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On the effective elasticity of a two-dimensional homogenised incompressible elastic composite

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Synopsis

The set of effective elasticity tensors for all two-dimensional mixtures of two isotropic incompressible elastic materials taken in prescribed proportion is described. In two dimensions the effective tensors are completely characterised by bounds on their eigenvalues.

1. Introduction

New bounds are established on the eigenvalues of the effective elasticity tensor for an N-dimensional homogenised mixture of two incompressible elastic solids taken in prescribed proportion. In two dimensions the eigenvalue bounds are used to describe the set of effective elasticity tensors.

The effective elasticity tensor is defined in the context of homogenisation. Mathematically, homogenisation corresponds to the theory of G-convergence [**17**, **18**, **19**, **23**] or H-convergence [**3**, **12**, **13**, **20**].

Let α, β be the Lame shear moduli for two isotropic incompressible elastic solids such that $0<\alpha<\beta<\infty$. A mixture of these two materials is characterised by the characteristic function $\chi_\alpha(x)$ of the α-material in $\mathbb{R}^N$. At any point x of $\mathbb{R}^N$ the elasticity tensor of the mixture is

$$A(x)=(2\alpha\chi_\alpha(x)+2\beta(1-\chi_\alpha(x)))I,$$

where I is the fourth order identity on the space of symmetric $N\times N$ trace-free matrices.

Consider the family of mixtures $\chi^\varepsilon_\alpha(x)$ contained in an open bounded set Ω of $\mathbb{R}^N$ with elasticity tensor

$$A^\varepsilon(x)=(2\alpha\chi^\varepsilon_\alpha(x)+2\beta(1-\chi^\varepsilon_\alpha(x)))I \tag{1.0}$$

such that

$$\chi^\varepsilon_\alpha \rightharpoonup \theta(x) \text{ in } L_\infty(\Omega) \text{ weak star} \tag{1.1}$$

as ε goes to zero. Here ε characterises the length scale of the mixture and θ is interpreted as the local average volume fraction of the α-material in the homogenised mixture.

THEOREM 1.0. *A subsequence of tensors $A^{\varepsilon'}$ of the sequence A^ε given by* (1.0)

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and (1.1) *H-converges to a tensor* A^0 *with* $\|A^0\|_{L_\infty(\Omega)} \leqq 2\beta$ *and* $A^0 \geqq 2\alpha I$ *almost everywhere in* Ω, *i.e. for any f in* $H^{-1}(\Omega)^N$ *the unique solution* $u^{\varepsilon'}$ *in* $H_0^1(\Omega)^N$, $P^{\varepsilon'}$ *in* $L^2(\Omega)/\mathbb{R}$ *of*

$$e_{ij}(u^{\varepsilon'}) = \tfrac{1}{2}(\partial_{x_j} u_i^{\varepsilon'} + \partial_{x_i} u_j^{\varepsilon'}),$$

$$\left.\begin{array}{c}\partial_{x_j}(A_{ijkl}^{\varepsilon'} e_{kl}(u^{\varepsilon'})) + \partial_{x_i} P^{\varepsilon'} = f_i \\ \operatorname{div} u^{\varepsilon'} = 0\end{array}\right\} \quad on\ \Omega$$

satisfies

$$u^{\varepsilon'} \to u^0 \text{ weakly in } H_0^1(\Omega)^N,$$

$$P^{\varepsilon'} \to P^0 \text{ weakly in } L^2(\Omega)/\mathbb{R}$$

as ε' *goes to zero, where* $u^0 \in H_0^1(\Omega)^N$, $P^0 \in L^2(\Omega)/\mathbb{R}$ *is the unique solution of*

$$\left.\begin{array}{c}\partial_{x_j}(A_{ijkl}^0(x) e_{kl}(u^0)) + \partial_{x_i} P^0 = f_i \\ \operatorname{div} u^0 = 0\end{array}\right\} \quad on\ \Omega.$$

The tensor A^0 is the effective elasticity tensor of the mixture and is a symmetric linear map on the $N(N+1)/2-1$ dimensional space of $N \times N$ symmetric trace-free matrices.

Remark 1.1. The theorems of S. Spagnolo **[18]** and L. Tartar **[13, 20]** applied to this setting would yield the H-convergence result given above.

