xi^{(v+1)}(w) = \xi^{(v+1)}(w) - \xi^{(v+1)}(w).$$

*Notation.* $d\tau_{w,v}d\xi$ is a (not necessarily finite) quadratic differential on $X$. Locally at $Q_j$, we may write $d\tau_{w,v}d\xi = h(t_j)d\tau_j^2$. We now introduce the notation $\tau'_{w,v}(Q_j)$ for $h(0)$.

Utilizing the techniques and formulas in [18] and [17], one can obtain the following proposition:

**Proposition 5.**

$$\Delta\xi^{(v+1)}(w) = v! \sum_{m=1}^n c_m \tau'_{w,v}(Q_m)\xi'(Q_m) + O(c^2)$$

where $c = \max_{1 \leq m \leq n} |c_m|$.

We will also want to use the following theorem, due to Patt [17]:

**Theorem 2.** One may choose $3g - 3$ points $Q_1, \cdots, Q_{3g-3}$ on $X$ such that, if $c_m$ is the variation parameter at $Q_m$, then a neighborhood of the origin in the $c_1, \cdots, c_{3g-3}$ space describes a complex-analytic structure for a neighborhood of $X$ in the Teichmüller space. Moreover, the set of collections of $3g - 3$ points with this property is open in $X^{3g-3}$.

**Proof.** The first assertion follows from Theorems 2 and 4 of [17]. Although Patt does not state the second assertion, his proofs demonstrate it, as was noted by Farkas [3, p. 885].

4. The equations which define the tangent space. Let $X$ be a compact Riemann surface of genus $g > 1$. Let $\{\gamma_j, \delta_j\}_{j=1}^g$ be a canonical homotopy basis and let $\{d\xi_k\}_{k=1}^g$ be a basis of the holomorphic differentials. Put

$$A_{jk} = \int_{\gamma_j} d\xi_k, \quad j, k = 1, \cdots, g.$$
Let $P$ be a point of $X$ and let $t$ be a local parameter on $X$ at $P$. Write

$$d\xi_k = \sum_{i=0}^{\infty} a_{k,i} t^i dt.$$ 

Fix a point $P_0$ different from $P$. Choose a point $(Q_1, \cdots, Q_{3g-3})$ from the open subset of $X^{3g-3}$ in Theorem 2 such that all the $Q_m$ are different from $P$ and $P_0$ and such that none of the $Q_m$ is a zero of any $d\xi_k$. Perform the variation described in §3, taking the disk about each $Q_m$ sufficiently small so that no two disks intersect and no disk contains $P, P_0$, or any zero of any $d\xi_k$. Let $c_m$ denote the variation parameter at $Q_m$, $m = 1, \cdots, 3g - 3$, as in §3.

Let $s_0 \in T_g$ be the module point of $X$ (i.e., $V_{s_0} = X$). By definition of the variation, there exists a complex-analytic neighborhood $U$ of $s_0$ in $T_g$ such that, for all $s' \in U$, the curves $\{\gamma_j, \delta_j\}_{j=1}^g$ are a canonical homotopy basis on $V_{s'}$, the points $P_0$ and $P$ are on $V_{s'}$, and $t$ is a local parameter on $V_{s'}$ at $P$. Choose holomorphic sections $d\xi_k^*, k = 1, \cdots, g$, of $h^*\Omega^1_{V/T_g}$ such that

$$\int_{\gamma_j} d\xi_k^*(s') = A_{jk}, \quad s' \in U, \quad j, k = 1, \cdots, g$$

(cf. [15, §3]).

**Proposition 6.** With notation as in §3 and above, if we define $a_{k,1}^*$ by $d\xi_k^* = \sum_{i=0}^{\infty} a_{k,i} t^i dt$, then we have

$$a_{k,1}^* = a_{k,1} + \sum_{m=1}^{3g-3} c_m \tau_{P,1}(Q_m) \xi_k(Q_m) + O(t^2).$$

**Proof.** The variational formula (Proposition 5) shows that this equality holds in a complex-analytic neighborhood of $(s_0, P)$ on $V$. This is the main import of the variational formula.

