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Optimal lower bounds on the local stress inside random thermoelastic composites

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Abstract A methodology is presented for bounding all higher moments of the local hydrostatic stress field inside random two phase linear thermoelastic media undergoing macroscopic thermomechanical loading. The method also provides a lower bound on the maximum local stress. Explicit formulae for the optimal lower bounds are found that are expressed in terms of the applied macroscopic thermal and mechanical loading, coefficients of thermal expansion, elastic properties, and volume fractions. These bounds provide a means to measure load transfer across length scales relating the excursions of the local fields to the applied loads and the thermal stresses inside each phase. These bounds are shown to be the best possible in that they are attained by the Hashin–Shtrikman coated sphere assemblage.

1 Introduction

Over the last century major strides have been made in the characterization of effective constitutive laws relating average fluxes to average gradients inside random heterogeneous media, see for example [6,10,18,19,21,22,25]. Unfortunately, much less is known about the point wise behavior of local fluxes and gradient fields inside random media. For most applications only statistical descriptions of the microstructure are available. Thus one is compelled to develop bounds and approximations for the local fields that are based upon the available statistical descriptors of the microgeometry. Bounds are useful as they provide a means to quantitatively assess load transfer across length scales relating the excursions of the local fields to applied macroscopic loads. Moreover, they provide explicit criteria on the applied loads that are necessary for failure initiation inside statistically defined heterogeneous media [1]. In this paper, we develop lower bounds on local field properties for statistically defined two phase microstructures when only the volume fraction of each phase is known. Here, the focus is on lower bounds since volume constraints alone do not preclude the existence of microstructures with rough interfaces for which the L^p norms of local fields are divergent, see [3,12,20].

We present a methodology for bounding the L^p norms, $2 \leq p \leq \infty$, of the local hydrostatic stress field inside random media made up of two thermoelastic materials. The method is used to obtain new optimal lower bounds that are given by explicit formulae expressed in terms of the applied thermal and mechanical loading, coefficients of thermal expansion, elastic properties, and volume fractions. We show that these bounds are the best possible in that they are attained by the local fields inside the coated sphere assemblage originally

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introduced in [7]. It has been known since 1963 that the coated spheres microstructure exhibits extreme effective elastic properties [8]. However, it was discovered only recently in [15,16] that this geometry supports extreme local fields that minimize the maximum local hydrostatic field over all two phase elastic mixtures in fixed volume fractions. More recently several scenarios are identified for which, in the absence of thermal stresses, these microstructures attain lower bounds on the total local stress field inside each material when the composite is subjected to mechanical loading, see [1].

In this paper, we consider mixtures of two thermoelastic materials with shear and bulk moduli specified by μ_1, k_1, μ_2, k_2 and coefficients of thermal expansion given by h_1 and h_2. New lower bounds are presented for elastically well-ordered phases for which $k_1 > k_2$ and $\mu_1 > \mu_2$ as well as for non well-ordered phases such that $k_2 > k_1$ and $\mu_1 > \mu_2$. For each of these cases we consider both macroscopic mechanical and thermal loads and present bounds that hold for $h_1 > h_2$ and $h_2 > h_1$. The set of bounds and optimal microstructures for the well-ordered case are listed in Sect. 3 and optimal lower bounds for the non well-ordered case are listed in Sect. 4. The methodology for deriving the bounds is presented in Sect. 5.

The optimal bounds and the associated coated sphere microstructures given in Sects. 3 and 4 show that there are combinations of applied stress and imposed temperature change for which the local hydrostatic stress inside the connected phase of the coated sphere assemblage vanishes identically. Other loading combinations are seen to cause the stress inside the included phase of the coated sphere assemblage to vanish identically. Thus for these cases the applied hydrostatic stress is converted into a pure local shear stress inside a preselected phase.

Recent related work provides optimal lower bounds on local fields in the absence of thermal loads. The work presented in [1] provides new optimal lower bounds on both the local shear stress and the local hydrostatic component of stress for random media subjected to a series of progressively more general applied macroscopic stresses. These bounds are explicit and given in terms of volume fractions, elastic constants of each phase, and the applied macroscopic stress. Earlier work considers random two phase elastic composites subject to imposed macroscopic hydrostatic stress and strain, see [15,16], as well as dielectric composites subjected to applied constant electric fields, see [14]. Those efforts deliver optimal lower bounds on the L^p norms for the hydrostatic components of local stress and strain fields as well as the magnitude of the local electric field for all p in the range $2 \leq p \leq \infty$. Other work examines the stress field around a single simply connected stiff elastic inclusion subjected to a remote constant stress at infinity [24] and provides optimal lower bounds for the supremum of the maximum principal stress. The work presented in [5] provides an optimal lower bound on the supremum of the maximum principal stress for two-dimensional periodic composites consisting of a single simply connected elastically stiff inclusion inside the period cell. The recent work of [9] builds on the earlier work of [15,16] and develops new lower bounds on the L^p norm of the local stress and strain fields inside statistically isotropic two-phase elastic composites. However, to date those bounds have been shown to be optimal for $p = 2$, see [9]. Their optimality for $p > 2$ remains to be seen. Optimal upper and lower bounds on the L^2 norm of local gradient fields are established using integral representation formulae in [17].

We conclude by providing the notation and summation conventions used in this article. Contractions of stress or strain fields σ and ϵ are defined by $\sigma : \epsilon = \sigma_{ij}\epsilon_{ij}$ and $|\sigma|^2 = \sigma : \sigma$, where repeated indices indicate summation. Products of fourth order tensors C and strain tensors ϵ are written as $C\epsilon$ and are given by $[C\epsilon]_{ij} = C_{ijkl}\epsilon_{kl}$; and products of stresses σ with vectors $\mathbf{v}$ are given by $[\sigma\mathbf{v}]_i = \sigma_{ij}v_j$. The fourth order identity map on the space of stresses or strains is denoted by $\mathbf{I}$ and $\mathbf{I}_{ijkl} = 1/2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. The projection onto the hydrostatic part of $\sigma(\mathbf{x})$ is denoted by $\mathbb{P}^H$ and is given explicitly by

$$\mathbb{P}^H_{ijkl} = \frac{1}{d}\delta_{ij}\delta_{kl} \quad \text{and} \quad \mathbb{P}^H\sigma(\mathbf{x}) = \frac{tr\,\sigma(\mathbf{x})}{d}I. \tag{1.1}$$

The projection onto the deviatoric part of $\sigma(\mathbf{x})$ is denoted by $\mathbb{P}^D$ and $\mathbf{I} = \mathbb{P}^H + \mathbb{P}^D$ with $\mathbb{P}^D\mathbb{P}^H = \mathbb{P}^H\mathbb{P}^D = 0$. The tensor product between two vectors $\mathbf{u}$ and $\mathbf{v}$ is the matrix $\mathbf{u} \otimes \mathbf{v}$ with elements$[\mathbf{u} \otimes \mathbf{v}]_{ij} = u_i v_j$. Last we denote the basis for the space of constant 3×3 symmetric strain tensors by $\bar{\epsilon}^{kl}$ where $\bar{\epsilon}^{kl}_{mn} = \frac{1}{2}\{\delta_{mk}\delta_{nl} + \delta_{ml}\delta_{nk}\}$.

