Proceedings of the Royal Society of Edinburgh, 114A, 71–79, 1990

Darcy's law for slow viscous flow past a stationary array of bubbles

Robert Lipton
Department of Mathematics, University of California, Berkeley, CA 94720, U.S.A.

and

Marco Avellaneda
Courant Institute of Mathematical Sciences, New York, NY 10012, U.S.A.

(MS received 4 January 1989. Revised MS received 23 June 1989)

Synopsis

We examine slow viscous flow past a concentrated bed of small stationary viscous bubbles of a second fluid, and derive Darcy's law relating the average fluid velocity to the overall pressure gradient and body force.

1. Introduction

In this paper, we consider a concentrated bubble bed or foam in which a continuous fluid phase flows past an array of liquid or gaseous bubbles. The fluid velocity at the centre of mass of each bubble is taken to be zero, so that there is no relative motion between bubbles. The bubbles are assumed to be small with respect to macroscopic length scales and we assume, as a first approximation, that surface tension keeps their shape spherical. The bubble concentration, although finite, is assumed to be low enough so that the bubbles are separated and retain their spherical shape. For the purposes of this study we suppose that fluids differ in viscosity only. The fluids are incompressible and inertial effects are neglected in both phases. Thus the stationary linearised Navier–Stokes equations apply everywhere in the flow regions. The viscosities of the bubbles and surrounding fluid are μ_1 and μ_2 respectively, with $0<\mu_1<\mu_2$. The case of gas bubbles is obtained in the limit $\mu_1 \ll 1$.

The local fluid velocity is denoted by $u(x)$. We consider the local strain rate tensor $e=(\nabla u+\nabla u^T)/2$ and the local stress tensor $\sigma = 2\mu e - IP$, where P is the local pressure and

$$\mu = \begin{cases} \mu_1 & \text{in the bubble,} \\ \mu_2 & \text{in the continuous fluid phase.} \end{cases} \tag{1.1}$$

For a prescribed body force f the equations of motion in each phase are

$$\operatorname{div} \sigma + f = 0 \tag{1.2}$$

and

$$\operatorname{div} u = 0. \tag{1.2'}$$

Following Taylor [7] and others [3, 4, 5], we assume that the velocity is continuous across bubble surfaces, that the normal component of the velocity vanishes along these surfaces

$$u \,.\, n = 0, \tag{1.3}$$

and that the stress tensor satisfies the jump condition

$$[\sigma]n = (n^T \,.\, [\sigma] \,.\, n)n, \tag{1.4}$$

where n is the exterior unit normal vector. Here the notation [ ] denotes the jump of the bracketed quantity across the bubble surface.

The conditions (1.3) and (1.4) are lowest order approximations to the kinematic and dynamic conditions on a bubble surface held nearly spherical by surface tension (cf. [5]). We motivate (1.3) and (1.4) by an argument along the lines of Cox [3] and Schowalter, Chaffey and Brenner [5]. To fix ideas, suppose that ε is the typical distance between bubbles and that for finite ε the bubbles are nearly spherical with radius $a\varepsilon$. Following [3, 5], the equation for the surface of a nearly spherical bubble with centre of mass at the point l is written

$$|x - l| = \varepsilon a + \varepsilon^2 g^\varepsilon(\theta, \phi, t), \tag{1.5}$$

where x is a point on the surface and $\varepsilon^2 g^\varepsilon$ is the small deformation at time t depending on the polar coordinates (θ, ϕ). If we denote the coefficient of surface tension by γ, the mean curvature by H^ε and the pressure by P^*, the dynamic condition on the bubble surface is given by

$$[2\mu e - IP^*]n = -\gamma n H_\varepsilon. \tag{1.6}$$

It follows from (1.5) that

$$H^\varepsilon = \frac{2}{\varepsilon a} - \frac{2g^\varepsilon}{a^2} + O(\varepsilon) \tag{1.7}$$

and hence

$$[2\mu e - IP^*]n = -\gamma n\left(\frac{2}{\varepsilon a} - \frac{2g^\varepsilon}{a^2} + O(\varepsilon)\right). \tag{1.8}$$