In order to study the map

$$u: f^*\Omega^1_{V/T_g} \longrightarrow \Omega^1_{V^{(n)}_{T_g}/T_g}$$

of §2, we first consider the divisor $nP$ on $X$. Let $t_1, \cdots, t_n$ be $n$ copies of $t$, and let $\sigma_1, \cdots, \sigma_n$ denote the $n$ elementary symmetric functions in $t_1, \cdots, t_n$.

**Proposition 7.** Local parameters on $V^{(n)}_{T_g}$ at $(s_0, nP)$ are given by $c_1, \cdots, c_{3g-3}, \sigma_1, \cdots, \sigma_n$. 

PROOF. By Theorem 2, local parameters on \( T_g \) at \( s_0 \) are given by \( c_1, \cdots, c_{3g-3} \). By [1], local parameters on \( X^{(n)} \) at \( nP \) are given by \( \sigma_1, \cdots, \sigma_n \). By the definition of the variation in §3, local parameters on \( (V^{(n)}_{T_g} s \)' at \( nP \), for \( s' \in U \), are also given by \( \sigma_1, \cdots, \sigma_n \). Thus, local parameters on \( V^{(n)}_{T_g} \) at \( (s_0, nP) \) are given by \( c_1, \cdots, c_{3g-3}, \sigma_1, \cdots, \sigma_n \).

Put

\[
\tau_j = t_1^j dt_1 + \cdots + t_n^j dt_n, \quad j = 0, 1, 2, \cdots.
\]

We have

PROPOSITION 8. The space of holomorphic 1-forms on \( X \) is naturally isomorphic to the space of holomorphic 1-forms on \( X^{(n)} \). Both these spaces are isomorphic to the space of symmetric holomorphic 1-forms on the Cartesian product \( X^n \). If \( d\tilde{\zeta} = \sum_{l=0}^{\infty} a_l \zeta^l d\zeta \) is a holomorphic 1-form on \( X \) and \( d\tilde{\zeta}^* \) is the corresponding symmetric holomorphic 1-form on \( X^n \), then \( d\tilde{\zeta}^* = \sum_{l=0}^{\infty} a_l \zeta^l \).

PROOF. [14, pp. 226–227].

This result is easily seen to relativize to the following proposition.

PROPOSITION 9. The space of relative holomorphic 1-forms on \( V^{(n)}_{T_g} \) over \( T_g \) and the space of relative holomorphic 1-forms on \( V \) over \( T_g \) are naturally isomorphic. Both spaces are isomorphic to the space of relative symmetric holomorphic 1-forms on \( V^n_{T_g} \), the product over \( T_g \) of \( n \) copies of \( V \), over \( T_g \).

If \( d\tilde{\zeta}^* \) is the relative symmetric holomorphic 1-form on \( V^n_{T_g} \) over \( T_g \) corresponding to \( d\zeta^* \) (cf. Proposition 6), then

\[
d\tilde{\zeta}^* = \sum_{l=0}^{\infty} a_{k,l}^* \zeta^l.
\]

We will identify relative symmetric holomorphic 1-forms on \( V^n_{T_g} \) over \( T_g \) and relative holomorphic 1-forms on \( V^{(n)}_{T_g} \) over \( T_g \).

Now, we can express \( d\tilde{\zeta}^* \) in terms of \( d\sigma_1, \cdots, d\sigma_n \) by using the following identities [14]:

\[
\sigma_k \tau_0 - \sigma_{k-1} \tau_1 + \cdots + (-1)^k \tau_k = d\sigma_{k+1}.
\]

(By convention, \( \sigma_k = 0 \) and \( d\sigma_k = 0 \) if \( k > n \).) Inverting these identities, and writing out only the linear terms, we obtain

\[
\tau_k = (-1)^k (d\sigma_{k+1} - \sigma_1 d\sigma_k - \cdots - \sigma_k d\sigma_1)
\]

+ higher order terms.