2 Stress and strain fields inside heterogeneous thermoelastic media with imposed macroscopic loading

Several distinct physical processes can generate prestress within heterogeneous media. In many cases it is generated by a mismatch between the coefficients of thermal expansion of the component materials. To fix ideas we present the physical model associated with this situation. The tensors of thermal expansion inside each phase are given by $\lambda_1 = h_1 I$ and $\lambda_2 = h_2 I$ where I is the 3×3 identity. The elastic properties of each component are specified by the elasticity tensors C_1 and C_2, respectively. In this treatment, we consider heterogeneous elastically isotropic materials, and the elasticity tensors of materials one and two are specified by

$$C_i = 3k_i \mathbb{P}^H + 2\mu_i \mathbb{P}^D, \quad i = 1, 2. \tag{2.1}$$

Without loss of generality we adopt the convention

$$\mu_1 > \mu_2. \tag{2.2}$$

The elastic displacement inside the composite is denoted by $\mathbf{u}$, and the associated strain tensor is denoted by $\epsilon(\mathbf{u})$. The position dependent elastic tensor and thermal expansion tensor for the heterogeneous medium are denoted by $C(\mathbf{x})$ and $\lambda(\mathbf{x})$, respectively, where $\mathbf{x}$ denotes a point inside the medium. The domain containing the composite is given by a cube Q of unit side length. Here it is supposed that Q is the period cell for an infinite elastic medium. In what follows the integral of a quantity q over the unit cube Q is denoted by $\langle q \rangle$.

A constant macroscopic stress $\overline{\sigma}$ and uniform change in temperature ΔT is imposed upon the heterogeneous material. The local stress inside the heterogeneous medium is expressed as the sum of a periodic mean zero fluctuation $\hat{\sigma}$ and $\overline{\sigma}$, i.e., $\sigma(\mathbf{x}) = \overline{\sigma} + \hat{\sigma}(\mathbf{x})$ with $\langle \hat{\sigma} \rangle = 0$. Elastic equilibrium inside each phase is given by:

$$\mathrm{div}\sigma = 0. \tag{2.3}$$

The local elastic strain $\epsilon(\mathbf{u})$ is related to the local stress through the constitutive law

$$\sigma(\mathbf{x}) = C(\mathbf{x})(\epsilon(\mathbf{u}(\mathbf{x})) - \lambda(\mathbf{x})\Delta T), \tag{2.4}$$

and the local elastic field is written in the form

$$\epsilon(\mathbf{u}) = \overline{\epsilon} + \epsilon(\mathbf{u}^{\mathrm{per}}), \tag{2.5}$$

where $\mathbf{u}^{\mathrm{per}}$ is Q periodic taking the same values on opposite sides of the period cell and $\langle \epsilon(\mathbf{u}^{\mathrm{per}}) \rangle = 0$. Perfect contact between the component materials is assumed, thus both the displacement $\mathbf{u}$ and traction $\sigma\mathbf{n}$ are continuous across the two phase interface, i.e.,

$$\mathbf{u}_{|_1} = \mathbf{u}_{|_2}, \tag{2.6}$$
$$\sigma_{|_1}\mathbf{n} = \sigma_{|_2}\mathbf{n}. \tag{2.7}$$

Here, the subscripts indicate the side of the interface that the displacement and traction fields are evaluated on and $\mathbf{n}$ denotes the normal vector to the interface pointing from material one into material two.

The effective "macroscopic" constitutive law for the heterogeneous medium is given by the constant effective elasticity tensor C^e and effective thermal stress tensor H^e that provide the linear relation between the imposed macroscopic stress $\bar{\sigma}$, uniform change in temperature ΔT, and the average strain $\bar{\epsilon}$ given by Milton [19],

$$\bar{\sigma} = C^e \bar{\epsilon} + H^e \Delta T. \tag{2.8}$$

Here, the components of C^e are given by

$$C^e_{ijkl} = \langle C_{ijmn}(x)(\epsilon(\varphi^{kl})_{mn} + \bar{\epsilon}^{kl}_{mn}) \rangle, \tag{2.9}$$

where the fields φ^{ij} are the periodic solutions of

$$\mathrm{div}(C(x)(\epsilon(\varphi^{kl}) + \bar{\epsilon}^{kl})) = 0 \quad \text{inside each phase}, \tag{2.10}$$

with the appropriate traction and continuity conditions along the two phase interface given by

$$\varphi^{kl}{}_{|_1} = \varphi^{kl}{}_{|_2}, \tag{2.11}$$
$$C_1 \left(\epsilon(\varphi^{kl}) + \bar{\epsilon}^{kl} \right)_{|_1} \mathbf{n} = C_2 \left(\epsilon(\varphi^{kl}) + \bar{\epsilon}^{kl} \right)_{|_2} \mathbf{n}. \tag{2.12}$$

The effective thermal stress tensor H^e is given by

$$H^e = \langle C(x)(\epsilon(\varphi^p) - \lambda(\mathbf{x}))\rangle, \tag{2.13}$$

where φ^p is the periodic solution of

$$\operatorname{div}(C(x)(\epsilon(\varphi^p) - \lambda(\mathbf{x}))) = 0 \quad \text{inside each phase}, \tag{2.14}$$

with the traction and continuity conditions along the two phase interface given by

$$\varphi^p{}_{|1} = \varphi^p{}_{|2}, \tag{2.15}$$

$$C_1 \left(\epsilon(\varphi^p) - \lambda_1\right)_{|1} \mathbf{n} = C_2 \left(\epsilon(\varphi^p) - \lambda_2\right)_{|2} \mathbf{n}. \tag{2.16}$$

From linearity it follows that the local fluctuating strain field is given by the sum

$$\epsilon(\mathbf{u}^{\text{per}}) = \epsilon(\varphi^{kl})\bar{\epsilon}^{kl} + \epsilon(\varphi^p)\Delta T. \tag{2.17}$$

We write $\varphi^e = \varphi^{kl}\bar{\epsilon}^{kl}$ and from linearity one has the expression for $C^e\bar{\epsilon}$ given by

$$C^e\bar{\epsilon} = \langle C(\mathbf{x})(\epsilon(\varphi^e) + \bar{\epsilon}\rangle. \tag{2.18}$$

In this article, the imposed mechanical stress is given by a constant hydrostatic stress

$$\bar{\sigma} = \sigma_0 I, \tag{2.19}$$

where σ_0 can assume any value in $-\infty < \sigma_0 < \infty$. We introduce a method for obtaining optimal lower bounds on the higher moments of the hydrostatic component of the local stress inside the composite when it is subjected to an imposed hydrostatic load $\sigma_0 I$ and temperature change ΔT. Here no restriction is placed on ΔT. The volume fractions of materials one and two are denoted by θ_1 and θ_2 and the average of a quantity q over material one is denoted by $\langle q\rangle_1$ and over material two by $\langle q\rangle_2$. In the following Section, we present optimal lower bounds on the following moments of the local hydrostatic stress $\mathbb{P}^H\sigma(\mathbf{x})$ over the domain occupied by each material given by

$$\langle|\mathbb{P}^H\sigma(\mathbf{x})|^p\rangle_1^{1/p} \quad \text{and} \quad \langle|\mathbb{P}^H\sigma(\mathbf{x})|^p\rangle_2^{1/p}, \tag{2.20}$$

for $1 < p \leq \infty$, as well as for the maximum local hydrostatic stress over the whole composite domain

$$\max_{\mathbf{x} \text{in Q}} \left\{|\mathbb{P}^H\sigma(\mathbf{x})|\right\}. \tag{2.21}$$