In view of the scaling in (1.8) we may decompose the pressure into two parts $P^* = P + P'$, where P' is constant in each fluid with a jump discontinuity of $\gamma 2/\varepsilon a$ across the bubble surface. We see from (1.8) that the part of the mean curvature that diverges as the radius of the bubble goes to zero is balanced by a pressure jump. The remaining jump in normal stress is of order one. Denoting the remainder by $[\sigma]n$ we have

$$[\sigma]n = -\gamma n\left(-\frac{2g^\varepsilon}{a^2} + O(\varepsilon)\right). \tag{1.9}$$

We approximate the dynamic and kinematic conditions on the bubble surface given by (1.5) to lowest order in the small deformation $\varepsilon^2 g^\varepsilon$. Following [3, 4, 5] we suppose that the small bubbles have rigid spherical surfaces $|x-l| = a\varepsilon$. Therefore the kinematic condition is given by (1.3). The dynamic condition (1.9) is approximated as in [3, 5] by (1.4) on the sphere $|x-l| = a\varepsilon$.

Lastly, we suppose that the region Ω is the unit cube in $\mathbb{R}^3$ centred at the origin and that u satisfies periodic boundary conditions on $\partial\Omega$. We note that our method applies to other boundary conditions. For example, the region Ω could be an arbitrary bounded open domain in $\mathbb{R}^3$ and $u = 0$ on $\partial\Omega$.

We break the local velocity and pressure into slowly varying quantities $\bar{u}$ and $\bar{p}$, varying on the scale of the macroscopic driving force, and oscillatory quantities u', p', varying on the scale of the bubble separation ε. In view of the classical Darcy's law for porous media, we expect that

$$\bar{u}_i = -\varepsilon^2 K_{ij}(\nabla_j \bar{P} - f_j), \tag{1.10}$$

where K is a tensor depending on the geometry of the bubble bed. This self-permeability tensor K is a measure of the ability of the surrounding fluid to slip by the stationary bubbles.

We provide justification of such a law using a simple model of a periodic array of spherical bubbles of radius εa. We suppose that the continuous fluid phase and bubbles are present in the relative volume fractions θ_2 and θ_1, respectively, where $\theta_1 + \theta_2 = 1$. Taking as period cell Q the unit cube centred at the origin, containing a spherical bubble of radius a centred at the origin, we consider the following local problem. Let e^k be a unit vector directed along the x_k axis, $k = 1, 2, 3$. There exist unique solutions ϕ^k and q^k (q^k unique up to an additive constant) of

$$0 = \mu(y)\Delta_y \phi^k - \nabla_y q^k + e^k, \quad g \in Q, \quad |y| \neq a, \tag{1.11}$$

$$\operatorname{div}_y \phi^k = 0, \tag{1.12}$$

$$\left.\begin{aligned} &\phi^k . n = 0 \text{ on } |y| = a, \\ &[2\mu e(\phi^k) - Iq^k]n = ([2\mu e(\phi^k) - Iq^k]n . n)n \text{ on } |y| = a, \end{aligned}\right\} \tag{1.13}$$

$$\left.\begin{aligned} &\phi^k \text{ is } Q \text{ periodic}, \\ &q^k \text{ is } Q \text{ periodic}, \end{aligned}\right\} \tag{1.14}$$

where

$$\mu = \begin{cases} \mu_1 & |y| < a, \\ \mu_2 & |y| > a. \end{cases}$$

The self-permeability tensor K_{ij} is defined by

$$K_{ij} = \frac{1}{|Q|} \int_Q \phi_j^i \, dy. \tag{1.15}$$

It is easily seen that the self-permeability tensor is symmetric and positive definite (cf. [2, 6]).

Let μ^ε be the local viscosity field of an ε–periodic array of bubbles. Given a square integrable momentum source $f(x)$, we define u^ε to be the local fluid velocity and P^ε to be the local pressure (unique up to an additive constant in each phase) that satisfy

$$\sigma^\varepsilon = 2\mu^\varepsilon e^\varepsilon - IP^\varepsilon, \tag{1.16}$$

$$\operatorname{div} \sigma^\varepsilon + f = 0, \tag{1.17}$$

$$\operatorname{div} u^\varepsilon = 0, \tag{1.18}$$

$$u^\varepsilon . n = 0 \quad \text{on bubble surfaces,} \tag{1.19}$$

$$[\sigma^\varepsilon]n = ([\sigma^\varepsilon]n . n)n \quad \text{on bubble surfaces,} \tag{1.20}$$

$$u^\varepsilon \text{ and } P^\varepsilon \text{ periodic on } \Omega. \tag{1.21}$$

We consider a security region S inside Q with smooth boundary γ containing the bubble volume V. The region between the bubble and γ is denoted by Y. The scaled security region about the lth drop in an ε-periodic array of bubbles is defined by $S(l, \varepsilon)$ and is homothetic to S with ratio ε. The regions $V(l, \varepsilon)$ and $Y(l, \varepsilon)$ for the lth bubble are homothetic to V and Y, respectively, with ratio ε.