Thus we may write
\[ d\tilde{\xi}_k = \sum_{l=0}^{\infty} (-1)^l \left[ (d\sigma_{l+1} - \sigma_1 d\sigma_l - \cdots - \sigma_l d\sigma_1) \right] + O(\sigma^2, c^2) \]

where \( O(\sigma^2, c^2) \) denotes higher order terms in the \( \sigma_i \) and the \( c_m \).

By definition of the map \( f: V^{(n)}_T \rightarrow J \) in §2, it is easy to see that \( f \) is given at \( (s_0, nP) \) by

\[ f(s_0, nP) = \left( s_0, \int_{P_0}^P d\tilde{\xi}_1(s_0), \cdots, \int_{P_0}^P d\tilde{\xi}_g(s_0) \right) \mod\text{periods} \]

where the integrals \( \int_{P_0}^P d\tilde{\xi}_k(s_0) \) are evaluated by recalling that \( t_1, \cdots, t_n \) are just copies of \( t \). Let \( \partial \tilde{\xi}_k / \partial \sigma_j \) be given by

\[ d\tilde{\xi}_k = \sum_{j=1}^n \frac{\partial \tilde{\xi}_k}{\partial \sigma_j} \, d\sigma_j. \]

Then we have

**Proposition 10.** The map

\[ u: f^*\Omega^1_J \rightarrow \Omega^1_{V^{(n)}_T} \]

is given locally at \( (s_0, nP) \) by the matrix

\[ [\partial \tilde{\xi}_k / \partial \sigma_j], \quad j = 1, \cdots, n, k = 1, \cdots, g. \]

**Proof.** This follows easily from the definitions of \( f \) and \( \partial \tilde{\xi}_k / \partial \sigma_j \). (Compare with [3] and [6].)

**Remark.** Let \( J \) denote the Jacobian variety of \( X \) and let \( f_0: X^{(n)} \rightarrow J \) be the classical map (i.e., the map \( f \otimes \kappa(s_0) \)). Then the matrix \( M = [(\partial \tilde{\xi}_k / \partial \sigma_j)(s_0, nP)] \) is the matrix of the map \( u_0: f^*\Omega^1_J \rightarrow \Omega^1_{X^{(n)}} \) at \( nP \) (cf. [3], [6]). It is then easy to see that \( (C'_n)_{s_0} \) is what was denoted by \( C'_n(X) \) in §0.

Now let \( \xi \) be a tangent vector to \( V^{(n)}_T \) at \( (s_0, nP) \). Let \( s_j \) and \( b_m \) in \( C \) be given by

\[ \xi(\sigma_j) = s_j e, \quad j = 1, \cdots, n, \]

\[ \xi(c_m) = b_m e, \quad m = 1, \cdots, 3g - 3. \]

Then, using Taylor's Theorem as in §1, we have
\[ \xi \left( \frac{\partial \tilde{\xi}^*_k}{\partial \sigma_j} \right) = \frac{\partial \tilde{\xi}^*_k}{\partial \sigma_j} (s_0, nP) + \epsilon \sum_{l=1}^{n} s_l \frac{\partial^2 \tilde{\xi}^*_k}{\partial \sigma_i \partial \sigma_j} (s_0, nP) + \epsilon \sum_{m=1}^{3g-3} b_m \frac{\partial^2 \tilde{\xi}^*_k}{\partial c_m \partial \sigma_j} (s_0, nP). \]

We will now use (\#) to compute the partial derivatives of \( \frac{\partial \tilde{\xi}^*_k}{\partial \sigma_j} \) with respect to \( \sigma_i \) and with respect to \( c_m \). (We remind the reader that the functions \( \sigma_i \) and \( c_m \) vanish at \( (s_0, nP) \).) We obtain

\[ \frac{\partial^2 \tilde{\xi}^*_k}{\partial \sigma_i \partial \sigma_j} (s_0, nP) = (-1)^{l+i} a_{k,i+l-1} \]

and

\[ \frac{\partial^2 \tilde{\xi}^*_k}{\partial c_m \partial \sigma_j} (s_0, nP) = \gamma_{P,j-1}(Q_m) \xi'_k(Q_m). \]