It is pointed out that the case corresponding to $p = \infty$ in (2.20) corresponds to lower bounds on the maximum local stress over each phase

$$\max_{\mathbf{x} \text{ in material 1}} \left\{|\mathbb{P}^H\sigma(\mathbf{x})|\right\}, \quad \max_{\mathbf{x} \text{ in material 2}} \left\{|\mathbb{P}^H\sigma(\mathbf{x})|\right\}. \tag{2.22}$$

The lower bounds are given in terms of the volume fractions θ_1 and θ_2, as well as the bulk and shear moduli k_1, k_2, μ_1, μ_2, and the coefficients of thermal expansion h_1 and h_2. The lower bounds are described by the following characteristic combinations of these parameters given by:

$$L_1 = \frac{k_1(k_2 + \frac{4}{3}\mu_2)}{k_1k_2 + (k_1\theta_1 + k_2\theta_2)\frac{4}{3}\mu_2}, \tag{2.23}$$

$$L_2 = \frac{k_2(k_1 + \frac{4}{3}\mu_1)}{k_1k_2 + (k_1\theta_1 + k_2\theta_2)\frac{4}{3}\mu_1}, \tag{2.24}$$

$$M_1 = \frac{k_1(k_2 + \frac{4}{3}\mu_1)}{k_1k_2 + (k_1\theta_1 + k_2\theta_2)\frac{4}{3}\mu_1}, \tag{2.25}$$

$$M_2 = \frac{k_2(k_1 + \frac{4}{3}\mu_2)}{k_1k_2 + (k_1\theta_1 + k_2\theta_2)\frac{4}{3}\mu_2}, \tag{2.26}$$

$$D = \Delta T\left(\frac{3k_1k_2(h_2 - h_1)}{k_2 - k_1}\right), \tag{2.27}$$

and

$$F = D\left(1 - \frac{1}{\frac{L_1+M_2}{2}}\right). \tag{2.28}$$

For elastically well-ordered materials, $k_1 > k_2$, one has $L_1 > 1 > L_2$, and $M_1 > 1 > M_2$; for the non well-ordered case $k_2 > k_1$, one has $L_2 > 1 > L_1$ and $M_2 > 1 > M_1$.

The lower bounds are shown to be obtained by the local fields inside the coated sphere assemblages introduced in [7]. The lower bounds presented here include the effects of thermal stresses due to thermal loads and reduce to the optimal bounds reported in [16] when $\Delta T = 0$.

3 Lower bounds on local stress for elastically well-ordered thermoelastic composite media

In this Section, it is assumed that the materials inside the heterogeneous medium are elastically well-ordered, i.e., $\mu_1 > \mu_2$ and $\kappa_1 > \kappa_2$. We present lower bounds that are optimal for the full range of imposed hydrostatic stresses, i.e., $-\infty < \sigma_0 < \infty$ as well as for unrestricted choices of ΔT. The configurations that attain the bounds are given by the coated sphere assemblages [7]. To fix ideas we describe the coated sphere assemblage made from a core of material one with a coating of material two. We first fill the cube Q with an assemblage of spheres with sizes ranging down to the infinitesimal. Inside each sphere one places a smaller concentric sphere filled with "core" material one, and the surrounding coating is filled with material two. The volume fractions of material one and two are taken to be the same for all of the coated spheres.

In what follows we list the lower bounds for the well ordered case. These bounds are derived in Sect. 5. Their optimality follows from explicit formulae for the moments of the local fields inside the coated sphere assemblage, these are discussed and presented in Sect. 5. The first set of bounds applies to all moments $\langle|\mathbb{P}^H\sigma|^p\rangle_2^{1/p}$ for $1 < p \le \infty$. We suppose that $h_2 > h_1$ fix ΔT and list the bounds as a function of the imposed macroscopic stress σ_0. The bounds are displayed in the following table where the optimal microstructures are given by the coated spheres construction. The coating and core phase of the optimal configuration is listed in the following Table.

Range	Lower bound	Optimal microstructure
$-\infty < \sigma_0 \le D$	$\langle\vert\mathbb{P}^H\sigma\vert^p\rangle_2^{1/p} \ge \sqrt{3}[(D-\sigma_0)L_2 - D]$	Core material 2 and coating material 1
$D \le \sigma_0 \le D(1-\frac{1}{M_2})$	$\langle\vert\mathbb{P}^H\sigma\vert^p\rangle_2^{1/p} \ge \sqrt{3}[(D-\sigma_0)M_2 - D]$	Core material 1 and coating material 2
$D(1-\frac{1}{M_2}) < \sigma_0 < D(1-\frac{1}{L_2})$	$\langle\vert\mathbb{P}^H\sigma\vert^p\rangle_2^{1/p} \ge 0$	Optimality undetermined
$D(1-\frac{1}{L_2}) \le \sigma_0 < \infty$	$\langle\vert\mathbb{P}^H\sigma\vert^p\rangle_2^{1/p} \ge \sqrt{3}[(\sigma_0 - D)L_2 + D]$	Core material 2 and coating material 1

Next, we suppose that $h_1 > h_2$ and present optimal lower bounds on $\langle|\mathbb{P}^H\sigma|^p\rangle_2^{1/p}$, for $1 < p \le \infty$. The bounds and associated optimal microstructures are given in the following Table.

Range	Lower bound	Optimal microstructure
$-\infty < \sigma_0 \le D(1-\frac{1}{L_2})$	$\langle\vert\mathbb{P}^H\sigma\vert^p\rangle_2^{1/p} \ge \sqrt{3}[(D-\sigma_0)L_2 - D]$	Core material 2 and coating material 1
$D(1-\frac{1}{L_2}) < \sigma_0 < D(1-\frac{1}{M_2})$	$\langle\vert\mathbb{P}^H\sigma\vert^p\rangle_2^{1/p} \ge 0$	Optimality undetermined
$D(1-\frac{1}{M_2}) \le \sigma_0 \le D$	$\langle\vert\mathbb{P}^H\sigma\vert^p\rangle_2^{1/p} \ge \sqrt{3}[(\sigma_0 - D)M_2 + D]$	Core material 1 and coating material 2
$D \le \sigma_0 < \infty$	$\langle\vert\mathbb{P}^H\sigma\vert^p\rangle_2^{1/p} \ge \sqrt{3}[(\sigma_0 - D)L_2 + D]$	Core material 2 and coating material 1

Lower bounds and the associated optimal microstructures for all moments $\langle|\mathbb{P}^H\sigma|^p\rangle_1^{1/p}$ for $1 < p \le \infty$ for the case $h_2 > h_1$ are given in the following Table.

Range	Lower bound	Optimal microstructure
$-\infty < \sigma_0 \le D$	$\langle\vert\mathbb{P}^H\sigma\vert^p\rangle_1^{1/p} \ge \sqrt{3}[(D-\sigma_0)L_1 - D]$	Core material 1 and coating material 2
$D \le \sigma_0 \le D(1-\frac{1}{M_1})$	$\langle\vert\mathbb{P}^H\sigma\vert^p\rangle_1^{1/p} \ge \sqrt{3}[(D-\sigma_0)M_1 - D]$	Core material 2 and coating material 1
$D(1-\frac{1}{M_1}) < \sigma_0 < D(1-\frac{1}{L_1})$	$\langle\vert\mathbb{P}^H\sigma\vert^p\rangle_1^{1/p} \ge 0$	Optimality undetermined
$D(1-\frac{1}{L_1}) \le \sigma_0 < \infty$	$\langle\vert\mathbb{P}^H\sigma\vert^p\rangle_1^{1/p} \ge \sqrt{3}[(\sigma_0 - D)L_1 + D]$	Core material 1 and coating material 2

Lower bounds and the associated optimal microstructures for all moments $\langle|\mathbb{P}^H\sigma|^p\rangle_1^{1/p}$ for $1 < p \le \infty$ for the case $h_1 > h_2$ are given in the following Table.