The pressure field P^ε is defined inside each bubble up to an additive constant. We introduce a *normalised* pressure field $\bar{P}^\varepsilon$ by modifying if necessary P^ε by a constant inside each bubble, so as to obtain

$$\frac{1}{|V(l, \varepsilon)|} \int_{V(l, \varepsilon)} \bar{P}^\varepsilon \, dx = \frac{1}{|Y(l, \varepsilon)|} \int_{Y(l, \varepsilon)} \bar{P}^\varepsilon \, dx \tag{1.22}$$

for all l.

THEOREM 1.1. *For any square integrable momentum source f, we have*

$$\frac{u^\varepsilon}{\varepsilon^2} \to u^0 \text{ weakly in } L^2(\Omega)^3, \tag{1.23}$$

$$\bar{P}^\varepsilon \to \bar{P} \text{ strongly in } L^2(\Omega)/\mathbb{R}, \tag{1.24}$$

where

$$\operatorname{div} u^0 = 0, \tag{1.25}$$

$$u^0 \text{ is } \Omega\text{-periodic} \tag{1.26}$$

and

$$u_i^0 = -K_{ij}(\nabla_j \bar{P} - f_j) \text{ in } \Omega. \tag{1.27}$$

The main difference between our problem and the one treated by Tartar in [6] lies in the different kinematic and dynamic conditions at the two phase boundary. We believe that we have clarified a point in [6], in that we show that the correct extension of the pressure inside solid inclusions is determined by the average pressure over a security region surrounding the inclusion (see equation (2.29)). Thus our analysis gives a simple expression for the extension which is needed to obtain L^2-convergence of the pressure as the radii of the inclusions go to zero. Our proof of Theorem 1.1 follows the same steps as given in [6]. We learned that Tartar's method has been extended recently to biconnected porous media by Allaire [1], as communicated to us by F. Murat. Lastly, we note that there is some flexibility with the choice of security region as the L^2 limit of the sequence of normalised pressures is the same for all security regions.

2. Convergence of the flow fields and pressures

We prove the convergence results (1.23), (1.24) of Theorem 1.1. We consider the unit cell Q and note that, by Stokes' theorem, the average of a divergence free flow field u over the drop volume V is equal to $\bar{u}_j = 1/|V| \int_{\partial V} u \,.\, nx_j \, dS$. Hence if $u \,.\, n = 0$ on ∂V, then $\bar{u}_j = 0$ and, by Poincaré's inequality,

$$\int_Q u^2 \, dy \leqq C \int_Q (\nabla_y u)^2 \, dy. \tag{2.1}$$

Therefore rescaling by $x = \varepsilon y$, one has $dx = \varepsilon^3 \, dy$, $\partial_y = \varepsilon \partial_z$ and hence

$$\int_{\varepsilon Q} u^2 \, dx \leqq \varepsilon^2 C \int_{\varepsilon Q} (\nabla_x u)^2 \, dx. \tag{2.2}$$

We observe that the constant C is the same for all periods εQ. Thus summing over all the periods, we obtain the estimate

$$\|u^\varepsilon\|^2_{L^2(\Omega)^3} \leqq \varepsilon^2 C \, \|\nabla u^\varepsilon\|^2_{L^2(\Omega)^{3\times 3}}. \tag{2.3}$$

Multiplication of (1.17) by u^ε, integration by parts and application of Korn's inequality gives

$$\min(\mu_1, \mu_2) \, \|\nabla u^\varepsilon\|^2_{L^2(\Omega)^{3\times 3}} \leqq \left| \int_\Omega f \,.\, u^\varepsilon \right| \leqq C \, \|u^\varepsilon\|_{L^2(\Omega)^3}. \tag{2.4}$$