Substituting these expressions for the partial derivatives evaluated at \( (s_0, nP) \) into (1) gives us

**Proposition 11.**

\[ \xi \left( \frac{\partial \tilde{\xi}^*_k}{\partial \sigma_j} \right) = \frac{\partial \tilde{\xi}^*_k}{\partial \sigma_j} (s_0, nP) + \epsilon \sum_{l=1}^{n} (-1)^{l+i} a_{k,i+l-1} \]

\[ + \epsilon \sum_{m=1}^{3g-3} b_m \gamma_{P,j-1}(Q_m) \xi'_k(Q_m). \]

Now on to the general case. Consider a divisor \( D \) on \( X \) of the form

\( D = m_1 P_1 + \cdots + m_d P_d \). Assume \( D \) is in \( G^r_n(X) \) and choose a basis \( \{ d\tilde{\xi}^*_k \}_{k=1}^{\xi} \) of the holomorphic differentials on \( X \) such that the last \( i = \dim H^1(X, O_X(D)) \) of them vanish on \( D \).

In performing the variation in \$3\$, choose a point \( (Q_1, \ldots, Q_{3g-3}) \) from the open set in \( X^{3g-3} \) in Theorem 2 so that each \( Q_m \) is different from \( P_0, P_1, \ldots, P_d \) and any other zero of any \( d\tilde{\xi}^*_k \). (The choice of this point will be further modified later.) Take the disk about each \( Q_m \) sufficiently small so that no two disks intersect and such that no disk contains \( P_0, P_1, \ldots, P_d \) or any other zero of any \( d\tilde{\xi}^*_k \).

Let \( f_j: V_{T_g}^{(m_j)} \rightarrow J \) be the map defined in \$2\$ and let

\[ u_j: f_j^* \Omega_{T_g}^{1} \rightarrow \Omega_{V_{T_g}}^{1} \]
be the map induced by $f_j$. The obvious map

$$V_{T_g}^{(m_1)} \times_{T_g} V_{T_g}^{(m_2)} \times_{T_g} \cdots \times_{T_g} V_{T_g}^{(m_d)} \rightarrow V_{T_g}^{(n)}$$

is a local analytic isomorphism by an argument analogous to that given in [14] in the case of a curve over a field. Locally, the map $f$ is the one induced by the $f_j$ and the map $u$ is the one induced by the $u_j$. Thus, the matrix of $u$ locally at $(s_0, D)$ is obtained by “stacking” the matrices of the $u_j$ locally at $(s_0, m_j P_j)$.

Let $\xi$ be a tangent vector to $V_{T_g}^{(n)}$ at $(s_0, D)$ and let $\xi_j$ be the tangent vector to $V_{T_g}^{(m_j)}$ at $(s_0, m_j P_j)$ induced by $\xi$, for $j = 1, \ldots, d$. Then the matrix of $\xi^* u$ is obtained by “stacking” the matrices of the $\xi_j^* u_j$, for $j = 1, \ldots, d$.

Let $M'$ denote the matrix of $\xi^* u$. Let $\mu$ denote the leading minor of order $n - r$ of $M$, the matrix $[(\partial \xi^* / \partial \sigma_j)(s_0, D)]$, and let $\mu'$ denote the leading minor of order $n - r$ of $M'$. Then we have $\mu' = \mu + c \epsilon$ for some $c$ in $C$. Now, by our choice of a basis of the holomorphic differentials on $X$, the last $i$ columns of $M$ are identically zero, hence the last $i$ columns of $M'$ contain “pure” $\epsilon$ terms (i.e., members of the maximal ideal of $C[\epsilon]/(\epsilon^2)$). Thus, in computing a minor of order $n - r + 1$ containing $\mu'$, any $\epsilon$’s in the first $n - r$ columns will be “killed” by the $\epsilon$ in the last column of the minor of order $n - r + 1$. Hence, we have established

**Lemma 3.** For purposes of computing the minors of order $n - r + 1$ of $M'$, we may replace the first $n - r$ columns of $M'$ by the first $n - r$ columns of $M$.