Range	Lower bound	Optimal microstructure
$-\infty < \sigma_0 \le D(1-\frac{1}{L_1})$	$\langle|\mathbb{P}^H\sigma|^p\rangle_1^{1/p} \ge \sqrt{3}[(D-\sigma_0)L_1 - D]$	Core material 1 and coating material 2
$D(1-\frac{1}{L_1}) < \sigma_0 < D(1-\frac{1}{M_1})$	$\langle|\mathbb{P}^H\sigma|^p\rangle_1^{1/p} \ge 0$	Optimality undetermined
$D(1-\frac{1}{M_1}) \le \sigma_0 \le D$	$\langle|\mathbb{P}^H\sigma|^p\rangle_1^{1/p} \ge \sqrt{3}[(\sigma_0 - D)M_1 + D]$	Core material 2 and coating material 1
$D \le \sigma_0 < \infty$	$\langle|\mathbb{P}^H\sigma|^p\rangle_1^{1/p} \ge \sqrt{3}[(\sigma_0 - D)L_1 + D]$	Core material 1 and coating material 2

Next, we display lower bounds on $\max_{\mathbf{x}\text{in Q}}\{|\mathbb{P}^H\sigma(\mathbf{x})|\}$. We start with the case $h_2 > h_1$, and the lower bounds and optimal geometries are given in the following Table. The phase in which the maximum is attained is denoted with an asterisk.

Range	Lower bound	Optimal microstructure
$-\infty < \sigma_0 \le D$	$\max_{\mathbf{x}\text{in Q}}\{|\mathbb{P}^H\sigma(\mathbf{x})|\} \ge \sqrt{3}[(D-\sigma_0)L_1 - D]$	Core material 1* and coating material 2
$D \le \sigma_0 \le F$	$\max_{\mathbf{x}\text{in Q}}\{|\mathbb{P}^H\sigma(\mathbf{x})|\} \ge \sqrt{3}[(D-\sigma_0)M_2 - D]$	Core material 1 and coating material 2*
$F \le \sigma_0 < \infty$	$\max_{\mathbf{x}\text{in Q}}\{|\mathbb{P}^H\sigma(\mathbf{x})|\} \ge \sqrt{3}[(\sigma_0 - D)L_1 + D]$	Core material 1* and coating material 2

Lower bounds and optimal microgeometries for the case $h_1 > h_2$ are given in the following Table. The phase in which the maximum is attained is denoted with an asterisk.

Range	Lower bound	Optimal microstructure
$-\infty < \sigma_0 \le F$	$\max_{\mathbf{x}\text{in Q}}\{|\mathbb{P}^H\sigma(\mathbf{x})|\} \ge \sqrt{3}[(D-\sigma_0)L_1 - D]$	Core material 1* and coating material 2
$F \le \sigma_0 \le D$	$\max_{\mathbf{x}\text{in Q}}\{|\mathbb{P}^H\sigma(\mathbf{x})|\} \ge \sqrt{3}[(\sigma_0 - D)M_2 + D]$	Core material 1 and coating material 2*
$D \le \sigma_0 < \infty$	$\max_{\mathbf{x}\text{in Q}}\{|\mathbb{P}^H\sigma(\mathbf{x})|\} \ge \sqrt{3}[(\sigma_0 - D)L_1 + D]$	Core material 1* and coating material 2

4 Lower bounds on local stress for non well-ordered thermoelastic composite media

In this Section, it is assumed that the materials inside the heterogeneous medium are elastically non well-ordered, i.e., $\mu_1 > \mu_2$ and $\kappa_2 > \kappa_1$. We fix ΔT and present lower bounds that are optimal for the full range of imposed hydrostatic stresses, i.e., $-\infty < \sigma_0 < \infty$. The configurations that attain the bounds for the non well-ordered case are also given by the coated sphere assemblages [7].

In what follows we list the lower bounds for the non well-ordered case. These bounds are derived in Sect. 5. Their optimality follows from explicit formulae for the moments of the local fields inside the coated sphere assemblage, these are discussed and presented in Sect. 5. The first set of bounds applies to all moments $\langle|\mathbb{P}^H\sigma|^p\rangle_2^{1/p}$ for $1 < p \le \infty$. We suppose that $h_2 > h_1$ and list the bounds as functions of the imposed macroscopic stress σ_0. The bounds are displayed in the following Table where the optimal microstructures are given by the coated spheres construction. The coating and core phase of the optimal coated sphere configuration is listed in the following Table.

Range	Lower bound	Optimal microstructure
$-\infty < \sigma_0 \le D(1-\frac{1}{M_2})$	$\langle|\mathbb{P}^H\sigma|^p\rangle_2^{1/p} \ge \sqrt{3}[(D-\sigma_0)M_2 - D]$	Core material 1 and coating material 2
$D(1-\frac{1}{M_2}) \le \sigma_0 \le D(1-\frac{1}{L_2})$	$\langle|\mathbb{P}^H\sigma|^p\rangle_2^{1/p} \ge 0$	Optimality undetermined
$D(1-\frac{1}{L_2}) < \sigma_0 < D$	$\langle|\mathbb{P}^H\sigma|^p\rangle_2^{1/p} \ge \sqrt{3}[(\sigma_0 - D)L_2 + D]$	Core material 2 and coating material 1
$D \le \sigma_0 < \infty$	$\langle|\mathbb{P}^H\sigma|^p\rangle_2^{1/p} \ge \sqrt{3}[(\sigma_0 - D)M_2 + D]$	Core material 1 and coating material 2

Next, we suppose that $h_1 > h_2$ and present optimal lower bounds on $\langle|\mathbb{P}^H\sigma|^p\rangle_2^{1/p}$, for $1 < p \le \infty$. The bounds and associated optimal microstructures are given in the following Table.

Range	Lower bound	Optimal microstructure
$-\infty < \sigma_0 \leq D$	$\langle\lvert\mathbb{P}^H\sigma\rvert^p\rangle_2^{1/p} \geq \sqrt{3}[(D-\sigma_0)M_2 - D]$	Core material 1 and coating material 2
$D < \sigma_0 < D(1-\frac{1}{L_2})$	$\langle\lvert\mathbb{P}^H\sigma\rvert^p\rangle_2^{1/p} \geq \sqrt{3}[(D-\sigma_0)L_2 - D]$	Core material 2 and coating material 1
$D(1-\frac{1}{L_2}) \leq \sigma_0 \leq D(1-\frac{1}{M_2})$	$\langle\lvert\mathbb{P}^H\sigma\rvert^p\rangle_2^{1/p} \geq 0$	Optimality undetermined
$D(1-\frac{1}{M_2}) \leq \sigma_0 < \infty$	$\langle\lvert\mathbb{P}^H\sigma\rvert^p\rangle_2^{1/p} \geq \sqrt{3}[(\sigma_0-D)M_2 + D]$	Core material 1 and coating material 2

Lower bounds and the associated optimal microstructures for all moments $\langle\lvert\mathbb{P}^H\sigma\rvert^p\rangle_1^{1/p}$ for $1 < p \leq \infty$ for the case $h_2 > h_1$ are given in the following Table.