We use (2.3) to estimate the right-hand side of (2.4) and obtain

$$\min(\mu_1, \mu_2) \, \|\nabla u^\varepsilon\|^2_{L^2(\Omega)^{3\times 3}} \leqq \varepsilon C \, \|\nabla u^\varepsilon\|_{L^2(\Omega)^{3\times 3}}. \tag{2.5}$$

It follows directly from inequalities (2.3), (2.5) that one has

LEMMA 2.1. *The sequence u^ε satisfies*

$$\|u^\varepsilon\|_{L^2(\Omega)^3} \leqq C\varepsilon^2, \tag{2.6}$$

$$\|\nabla u^\varepsilon\|_{L^2(\Omega)^{3\times 3}} \leqq C\varepsilon. \tag{2.7}$$

The weak convergence in $L^2(\Omega)^3$ of a subsequence $u^{\varepsilon'}/\varepsilon'^2$ follows immediately.

We now show the convergence of the normalised pressure (see equation (1.22)) in $L^2(\Omega)/\mathbb{R}$. We introduce the spaces $H^1_{\text{per}}(\Omega)^3 = \{\delta \in H^1(\Omega)^3 \mid \delta \text{ is } \Omega\text{-periodic}\}$, and $\tilde{V}^\varepsilon = \{\delta \in H^1_{\text{per}}(\Omega)^3 \mid \delta \,.\, n = 0, \text{ on bubble surfaces}\}$. Following Tartar [6], we construct a local restriction operator R^ε such that

$$R_\varepsilon \colon H^1_{\text{per}}(\Omega)^3 \to \tilde{V}^\varepsilon, \tag{2.8}$$

$$\delta \text{ in } \tilde{V}^\varepsilon \quad \text{implies} \quad R_\varepsilon \delta = \delta, \tag{2.9}$$

$$\operatorname{div} \delta = 0 \quad \text{implies} \quad \operatorname{div} R_\varepsilon \delta = 0, \tag{2.10}$$

$$\|R\varepsilon\delta\|_{L^2(\Omega)^3} \leqq c \, \|\delta\|_{L^2(\Omega)^3} + C\varepsilon \, \|\nabla\delta\|_{L^2(\Omega)^{3\times 3}}, \tag{2.11}$$

and

$$\|\nabla(R_\varepsilon \delta)\|_{L^2(\Omega)^3} \leqq \frac{C}{\varepsilon} \, \|\delta\|_{L^2(\Omega)^3} + C \, \|\nabla\delta\|_{L^2(\Omega)^{3\times 3}}. \tag{2.12}$$

The construction of R_ε proceeds as follows.

LEMMA 2.2. *Given the subregions S, V, and Y of the unit cell Q and given u in $H^1(Q)^3$ there exist solutions v in $H^1(Y)^3$, q in $L^2(Y)/\mathbb{R}$ of*

$$\Delta v = \Delta u - \nabla q, \tag{2.13}$$

$$\operatorname{div} v = \operatorname{div} u + \frac{1}{|Y|}\int_{\partial V} u \, . \, n \, ds, \tag{2.14}$$

$$v\,|_{Y} = u, \quad v \, . \, n\,|_{\partial V} = 0, \quad v \, . \, \tau\,|_{\partial V} = u \, . \, \tau\,|_{\partial V}, \tag{2.15}$$

and there exist solutions w in $H^1(V)^3$, p in $L^2(V)/\mathbb{R}$ of

$$\Delta w = \Delta u - \nabla p, \tag{2.16}$$

$$\operatorname{div} w = \operatorname{div} u - \frac{1}{|V|}\int_{\partial V} u \, . \, n \, dS, \tag{2.17}$$

$$w \, . \, n\,|_{\partial V} = 0, \quad w \, . \, \tau\,|_{\partial V} = u \, . \, \tau\,|_{\partial V}, \tag{2.18}$$

where n is the outward unit normal to ∂V and τ is any unit tangent vector to ∂V.