Let $M$ denote the resulting matrix.

$M$ has a particularly nice form in the case that $D = P_1 + P_2 + \cdots + P_n$, with all points distinct. Let $t_j$ be a local parameter at $P_j$ and write $dx_k = \varphi_{j,k} dt_j$. Then we have

$$M = \begin{bmatrix}
\varphi_{j,k}(P_j) & \epsilon \left( s_j \varphi_{j,k}'(P_j) + \sum_{m=1}^{3g-3} b_m \tau_{P_j,0}(Q_m) \kappa_k'(Q_m) \right) \\
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& & & & & & \left. \begin{array}{c}
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\epsilon \left( s_j \varphi_{j,k}'(P_j) + \sum_{m=1}^{3g-3} b_m \tau_{P_j,0}(Q_m) \kappa_k'(Q_m) \right) \\
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\end{array} \right)
\end{bmatrix}
$$

Going back to the general case, recall that, by Proposition 1, $\xi$ will be a tangent vector to $G_n'$ at $(s_0, D)$ iff the minors of order $n - r + 1$ of the matrix $M$ all vanish. Assume $D = m_1 P_1 + \cdots + m_d P_d$ is in $G_n'(X) - G_n^{r+1}(X)$. Then the matrix $M$ has rank precisely $n - r$. Hence, by permuting the rows of
$M$, if necessary, we end up with a matrix whose leading minor of order $n - r$, which we will denote by $\mu$, is nonzero. We will continue to denote this matrix by $M$, although its form may differ slightly from that specified earlier.

Perform the same row permutations as above on the matrix $M$ and denote the resulting matrix also by $M$. Then $\mu$ is also the leading minor of order $n - r$ of $M$, so we may apply Lemma 2. Thus, for all the minors of order $n - r + 1$ of $M$ to vanish, it is sufficient that every minor of order $n - r + 1$ which contains $\mu$ vanishes. The vanishing of each of these minors gives rise to a linear equation in the $s_j$ and the $b_m$.

Let $\mu_{j,k}$ denote the minor of order $n - r + 1$ of $M$ obtained by adjoining to $\mu$ the first $n - r$ elements of the $(n - r + j)$th row of $M$ and the first $n - r$ elements and the $(n - r + j)$th element of the $(n - r + k)$th column of $M$ (thus $j$ runs from 1 through $r$ and $k$ runs from 1 through $i$). The equation $\mu_{j,k} = 0$ is of the form $eE_{j,k} = 0$ where $E_{j,k}$ is a linear equation in the $s_j$ and the $b_m$ with coefficients in $C$.

We will now view the $s_j$ and the $b_m$ as being unknowns (as in §1). Thus, $E_{j,k}$ is an equation in $3g - 3 + n$ unknowns. By the discussion after Proposition 1, the dimension of the tangent space to $G_n'$ at $(s_0, D)$ is

$$3g - 3 + n - (\text{the number of } E_{j,k} \text{ which are linearly independent}).$$

Consider the coefficient of $b_m$ in $E_{j,k}$. This coefficient will be a linear combination of certain of the $\tau_{p_j,v}(Q_m)s_{k}(Q_m)$. That is, the coefficient of $b_m$ will be a certain quadratic differential (the above linear combination of certain of the $d\tau_{p_j,v}d\xi_k$) evaluated at the point $Q_m$. It should be noted that, by the symmetry of the matrix $M$ in the $b_m$, this quadratic differential does not depend on $m$, but only on $j$ and $k$. The coefficient of $b_1$ in $E_{j,k}$ is the value of this quadratic differential at $Q_1$, the coefficient of $b_2$ in $E_{j,k}$ the value at $Q_2$, etc. Put $\alpha_{j,k}$ equal to the above linear combination of certain of the $d\tau_{p_j,v}d\xi_k$. Then $\alpha_{j,k}$ is a (not necessarily finite) quadratic differential.