Range	Lower bound	Optimal microstructure
$-\infty < \sigma_0 \leq D(1-\frac{1}{M_1})$	$\langle\lvert\mathbb{P}^H\sigma\rvert^p\rangle_1^{1/p} \geq \sqrt{3}[(D-\sigma_0)M_1 - D]$	Core material 2 and coating material 1
$D(1-\frac{1}{M_1}) \leq \sigma_0 \leq D(1-\frac{1}{L_1})$	$\langle\lvert\mathbb{P}^H\sigma\rvert^p\rangle_1^{1/p} \geq 0$	Optimality undetermined
$D(1-\frac{1}{L_1}) < \sigma_0 < D$	$\langle\lvert\mathbb{P}^H\sigma\rvert^p\rangle_1^{1/p} \geq \sqrt{3}[(\sigma_0-D)L_1 + D]$	Core material 1 and coating material 2
$D \leq \sigma_0 < \infty$	$\langle\lvert\mathbb{P}^H\sigma\rvert^p\rangle_1^{1/p} \geq \sqrt{3}[(\sigma_0-D)M_1 + D]$	Core material 2 and coating material 1

Lower bounds and the associated optimal microstructures for all moments $\langle\lvert\mathbb{P}^H\sigma\rvert^p\rangle_1^{1/p}$ for $1 < p \leq \infty$ for the case $h_1 > h_2$ are given in the following Table.

Range	Lower bound	Optimal microstructure
$-\infty < \sigma_0 \leq D$	$\langle\lvert\mathbb{P}^H\sigma\rvert^p\rangle_1^{1/p} \geq \sqrt{3}[(D-\sigma_0)M_1 - D]$	Core material 2 and coating material 1
$D < \sigma_0 < D(1-\frac{1}{L_1})$	$\langle\lvert\mathbb{P}^H\sigma\rvert^p\rangle_1^{1/p} \geq \sqrt{3}[(D-\sigma_0)L_1 - D]$	Core material 1 and coating material 2
$D(1-\frac{1}{L_1}) \leq \sigma_0 \leq D(1-\frac{1}{M_1})$	$\langle\lvert\mathbb{P}^H\sigma\rvert^p\rangle_1^{1/p} \geq 0$	Optimality undetermined
$D(1-\frac{1}{M_1}) \leq \sigma_0 < \infty$	$\langle\lvert\mathbb{P}^H\sigma\rvert^p\rangle_1^{1/p} \geq \sqrt{3}[(\sigma_0-D)M_1 + D]$	Core material 2 and coating material 1

Next, we display lower bounds on $\max_{\mathbf{x}\text{in Q}}\left\{\lvert\mathbb{P}^H\sigma(\mathbf{x})\rvert\right\}$. We start with the case $h_2 > h_1$, and the lower bounds and optimal geometries are given in the following Table. The phase in which the maximum is attained is denoted by an asterisk.

Range	Lower bound	Optimal microstructure
$-\infty < \sigma_0 \leq F$	$\max_{\mathbf{x}\text{in Q}}\left\{\lvert\mathbb{P}^H\sigma(\mathbf{x})\rvert\right\} \geq \sqrt{3}[(D-\sigma_0)M_2 - D]$	Core material 1 and coating material 2*
$F \leq \sigma_0 \leq D$	$\max_{\mathbf{x}\text{in Q}}\left\{\lvert\mathbb{P}^H\sigma(\mathbf{x})\rvert\right\} \geq \sqrt{3}[(\sigma_0-D)L_1 + D]$	Core material 1* and coating material 2
$D \leq \sigma_0 < \infty$	$\max_{\mathbf{x}\text{in Q}}\left\{\lvert\mathbb{P}^H\sigma(\mathbf{x})\rvert\right\} \geq \sqrt{3}[(\sigma_0-D)M_2 + D]$	Core material 1 and coating material 2*

Lower bounds and optimal microgeometries for the case $h_1 > h_2$ are given in the following Table. The phase in which the maximum is attained is denoted by an asterisk.

Range	Lower bound	Optimal microstructure
$-\infty < \sigma_0 \leq D$	$\max_{\mathbf{x}\text{in Q}}\left\{\lvert\mathbb{P}^H\sigma(\mathbf{x})\rvert\right\} \geq \sqrt{3}[(D-\sigma_0)M_2 - D]$	Core material 1 and coating material 2*
$D \leq \sigma_0 \leq F$	$\max_{\mathbf{x}\text{in Q}}\left\{\lvert\mathbb{P}^H\sigma(\mathbf{x})\rvert\right\} \geq \sqrt{3}[(D-\sigma_0)L_1 - D]$	Core material 1* and coating material 2
$F \leq \sigma_0 < \infty$	$\max_{\mathbf{x}\text{in Q}}\left\{\lvert\mathbb{P}^H\sigma(\mathbf{x})\rvert\right\} \geq \sqrt{3}[(\sigma_0-D)M_2 + D]$	Core material 1 and coating material 2*

5 Derivation of the lower bounds on $\langle\lvert\mathbb{P}_H\sigma\rvert^p\rangle i^{\frac{1}{p}}$

In this Section, we outline the methodology for proving optimal lower bounds. The bounds are derived using duality relations. We use the following duality relation posed over the space of square integrable symmetric matrix fields η that holds for $p > 1$ given by

$$\frac{1}{p}\langle\lvert\mathbb{P}^H\sigma\rvert^p\rangle_i = \sup_{\eta}\left\{\langle\mathbb{P}^H\sigma : \eta\rangle_i - \frac{1}{p'}\langle\lvert\eta\rvert^{p'}\rangle_i\right\}, \quad \text{for } i = 1, 2, \tag{5.1}$$

where p' is the conjugate exponent to p given by $p' = \frac{p}{p-1}$. This relation follows immediately from standard duality relations, see [2]. Restricting η to the set of all constant matrices and taking the supremum delivers the basic bounds:

$$\langle |\mathbb{P}^H \sigma|^p \rangle_i \geq |\langle \mathbb{P}^H \sigma \rangle_i|^p, \quad p > 1 \text{ and for } i = 1, 2. \tag{5.2}$$

We point out that equality holds in (5.2) if and only if $\mathbb{P}^H \sigma$ is identically constant inside the ith material. In what follows we outline the method for obtaining bounds on the moments in material two noting that the identical procedure delivers bounds on the moments in material one. We introduce the indicator function of material one χ_1 taking the value 1 in material one and zero outside. The indicator function corresponding to material two is denoted by χ_2 and $\chi_2 = 1 - \chi_1$. To proceed we rewrite the right hand side of (5.2) in terms of the effective elastic properties and thermal expansion coefficient. To do this we use the following identity given by

$$tr\langle \chi_2 \sigma \rangle = \frac{3k_2}{k_2 - k_1}(\sigma_0 - k_1 \sigma_0 (C^e)^{-1} I : I + k_1 \Delta T (C^e)^{-1} H^e : I + k_1 \Delta T \langle \lambda \rangle : I). \tag{5.3}$$