The proof of this lemma is analogous to the proof of [6, Lemma 3]. □

We define the operator R acting on the space $H^1(Q)^3$ by

$$Ru(y) = \begin{cases} u(y), & u \in Q/S, \\ v(y), & y \in Y, \\ w(y), & y \in V. \end{cases}$$

It is evident that

$$\|Ru\|_{H^1(Q)^3} \leqq \|u\|_{H^1(Q)^3}, \tag{2.19}$$

$$Ru = u \quad \text{if} \quad u \, . \, n\,|_{\partial V} = 0, \tag{2.20}$$

and that

$$\operatorname{div} u = 0 \quad \text{implies} \quad \operatorname{div} Ru = 0. \tag{2.21}$$

We define R_ε by applying R to every εQ period. It is easily seen that (2.8), (2.9) and (2.10) hold. We estimate $\|R_\varepsilon \delta\|_{H^1(\Omega)^3}$ by rescaling and applying (2.1) to obtain

$$\|R_\varepsilon \delta\|^2_{H^1(\Omega)^3} \leqq C\{\varepsilon^{-2}\,\|\delta\|^2_{L^2(\Omega)^3} + \|\nabla\delta\|^2_{L^2(\Omega)^{3\times 3}}\}. \tag{2.22}$$

Arguing as in Lemma 2.1 we obtain

$$\|R_\varepsilon \delta\|^2_{L^2(\Omega)^3} \leqq C\varepsilon^2\,\|\nabla R_\varepsilon \delta\|_{L^2(\Omega)^{3\times 3}}. \tag{2.23}$$

From (2.22) and (2.23) it is evident that $\|R_\varepsilon\delta\|^2_{L^2(\Omega)^3} + \varepsilon^2\,\|\nabla R_\varepsilon \delta\|^2_{L^2(\Omega)^3} \leqq C\{\|\delta\|^2_{L^2(\Omega)^3} + \|\nabla\delta\|^2_{L^2(\Omega)^{3\times 3}}\}$ and (2.11) and (2.12) follow.

LEMMA 2.3. *For all δ in $H^1_{\text{per}}(\Omega)^3$ we have*

$$\int_\Omega \bar{P}^\varepsilon \operatorname{div} \delta \, dx = \int_\Omega \bar{P}^\varepsilon \operatorname{div} R_\varepsilon \delta \, dx. \tag{2.24}$$

Proof. From the definition of the local restriction operator R_ε and equation

(1.22) we have

$$\int_\Omega \bar{P}^\varepsilon \operatorname{div} \delta \, dx - \int_\Omega \bar{P}^\varepsilon \operatorname{div} R_\varepsilon \delta \, dx = \sum_{\text{bubbles}} \left\{ -\frac{1}{|Y(l,\varepsilon)|} \int_{Y(l,\varepsilon)} \bar{P}^\varepsilon \, dx \left(\int_{\partial V(l,\varepsilon)} \delta \, . \, n \, dS \right) + \frac{1}{|V(l,\varepsilon)|} \int_{V(l,\varepsilon)} \bar{P}^\varepsilon \, dx \left(\int_{\partial V(l,\varepsilon)} \delta \, . \, n \, dS \right) \right\} = 0. \quad \square$$

Using Lemma 2.3, we follow the arguments of Tartar [6] to obtain strong convergence of the normalised pressures. We multiply (1.17) by $R_\varepsilon \delta$ and integrate by parts to obtain

$$\int_\Omega \bar{P}^\varepsilon \operatorname{div} R_\varepsilon \delta \, dx = \int_\Omega 2\mu^\varepsilon e^\varepsilon : e(R_\varepsilon \delta) \, dx - \int_\Omega f \, . \, R_\varepsilon \delta \, dx. \tag{2.25}$$

By using inequalities (2.11), (2.12), (2.24), and (2.25) we obtain the estimate $|\int_\Omega \bar{P}^\varepsilon \operatorname{div} \delta \, dx| \leqq C \, \|\delta\|_{H^1(\Omega)^3}$. Thus,

$$\|\nabla \bar{P}^\varepsilon\|_{H^{-1}(\Omega)^3} \leqq C. \tag{2.26}$$

It follows from [8, Chapter 1] that

$$\|\bar{P}^\varepsilon\|_{L^2(\Omega)/\mathbb{R}} \leqq C \, \|\nabla \bar{P}^\varepsilon\|_{H^{-1}(\Omega)^3}. \tag{2.27}$$

Hence $\bar{P}^\varepsilon$ is bounded uniformly in $L^2(\Omega)/\mathbb{R}$ and there exists a subsequence $\bar{P}^{\varepsilon'}$ converging to $\bar{P}$ such that

$$\begin{aligned} &\bar{P}^{\varepsilon'} \rightharpoonup \bar{P} \quad \text{weakly in } L^2(\Omega)/\mathbb{R}, \\ \text{and} \quad &\nabla \bar{P}^{\varepsilon'} \rightharpoonup \nabla \bar{P} \quad \text{weakly in } H^{-1}(\Omega)^3. \end{aligned} \tag{2.28}$$