**Notation.** Choose a local parameter $u_m$ on $X$ at $Q_m$ and write $\alpha_{j,k} = g(u_m)du_m^2$. Then we will write $\alpha_{j,k}(Q_m)$ for $g(0)$. Hence, by the above discussion, $\alpha_{j,k}(Q_m)$ is the coefficient of $b_m$ in $E_{j,k}$.

Our aim now is to show that, in certain situations, by suitably choosing the point $(Q_1, \cdots, Q_{3g-3})$, we may conclude that the $E_{j,k}$ are linearly independent. Assume that $ri \leq 3g - 3$. By elementary linear algebra, to conclude that the $E_{j,k}$ are linearly independent, it is sufficient to show that the matrix of coefficients

$$A = [\alpha_{j,k}(Q_m)], \quad j = 1, \cdots, r; k = 1, \cdots, i; m = 1, \cdots, ri,$$

is nonsingular.
Lemma 4. Assume that the $\alpha_{j,k}$ for $j = 1, \cdots, r$ and $k = 1, \cdots, i$, are linearly independent and that $r \leq 3g - 3$. Then we may choose a point $(Q_1, \cdots, Q_{3g-3})$ from the open set in $X^{3g-3}$ in Theorem 2 such that each $Q_m$ is different from $P_0$ and no $Q_m$ is a zero of $d\xi_j, \cdots, d\xi_g$ and such that the above matrix $A$ is nonsingular.

Proof. The lemma will follow readily from the following

Sublemma. Let $\beta_1, \cdots, \beta_n$ be $n$ linearly independent quadratic differentials on $X$. Let $U$ be an open set contained in $X^n$. Then we may choose a point $(P_1, \cdots, P_n) \in U$ such that each $P_m$ is different from a finite set of points of $X$ and such that the matrix $[\beta_j(P_k)]$ ($j = 1, \cdots, n; k = 1, \cdots, n$) is nonsingular.

Proof. By induction on $n$. If $n = 1$, then $\beta_1$ is a nontrivial quadratic differential. Hence, $\beta_1$ is nonzero and finite on a dense open set of $X$. So, given any open set in $X$, there exists a point in that set satisfying the requirements of the Sublemma.

Now suppose $U$ is an open set contained in $X^n$. Let $V$ be the projection of $U$ onto $X^{n-1}$. Then $V$ is open and, by induction, we may choose a point $(P_0, \cdots, P_{n-1}) \in V$ such that each $P_m$ is different from a finite set of points of $X$ and such that the leading subdeterminant of order $n - 1$ of the determinant

$$
\begin{vmatrix}
\beta_1(P_1) & \cdots & \beta_1(P_{n-1}) & \beta_1 \\
\vdots & & \vdots & \vdots \\
\vdots & & \vdots & \vdots \\
\beta_n(P_1) & \cdots & \beta_n(P_{n-1}) & \beta_n
\end{vmatrix}
$$

is nonzero. Expanding the full determinant by the last column, we obtain a nontrivial linear combination of $\beta_1, \cdots, \beta_n$. By the linear independence of these quadratic differentials, this linear combination is a nontrivial quadratic differential, hence is nonzero and finite on an open dense set $W$ contained in $X$. Since $U$ is open in $X^n$ and $W$ is dense in $X$, we may choose a point in the intersection of $U$ and $\{(P_0, \cdots, P_{n-1})\} \times W$ which satisfies the requirements of our Sublemma.

Now, since the set of points in $X^{3g-3}$ in Theorem 2 is open, it is easy to see that we may choose a point $(Q_1, \cdots, Q_{3g-3})$ in this set such that each $Q_m$ is different from $P_0$ and the zeros of $d\xi_1, \cdots, d\xi_g$ and so that $Q_1, \cdots, Q_{rt}$ make the matrix $A$ nonsingular. This completes the proof of the lemma.