This identity is obtained in the following way. Taking averages on both sides of (2.4) gives $\langle \sigma \rangle = \langle C(x)(\epsilon(\mathbf{u}(x)) - \Delta T \lambda(x)) \rangle$. On writing $C(\mathbf{x}) = C_1 + (C_2 - C_1)\chi_2(\mathbf{x})$ we see that

$$\langle \sigma \rangle = C_1(\bar{\epsilon} - \Delta T \langle \lambda \rangle) + (C_2 - C_1) C_2^{-1} \langle \chi_2 \sigma \rangle. \tag{5.4}$$

From (2.8) we see that $\bar{\epsilon} = (C^e)^{-1}(\langle \sigma \rangle - \Delta T H^e)$ and for $\langle \sigma \rangle = \sigma_0 I$ we obtain

$$\langle \chi_2 \sigma \rangle = C_2 (C_2 - C_1)^{-1} (\sigma_0 I - C_1((C^e)^{-1} \sigma_0 I - \Delta T (C^e)^{-1} H^e - \Delta T \langle \lambda \rangle)). \tag{5.5}$$

The identity (5.3) now follows by applying the hydrostatic projection $\mathbb{P}_H$ to both sides of (5.5).

We now derive the lower bound. Applying the basic bound (5.2) to $\langle |\mathbb{P}_H \sigma|^p \rangle_2$ and (5.3) gives

$$\begin{aligned} &\langle |\mathbb{P}_H \sigma|^p \rangle_2 \\ &\geq |\langle \mathbb{P}_H \sigma \rangle_2|^p = \left(\frac{tr(\langle \chi_2 \sigma \rangle)}{\sqrt{3} \theta_2} \right)^p \\ &= 3^{p/2} \theta_2^{-p} |\frac{k_2}{k_2 - k_1}|^p |k_1 \sigma_0 \left(\frac{1}{k_1} - (C^e)^{-1} I : I \right) + k_1 \Delta T (C^e)^{-1} H^e I : I + k_1 \Delta T \langle \lambda \rangle : I|^p. \end{aligned} \tag{5.6}$$

We note that equality holds in (5.6) when $\mathbb{P}_H \sigma$ is constant inside material two.

We now employ an exact relation that relates the contraction $(C^e)^{-1} H^e : I$ involving the effective thermal stress tensor H^e to the quantity $(C^e)^{-1} I : I$. The exact relation used here is given by

$$(C^e)^{-1} H^e : I = \frac{3(h_2 - h_1)(C^e)^{-1} I : I + 3(\frac{h_1}{k_2} - \frac{h_2}{k_1})}{\frac{1}{k_1} - \frac{1}{k_2}}. \tag{5.7}$$

This exact relation is a direct consequence of the exact relation developed by [23], see also [13], for the effective thermal expansion tensor $\alpha^e = -(C^e)^{-1} H^e$.

Substitution of (5.7) into (5.6) and algebraic manipulation gives

$$\langle |\mathbb{P}_H \sigma|^p \rangle_2^{1/p} \geq \sqrt{3}\, |(\sigma_0 - D) X + D|\,, \tag{5.8}$$

where

$$X = \frac{\theta_2^{-1}}{k_1^{-1} - k_2^{-1}} \left(\frac{1}{k_1} - (C^e)^{-1} I : I \right). \tag{5.9}$$

As before we point out that equality holds in (5.8) when $\mathbb{P}_H \sigma$ is identically constant inside material two.

Identical arguments deliver the lower bound on the moments over material one given by

$$\langle |\mathbb{P}_H \sigma|^p \rangle_1^{1/p} \geq \sqrt{3}\, |(\sigma_0 - D) Y + D|\,, \tag{5.10}$$

where

$$Y = \frac{\theta_1^{-1}}{k_2^{-1} - k_1^{-1}} \left(\frac{1}{k_2} - (C^e)^{-1} I : I \right), \tag{5.11}$$

where (5.10) holds with equality when $\mathbb{P}^H \sigma$ is a constant in material one.

The variables X and Y are constrained to lie within intervals set by bounds on the contraction $(C^e)^{-1} I : I$. These bounds follow immediately from the work of Kantor and Bergman [11] and are given by

$$(K_{HS}^+)^{-1} \leq (C^e)^{-1} I : I \leq (K_{HS}^-)^{-1}, \tag{5.12}$$

where K_{HS}^- and K_{HS}^+ are the Hashin and Shtrikman bulk modulus bounds [8] given by

$$K_{HS}^+ = k_1\theta_1 + k_2\theta_2 - \left(\frac{\theta_1\theta_2(k_2 - k_1)^2}{k_1\theta_2 + k_2\theta_1 + \frac{4}{3}\mu_1} \right), \tag{5.13}$$

and

$$K_{HS}^- = k_1\theta_1 + k_2\theta_2 - \left(\frac{\theta_1\theta_2(k_2 - k_1)^2}{k_1\theta_2 + k_2\theta_1 + \frac{4}{3}\mu_2} \right). \tag{5.14}$$

These bounds hold both for elastically well-ordered materials and elastically non well-ordered materials. When the materials are well-ordered (5.12) implies that X and Y lie in the intervals

$$L_2 \leq X \leq M_2, \tag{5.15}$$

$$L_1 \leq Y \leq M_1, \tag{5.16}$$

while for non well-ordered materials

$$M_2 \leq X \leq L_2, \tag{5.17}$$

$$M_1 \leq Y \leq L_1. \tag{5.18}$$

A straightforward calculation in Sect. 6 shows that the hydrostatic component of the local stress is constant inside each phase of the coated sphere construction. Hence (5.8) and (5.10) hold with equality for the coated spheres construction, and we obtain explicit formulae for the moments of the hydrostatic component of the local stresses for these composites. For coated spheres with core phase 1 and coating phase 2, $(C^e)^{-1} I : I = (K_{HS}^-)^{-1}$ and substitution of (5.14) into (5.9) and (5.11) together with (5.8) and (5.10) shows that the moments are given by

$$\langle |\mathbb{P}_H \sigma|^p \rangle_2^{1/p} = \sqrt{3}\, |(\sigma_0 - D) M_2 + D|, \quad \text{and} \tag{5.19}$$

$$\langle |\mathbb{P}_H \sigma|^p \rangle_1^{1/p} = \sqrt{3}\, |(\sigma_0 - D) L_1 + D|. \tag{5.20}$$

For coated spheres with core phase 2 and coating phase 1, $(C^e)^{-1} I : I = (K_{HS}^+)^{-1}$, and substitution of (5.13) into (5.9) and (5.11) together with (5.8) and (5.10) shows that the moments are given by

$$\langle |\mathbb{P}_H \sigma|^p \rangle_2^{1/p} = \sqrt{3}\, |(\sigma_0 - D) L_2 + D|, \quad \text{and} \tag{5.21}$$

$$\langle |\mathbb{P}_H \sigma|^p \rangle_1^{1/p} = \sqrt{3}\, |(\sigma_0 - D) M_1 + D|. \tag{5.22}$$

We collect results and state the lower bounds and indicate when they are optimal.