We conclude, following the arguments of [6], that $\nabla \bar{P}^{\varepsilon'} \to \nabla \bar{P}$ strongly in $H^{-1}(\Omega)^3$, and, from (2.27), we have $\bar{P}^{\varepsilon'} \to \bar{P}$ strongly in $L^2(\Omega)/\mathbb{R}$. $\square$

We observe that the extension of the pressure for the case of solid inclusions given in [6, equation (33)] amounts to the extension of the pressure by an appropriate constant inside each inclusion. Indeed, for solid inclusions the pressure is initially defined only in the fluid region $\Omega_\varepsilon = \{\Omega / \bigcup_l V(l,\varepsilon)\}$. If we set

$$\bar{P}^\varepsilon = \begin{cases} P^\varepsilon, & x \in \Omega_\varepsilon, \\ \dfrac{1}{|Y^{(l,\varepsilon)}|} \displaystyle\int_{Y(l,\varepsilon)} P^\varepsilon \, dx, & x \in V(l,\varepsilon), \end{cases} \tag{2.29}$$

then arguing as in Lemma 2.3, we have $\int_\Omega \bar{P}^\varepsilon \operatorname{div} \delta \, dx = \int_\Omega P^\varepsilon \operatorname{div} R_\varepsilon \delta \, dx$ for all δ in $H_0^1(\Omega)^3$, where R_ε is defined by [6, Lemma 4]. Thus the extended pressure given in [6] is given by (2.29).

3. Identification of u^0 and $\bar{P}$

In this section we follow the energy method of [6] to obtain the constitutive relation between u^0 and $\bar{P}$. We rescale by $x = \varepsilon y$ in the cell problem (1.11)–(1.14) to obtain

$$\phi_\varepsilon^k(x) = \phi^k\left(\frac{x}{\varepsilon}\right), \quad q_\varepsilon^k(x) = q^k\left(\frac{x}{\varepsilon}\right), \tag{3.1}$$

$$0 = \varepsilon^2 \mu^\varepsilon(x) \Delta_x \phi_\varepsilon^k - \varepsilon \nabla_x q_\varepsilon^k + e^k, \quad x \notin \partial V(l, \varepsilon), \tag{3.2}$$

$$\begin{gathered} \phi_\varepsilon^k . n = 0 \quad \text{on } \partial V(l, \varepsilon), \\ [2\mu^\varepsilon e_x(\phi_\varepsilon^k) - I q_\varepsilon^k]_2^1 n = ([2\mu^\varepsilon e_x(\phi_\varepsilon^k) - I q_\varepsilon^k]_2^1 n . n) n \quad \text{on } \partial V(l, \varepsilon), \end{gathered} \tag{3.3}$$

and

$$\operatorname{div}_x \phi_\varepsilon^k = 0. \tag{3.4}$$

It is easy to see that

$$\|\phi_\varepsilon^k\|_{L^2(\Omega)^3} \leqq C, \quad \|q_\varepsilon^k\|_{L^2(\Omega)} \leqq C \tag{3.5}$$

and

$$\|\nabla_x \phi_\varepsilon^k\|_{L^2(\Omega)^{3\times 3}} \leqq C\varepsilon^{-1}. \tag{3.6}$$

We choose δ in $C^\infty(\Omega)$ and multiply (3.2) by δu^ε and integrate over Ω to obtain

$$\int_\Omega \delta u^\varepsilon . (\mu^\varepsilon(x) \Delta \phi_\varepsilon^k) \, dx - \int_\Omega \delta u^\varepsilon . \varepsilon \nabla q_\varepsilon^k \, dx + \int_\Omega \delta u^\varepsilon . e^k \, dx = 0. \tag{3.7}$$

Integration by parts in (3.7) and application of (1.19) and (3.3) yields

$$\int_\Omega 2\mu^\varepsilon e(\phi_\varepsilon^k) : e(\delta u^\varepsilon) \, dx = -\frac{1}{\varepsilon} \int_\Omega I q_\varepsilon^k : e(\delta u^\varepsilon) \, dx + \frac{1}{\varepsilon^2} \int_\Omega \delta u^\varepsilon . e^k \, dx. \tag{3.8}$$