We then have
Proposition 12. Suppose $D$ is in $G_n^r(X) - G_n^{r+1}(X)$ and that $ri \leq 3g - 3$. Then if all the $\alpha_{j,k}$ are linearly independent, the dimension of the tangent space to $G_n^r$ at $(s_0, D)$ is $3g - 3 + \tau + r$.

Proof. By Lemma 4, we may choose a point $(Q_1, \cdots, Q_{3g-3})$ from the open set in Theorem 2 such that each of the $Q_m$ is different from $P_0$ and the zeros of $dk_1, \cdots, dk_g$ (note that this latter set includes the points of $D$), and such that the equations $E_{j,k}$ are linearly independent. Thus the dimension of the tangent space to $G_n^r$ at $(s_0, D)$ is $3g - 3 + n - ir = 3g - 3 + \tau + r$.

In the next section, we show that if $D$ is in $G_n^1(X) - G_n^2(X)$, then the $\alpha_{j,k}$ are linearly independent. (Note that we have $i < 3g - 3$ if $g > 1$.)

5. The dimension of $G_n^1 - G_n^2$. For simplicity, we will first treat a divisor consisting of $n$ distinct points. So assume $D = P_1 + P_2 + \cdots + P_n$, all points distinct, is in $G_n^1(X) - G_n^2(X)$. Recall that the matrix $M$ is

$$M = \begin{bmatrix} \varphi_{j,k}(P_j) & \vdots & e \left(s_j \varphi_{j,k}(P_j) + \sum_{m=1}^{3g-3} b_m \tau_{P_j,0}^r(Q_m) \xi_k^r(Q_m) \right) \\ j = 1, \cdots, n \\ k = 1, \cdots, g-i \end{bmatrix}.$$ 

Let $|j|$ denote the minor of order $n - 1$ obtained by omitting the $j$th row from the matrix $[\varphi_{j,k}(P_j)]$ $(j = 1, \cdots, n, k = 1, \cdots, g-i)$. Then we have

$$\alpha_{1,k}(Q) = \sum_{j=1}^{n} (-1)^{j-1} |j| \tau_{P_j,0}^r(Q) \xi_{n+k-1}^r(Q)$$

for $k = 1, 2, \cdots, i$. Suppose we had a linear relation of the form $\sum_{k=1}^{n} a_k \alpha_{1,k} = 0$ with some $a_i$ nonzero. Then this would imply that

$$(*) \quad \left( \sum_{j=1}^{n} (-1)^{j-1} |j| \tau_{P_j,0}^r(Q) \right) \left( \sum_{k=1}^{i} a_k \xi_{n+k-1}^r(Q) \right) = 0.$$ 

But the $d\tau_{P_j,0}$, $j = 1, \cdots, n$, are linearly independent, since they have poles at different points. This, together with the fact that $|j| \neq 0$, implies that there is a dense open set of points of $X$ where the expression $\sum_{j=1}^{n} (-1)^{j-1} |j| \tau_{P_j,0}^r(Q)$ is nonzero.

And the linear independence of $dk_1, \cdots, dk_g$, together with the fact that some $a_i$ is nonzero, implies that the expression $\sum_{k=1}^{i} a_k \xi_{n+k-1}^r(Q)$ is nonzero on a dense open set of points of $X$. Hence, we may choose a point $Q$ such that

$(*)$ is nonzero, contradicting the assumption that $\alpha_{1,1}, \cdots, \alpha_{1,i}$ are linearly dependent.
Now suppose \( D = m_1P_1 + \cdots + m_dP_d \) is in \( G_n^1(X) - G_n^2(X) \). Then we have
\[
\alpha_{1,k}(Q) = \xi'_{n+k-1}(Q)(|1|_{\tau',0}(Q) + \cdots + (-1)^{n-1}|\hat{n}|_{\tau',m_d-1}(Q))
\]
Hence, if there existed a linear relation \( \sum_{k=1}^l a_k\alpha_{1,k} = 0 \), we would have
\[
\left( \sum_{k=1}^l a_k\xi'_{n+k-1}(Q) \right)(|1|_{\tau',0}(Q) + \cdots + (-1)^{n-1}|\hat{n}|_{\tau',m_d-1}(Q)) = 0.
\]

The same reasoning as in the case of simple points applies, since \( d\tau_{P_1,0}, \ldots, d\tau_{P_d,m_d-1} \) are easily seen to be linearly independent (they either have poles at different points or have poles of differing orders at the same point).