Theorem 5.1 Bounds for well-ordered composites, $k_1 > k_2$. *For $1 < p \leq \infty$, any choice of ΔT, and $-\infty < \sigma_0 < \infty$ the lower bounds are given by the following formulae:*

$$\langle |\mathbb{P}_H \sigma|^p \rangle_2^{1/p} \geq \min_{L_2 \leq X \leq M_2} \left\{ \sqrt{3}\, |(\sigma_0 - D) X + D| \right\}, \tag{5.23}$$

and when the minimum is realized for $X = L_2$ the bound is attained by the fields inside the core phase of a coated sphere construction with core material 2 and coating 1; when the minimum is realized for $X = M_2$ the *bound is attained by the fields inside the coating phase of a coated sphere construction with core material 1 and coating 2,*

$$\langle|\mathbb{P}_H\sigma|^p\rangle_1^{1/p} \geq \min_{L_1 \leq Y \leq M_1} \left\{\sqrt{3}\,|(\sigma_0 - D)Y + D|\right\}, \tag{5.24}$$

and when the minimum is realized for $Y = L_1$ the bound is attained by the fields inside the core phase of a coated sphere construction with core material 1 and coating 2, and when the minimum is realized for $Y = M_1$ the bound is attained by the fields inside the coating phase of a coated sphere construction with core material 2 and coating 1.

Theorem 5.2 Bounds for non well-ordered composites, $k_2 > k_1$. *For $1 < p \leq \infty$, any choice of ΔT, and $-\infty < \sigma_0 < \infty$ the lower bounds are given by the following formulae:*

$$\langle|\mathbb{P}_H\sigma|^p\rangle_2^{1/p} \geq \min_{M_2 \leq X \leq L_2} \left\{\sqrt{3}\,|(\sigma_0 - D)X + D|\right\}, \tag{5.25}$$

and when the minimum is realized for $X = L_2$ the bound is attained by the fields inside the core phase of a coated sphere construction with core material 2 and coating 1; when the minimum is realized for $X = M_2$ the bound is attained by the fields inside the coating phase of a coated sphere construction with core material 1 and coating 2,

$$\langle|\mathbb{P}_H\sigma|^p\rangle_1^{1/p} \geq \min_{M_1 \leq Y \leq L_1} \left\{\sqrt{3}\,|(\sigma_0 - D)Y + D|\right\}, \tag{5.26}$$

and when the minimum is realized for $Y = L_1$ the bound is attained by the fields inside the core phase of a coated sphere construction with core material 1 and coating 2, and when the minimum is realized for $Y = M_1$ the bound is attained by the fields inside the coating phase of a coated sphere construction with core material 2 and coating 1.

These bounds are stated explicitly in the first four Tables of Sects. 3 and 4.

We conclude by outlining the steps behind the derivation of the lower bounds on the maximum values of the local fields inside thermally stressed composites. For the well-ordered case we use the simple lower bound given by

$$\max_{\mathbf{x} \text{ in } Q} \{|\,\mathbb{P}^H\sigma(\mathbf{x})\} \geq \max\{A, B\}, \tag{5.27}$$

where

$$A = \min_{L_2 \leq X \leq M_2} \left\{\sqrt{3}\,|(\sigma_0 - D)X + D|\right\},$$
$$B = \min_{L_1 \leq Y \leq M_1} \left\{\sqrt{3}\,|(\sigma_0 - D)Y + D|\right\}. \tag{5.28}$$

For the non well-ordered case we use

$$\max_{\mathbf{x} \text{ in } Q} \{|\,\mathbb{P}^H\sigma(\mathbf{x})\} \geq \max\{C, D\}, \tag{5.29}$$

where

$$C = \min_{M_2 \leq X \leq L_2} \left\{\sqrt{3}\,|(\sigma_0 - D)X + D|\right\},$$
$$D = \min_{M_1 \leq Y \leq L_1} \left\{\sqrt{3}\,|(\sigma_0 - D)Y + D|\right\}. \tag{5.30}$$

The bounds given in the last two Tables presented in Sects. 3 and 4 follow from straight forward but tedious calculation of the explicit formulae corresponding to (5.27) and (5.29). A delicate but straight forward computation shows that these lower bounds are attained by the fields inside the coated sphere assemblage.

6 Local stress and strain fields inside thermally stressed coated sphere geometries

In this Section, we summarize the properties of local fields inside the coated sphere assemblage in the presence of thermal stress due to a mismatch in the coefficients of thermal expansion. From linearity the local stress can be split into the sum of two components; one component arising from imposed mechanical stress and a second component associated with thermal stress. It is known that the local stress due to an imposed hydrostatic stress has constant hydrostatic part inside each phase, this follows from explicit solution, see for example [19]. Here we display the explicit solution for the local stress due to mismatch in the coefficients of thermal expansion and show that it has a constant hydrostatic component inside each phase. From this we conclude that the total local stress inside the coated sphere assemblage has a constant hydrostatic component inside each phase.

In what follows we identify properties of the stress field inside the unit cube Q filled with the coated sphere assemblage made with core material two and coating material one. This is done by explicit solution of the boundary value problem for the elastic displacement field φ^p when the assemblage is subjected to a unit thermal load, i.e., $\Delta T = 1$. In what follows we find an explicit solution such that $\varphi^p = 0$ on the boundary of Q and this is the solution of the equilibrium problem given by (2.14), (2.15), (2.16), presented in Sect. 2. The elastic displacement field is constructed explicitly with the solution of the displacement field inside a prototypical coated sphere composed of a spherical core of material two with radius a, surrounded by a concentric shell of material one with an outer radius b. The ratio $(a/b)^3$ is fixed and equal to the inclusion volume fraction θ_2. Here, the coefficients of thermal expansion for the core and coating are given by h_2 and h_1, respectively. The local elastic displacement $\tilde{\varphi}$ satisfies the equations of elastic equilibrium which are given by:

$$\begin{array}{ll}
\mathrm{div}(C_2(\epsilon(\tilde{\varphi}) - h_2 I)) = 0 & 0 < r < a,\\
\mathrm{div}(C_1(\epsilon(\tilde{\varphi}) - h_1 I)) = 0 & a < r < b,\\
C_1(\epsilon(\tilde{\varphi}) - h_1 I)\mathbf{n}|_1 = C_2(\epsilon(\tilde{\varphi}) - h_2 I)\mathbf{n}|_2 & \text{continuity of traction at } r = a,\\
\tilde{\varphi} \quad \text{is continuous} & \text{on } 0 < r < b,\\
\tilde{\varphi} = 0 & \text{on the boundary } r = b.
\end{array}$$

We assume a general form of the solution given by

$$\tilde{\varphi} = \begin{cases} C\mathbf{r} & 0 < r < a,\\ A\mathbf{r} + B\frac{\mathbf{n}}{r^2} & a < r < b,\\ 0 & r \geq b, \end{cases}$$

where $r = |\mathbf{r}|$, $\mathbf{n} = \mathbf{r}/r$, and A, B, C are unknowns. The corresponding strain field $\epsilon(\tilde{\varphi})$ is given by

$$(\epsilon(\tilde{\varphi}))_{ij} = \frac{1}{2}(\tilde{\varphi}_{i,j} + \tilde{\varphi}_{j,i}) = \begin{cases} C\delta_{ij} & 0 < r < a,\\ A\delta_{ij} + \frac{B}{r^3}(\delta_{ij} - 3n_i n_j) & a < r < b,\\ 0 & r \geq b. \end{cases} \tag{6.1}$$