However, from (1.18) and (2.6)

$$\lim_{\varepsilon \to 0} \left\{ \frac{1}{\varepsilon} \int_\Omega I q_\varepsilon^k : e(\delta u^\varepsilon) \, dx \right\} = 0.$$

Therefore passing to the limit in (3.8) gives

$$\lim_{\varepsilon \to 0} \int_\Omega 2\mu^\varepsilon e_x(\phi_\varepsilon^k) : e^k(\delta u^\varepsilon) \, dx = \int_\Omega \delta u^0 . e \, dx. \tag{3.9}$$

By multiplying (1.17) by $\delta\phi_\varepsilon^k$ and performing integration by parts, we obtain

$$\int_\Omega 2\mu^\varepsilon e(u^\varepsilon) : e(\delta \phi_\varepsilon^k) \, dx = \int_\Omega \bar{P}^\varepsilon \operatorname{div}(\delta \phi_\varepsilon^k) \, dx + \int_\Omega f . \delta \phi_\varepsilon^k \, dx. \tag{3.10}$$

Noting that $\bar{P}^\varepsilon$ converges strongly in $L^2(\Omega)/\mathbb{R}$ to $\bar{P}$ and that ϕ_ε^k converges weakly to its average, we pass to the limit in (3.10) to obtain

$$\lim_{\varepsilon \to 0} \int_\Omega 2\mu^\varepsilon e(u^\varepsilon) : e(\delta \phi_\varepsilon^k) \, dx = K_{ki} \int_\Omega \bar{P} \partial_{x_i} \delta \, dx + K_{ki} \int_\Omega f_i \delta \, dx. \tag{3.11}$$

We observe as in [6] that the difference between the left-hand sides of (3.9) and (3.11) is of order ε, thus in the limit we have

$$\int_\Omega \delta u^0 . e_x \, dx = K_{ki} \left(\int_\Omega (\bar{P} \partial_{x_i} \delta + f_i \delta) \, dx \right)$$

for all test functions δ in $C_0^\infty(\Omega)$, and Darcy's law $u_k^0 = K_{ki}(f_i - \partial_{x_i} \bar{P})$. □

Robert Lipton was s
Mathematical Sciences

1 G. Allaire. Homogenizat
(to appear).
2 A. Bensoussan, J. L. Li
(Amsterdam: North-Holla
3 R. G. Cox. The deformati
(1969), 601–623.
4 J. B. Keller, L. A. Rube
with application to suspen
5 W. R. Schowalter, C. E.
Colloid Interface Sci. 26 (
6 L. Tartar. Appendix. In A
Lecture Notes in Physics 1
7 G. I. Taylor. The viscosi
London Ser. A. 138 (1932
8 R. Temam. *Navier–Stokes*

Acknowledgment

Robert Lipton was supported by the U.S. Army Research Office through the Mathematical Sciences Institute of Cornell University.

References

1 G. Allaire. Homogenization of Stokes flow in a connected porous medium. *Asymptotic Analysis* (to appear).
2 A. Bensoussan, J. L. Lions and G. Papanicolaou. *Asymptotic Analysis for Periodic Structures* (Amsterdam: North-Holland, 1978).
3 R. G. Cox. The deformation of a drop in a general time-dependent fluid flow. *J. Fluid Mech.* 37(3) (1969), 601–623.
4 J. B. Keller, L. A. Rubenfield and J. E. Molyneux. Extremum principles for slow viscous flows with application to suspensions. *J. Fluid Mech.* **30** (1967), 97–125.
5 W. R. Schowalter, C. E. Chaffey and H. Brenner. Rheological behavior of a dilute emulsion. *J. Colloid Interface Sci.* **26** (1968), 152–160.
6 L. Tartar. Appendix. In *Nonhomogeneous Media and Vibration Theory*, ed. E. Sanchez-Palencia, Lecture Notes in Physics 127 (Berlin: Springer-Verlag, 1980).
7 G. I. Taylor. The viscosity of a fluid containing small drops of another fluid. *Proc. Roy. Soc. London Ser. A.* **138** (1932), 41–48.
8 R. Temam. *Navier–Stokes Equations* (Amsterdam: North-Holland, 1984).

(*Issued* 20 *April* 1990)