**Remark.** The above reasoning shows that if \( D \in G^r_n - G^{r+1}_n \), then the \( \alpha_{j,k} \) for a fixed \( j \) are linearly independent.

**Theorem 3.** \( G_n^1 - G_n^2 \), if nonempty, is smooth of pure dimension \( 3g - 3 + \tau + 1 \).

**Proof.** Let \( (s_0, D) \) be any point of \( G_n^1 - G_n^2 \). By Proposition 12 and the work of this section, we may conclude that the dimension of the tangent space to \( G_n^1 \) at \( (s_0, D) \) is \( 3g - 3 + \tau + 1 \). By Proposition 4, the dimension of \( G_n^1 \) at \( (s_0, D) \) is at least \( 3g - 3 + \tau + 1 \), hence \( G_n^1 \) is smooth at \( (s_0, D) \) and has dimension precisely \( 3g - 3 + \tau + 1 \).

**Remark.** Theorem 3 does not depend upon \( \tau \) being nonnegative.

**Theorem 4.** Suppose that \( G_n^1(X) - G_n^2(X) \) is nonempty for a generic curve \( X \). Then \( G_n^1(X) - G_n^2(X) \), for a generic \( X \), is smooth of pure dimension \( \tau + 1 \).

**Proof.** Under our assumption, the image of \( G_n^1 - G_n^2 \) in \( T_g \) would be a dense open subspace \( U \). By Sard's Theorem, since \( G_n^1 - G_n^2 \) is smooth, the generic fiber of the map \( G_n^1 - G_n^2 \to U \) is smooth. And since \( U \) has dimension \( 3g - 3 \) and \( G_n^1 - G_n^2 \) has dimension \( 3g - 3 + \tau + 1 \), the generic fiber has dimension \( \tau + 1 \). Thus, for a generic curve, \( G_n^1(X) - G_n^2(X) \) is smooth of dimension \( \tau + 1 \).

**Remark.** If \( \tau \geq 0 \), then by [10] we know that \( G_n^r(X) \) is nonempty. If we knew that \( G_n^r(X) \) were reduced for a generic \( X \), then, since the points of \( G_n^{r+1} \) are singular points of \( G_n^r \), we could conclude that \( G_n^r(X) - G_n^{r+1}(X) \) is nonempty for generic \( X \) if \( \tau \geq 0 \).

6. Moduli of trigonal curves. A trigonal curve is a curve \( X \) such that \( G_3^1(X) \) is nonempty. We can use Theorem 3 to compute the moduli of trigonal curves. By Clifford's Theorem, \( G_3^2 \) is empty hence, by Theorem 3, \( G_3^1 \) if
nonempty, is smooth of pure dimension $3g - 3 + \tau + 1$. Now $\tau = 2(3 - 1) - g = 4 - g$, so $G_3^1$, if nonempty, has dimension $2g + 2$.

By Theorem 1 of [12], we have that, for $g \geq 4$, if $G_3^1(X)$ is nonempty, then every component has dimension at least $5 - g$ and at most 2, with the upper bound occurring if and only if $X$ is hyperelliptic. So, if there exists a nonhyperelliptic trigonal curve of genus $g$, then we must have that the dimension of the generic fiber of the map $G_3^1 \to T_g$ is 1. Examples of such curves (for every $g \geq 3$) are given in [1a, p. 196]. Hence, the dimension of the subvariety of $T_g$, for $g \geq 4$, of trigonal curves is $2g + 2 - 1 = 2g + 1$. This agrees with the number which appears in Segre [20] and Severi [22].

BIBLIOGRAPHY


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