The homogeneous boundary condition at $r = b$ together with the continuity of displacement and traction at $r = a$ gives three linear equations for the determination of A, B, and C, and we find that

$$A = \frac{3\theta_2(k_1h_1 - k_2h_2)}{3k_1\theta_2 + 4\mu_1 + 3k_2(1-\theta_2)},$$
$$B = \frac{-3a^3(k_1h_1 - k_2h_2)}{3k_1\theta_2 + 4\mu_1 + 3k_2(1-\theta_2)},$$
$$C = \frac{-3(1-\theta_2)(k_1h_1 - k_2h_2)}{3k_1\theta_2 + 4\mu_1 + 3k_2(1-\theta_2)}.$$

Computation of the radial component of the stress at $r = b$ gives

$$C_1(\epsilon(\tilde{\varphi}) - h_1 I)\mathbf{n} = H^*\mathbf{n}, \tag{6.2}$$

where H^*

$$H^* = \frac{3\theta_2(3k_1 + 4\mu_1)(k_1h_1 - k_2h_2)}{3k_1\theta_2 + 4\mu_1 + 3k_2(1-\theta_2)} I - 3k_1h_1 I. \tag{6.3}$$

In this way, we have constructed a solution $\tilde{\varphi}$ for the elastic field inside every coated sphere in the assemblage. We now define φ^p on the whole domain Q to be given by $\tilde{\varphi}$ inside each coated sphere and zero outside. It easily follows on integrating by parts using (6.2) together with the fact that φ^p vanishes on the boundary of each coated sphere that φ^p is the weak solution [4] of $\mathrm{div}(C(\epsilon(\varphi^p - \lambda)) = 0$ over the full domain Q, i.e.,

$$\langle C(\epsilon(\varphi^p) - \lambda) : \epsilon(\phi)\rangle = 0,$$

for every periodic test function ϕ. Equation (6.1) implies that the hydrostatic component of stress is constant inside each phase.

Last, we show that the effective thermal stress H^e for the coated sphere assemblage is given by H^*. Inside each coated sphere S_i, $i = 1, 2, \ldots$ we consider $\sigma = C(\epsilon(\varphi^p) - \lambda)$ and integrate by parts and apply (6.2) to find that

$$\int_{S_i} \sigma \, d\mathbf{x} = \int_{\partial S_i} (\sigma \mathbf{n}) \otimes \mathbf{x} ds, \tag{6.4}$$

$$= \int_{\partial S_i} (H^* \mathbf{n}) \otimes \mathbf{x} ds = |S_i| H^*, \tag{6.5}$$

where ds is an element of surface area on the outer surface of the coated sphere ∂S_i, and $|S_i|$ is the volume of S_i. Substitution of (6.5) into (2.13) gives the required identity

$$H^e = \langle \sigma \rangle = \sum_{i=1}^{\infty} \int_{S_i} \sigma \, d\mathbf{x} = H^*. \tag{6.6}$$

7 Conclusion

We have introduced a variational method for bounding the local point wise hydrostatic stress inside statistically defined composites subjected to macroscopic thermomechanical loads. When only volume fraction information is known these bounds have been shown to be the best possible as they are attained by the Hashin–Shtrikman coated sphere assemblages. This paper considers thermomechanical loads together with applied hydrostatic stresses. The overall approach developed here applies to problems involving other load cases such as applied shear stresses. The analysis for applied thermomechanical shear loading will be presented in future work.

References

1. Alali, B., Lipton, R.: Optimal lower bounds on local stress inside random media. SIAM J. Appl. Math. **70**, 1260–1282 (2009)
2. Dacorogna, B.: Direct Methods in the Calculus of Variations. Springer, Berlin (1989)
3. Faraco, D.: Milton's conjecture on the regularity of solutions to isotropic equations. Annales de L'Institute Henri Poincare (C) Nonlinear Analysis **20**, 889–909 (2003)
4. Gilbarg, D., Trudinger, N.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (2001)
5. Grabovsky, Y., Kohn, R.V.: Microstructures minimizing the energy of a two phase elastic composite in two space dimensions. II: The Vigdergauz microstructure. J. Mech. Phys. Solids **43**, 949–972 (1995)
6. Hashin, Z.: Analysis of composite materials—a survey. J. Appl. Mech. **50**, 481–505 (1983)
7. Hashin, Z.: The elastic moduli of heterogeneous materials. J. Appl. Mech. **29**, 143–150 (1962)
8. Hashin, Z., Shtrikman, S.: A variational approach to the theory of the elastic behaviour of multiphase materials. J. Mech. Phys. Solids **11**, 127–140 (1963)
9. He, Q.C.: Lower bounds in the stress and strain fields inside random two-phase elastic media. Acta Mech. **188**, 123–137 (2007)
10. Hill, R.: Elastic properties of reinforced solids: some theoretical principles. J. Mech. Phys. Solids **11**, 357–372 (1963)
11. Kantor, Y., Bergman, D.J.: Improved rigorous bounds on the effective elastic moduli of a composite material. J. Mech. Phys. Solids **32**, 41–62 (1984)
12. Leonetti, F., Nesi, V.: Quasiconformal solutions to certain first order systems and the proof of a conjecture of G.W. Milton. J. Math. Pures. Appl. **76**, 109–124 (1997)
13. Levin, V.M.: Thermal expansion coefficients of heterogeneous materials. Mech. Solids **21**, 9–17 (1967)
14. Lipton, R.: Optimal lower bounds on the electric-field concentration in composite media. J. Appl. Phys. **96**, 2821–2827 (2004)
15. Lipton, R.: Optimal lower bounds on the dilatational strain inside random two-phase elastic composites subjected to hydrostatic loading. Mech. Mater. **38**, 833–839 (2006)

16. Lipton, R.: Optimal lower bounds on the hydrostatic stress amplification inside random two-phase elastic composites. J. Mech. Phys. Solids **53**, 2471–2481 (2005)
17. Lipton, R.: Optimal inequalities for gradients of solutions of elliptic equations occurring in two-phase heat conductors. SIAM J. Math. Anal. **32**, 1081–1093 (2001)
18. Maxwell Garnett, J.C.: Colours in metal glasses and in metallic films. Philos. Trans. R. Soc. Lond. **203**, 385–420 (1904)
19. Milton, G.W.: The Theory of Composites. Cambridge University Press, Cambridge (2002)
20. Milton, G.W.: Modeling the properties of composites by laminates. In: Erickson, J., Kinderleher, D., Kohn, R.V., Lions, J.L. Homogenization and Effective Moduli of Materials and Media. IMA Volumes in Mathematics and Its Applications, vol. 1, pp. 150–174. Springer, New York (1986)
21. Nemat-Nasser, S., Hori, M.: Micromechanics: Overall Properties of Heterogeneous Materials. Elsevier, Amsterdam (1999)
22. Rayleigh, L.: On the influence of obstacles arranged in rectangular order upon the properties of a medium. Philos. Mag. **34**, 481–502 (1892)
23. Rosen, B.W., Hashin, Z.: Effective thermal expansion coefficients and specific heats of composite materials. Int. J. Eng. Sci. **8**, 157–173 (1970)
24. Wheeler, L.T.: Inhomogeneities of minimum stress concentration. In: Angel, Y.C. (ed.) Anisotropy and Inhomogeneity in Elasticity and Plasticity, AMD-vol. 158, pp. 1–6. ASME, New York (1993)
25. Willis, J.R.: The overall elastic response of composite materials. J. Appl. Mech. **50**, 1202–1209 (1983)