A special class of effective tensors which are limits of H-converging sequences of tensors corresponding to periodic mixtures is considered. Let χ_α^ε represent a mixture such that χ_α is periodic in $\mathbb{R}^N$ with period $Q = (0, 1)^N$, i.e.

$$\chi_\alpha^\varepsilon(x) = \chi_\alpha\left(\frac{x}{\varepsilon}\right)$$

and

$$\theta = \int_Q \chi_\alpha(x)\, dx$$

and $0 \leqq \theta \leqq 1$. The fundamental theorem of periodic homogenisation is stated in the following definition and corollary of Theorem 1.0.

Definition 1.2. The sequence of tensors A^ε given by

$$A^\varepsilon(x) = \left(2\alpha\chi_\alpha\left(\frac{x}{\varepsilon}\right) + 2\beta\left(1 - \chi_\alpha\left(\frac{x}{\varepsilon}\right)\right)\right)I$$

is called a *PC* sequence of tensors.

Corollary 1.3. *The whole PC sequence* A^ε *H-converges to a unique constant tensor* A^0.

Remark 1.4. The sequence $\chi_\alpha\left(\frac{x}{\varepsilon}\right)$ converges to θ in $L_\infty(\Omega)$ weak star as ε tends to zero.

Remark 1.5. The convergence result given above is well known **[2, 14, 16]**.

Moreover, the limit of a *PC* sequence may be written explicitly in terms of a variational principle [**6, 8**] easily obtained from the results of [**14**]:

$$(A^0\xi, \xi) = \inf_{\phi} \int_Q 2a(y)(e_{ij}(\phi) + \xi_{ij})^2\, dy \tag{1.2}$$

for any $N \times N$ trace-free symmetric matrix ξ, where $a(y)$ is the Q periodic Lame shear modulus

$$a(y) = \alpha\chi_\alpha(y) + \beta(1 - \chi_\alpha(y)),$$

and ϕ ranges over all divergence-free, Q-periodic N-dimensional vector fields.

Definition 1.6. $P_\theta\{\alpha, \beta\}$ is the set of all H-limits of *PC* sequences for a specific choice of the parameters θ, α, β.

In this paper the set $P_\theta\{\alpha, \beta\}$ is described for the two-dimensional case. Although attention is restricted to the periodic setting, there is no loss of generality and the results can be extended to the non-periodic case (see Remark 2.6 in Section 2).

The results of the analysis are stated in Section 2. In Section 3 the methods of Kohn and Milton [**7**] are applied to the case of N-dimensional incompressible elasticity ($N \geqq 2$) and bounds are derived on the eigenvalues of the effective tensor equations (2.1), (2.2)) using the Hashin–Shtrikman variational principles [**4**]. In Section 4 optimality is proved in two dimensions by constructing a rank $\leqq 3$ laminar mixture for each point (λ_1, λ_2) in the region bounded by (2.1) and (2.2) such that its effective elasticity has eigenvalues λ_1, λ_2. This optimality proof is based on the optimality proof for the conductivity problem given by Murat and Tartar (see Tartar [**21**]). It is noted that the formulae (Theorem 4.3 (4.2), (4.3)) for the effective elasticity of a laminar mixture of two anisotropic incompressible materials are a modified version of [**3**, Theorem 4.1] which applies to the general elastic case. For completeness, a proof of Theorem 4.3 based on the ideas developed in [**3**, Theorem 4.1] is provided. The fact that optimality is proved using laminar mixtures corresponds to the recent results of Avellaneda [**1**], which state that for any two-phase elastic mixture with prescribed volume fractions both stronger and weaker laminar mixtures can be found.

2. Statement of results

In this section eigenvalue bounds are presented which completely characterise the set of effective tensors $P_\theta\{\alpha, \beta\}$ in two dimensions and give a partial characterisation for higher dimensions. The classical bounds on the eigenvalues of the effective elasticity tensor are the arithmetic mean-harmonic mean bounds derived in [**15**]. These bounds are independent of dimension and are given by

$$h_\theta \leqq \lambda \leqq m_\theta, \tag{2.0}$$

where λ is any eigenvalue of the effective tensor and

$$h_\theta = 2(\alpha^{-1}\theta + \beta^{-1}(1-\theta))^{-1}, \quad m_\theta = 2(\alpha\theta + \beta(1-\theta)).$$

Remark 2.0. Let $T_{N\times N}$ denote the space of all symmetric trace-free $N \times N$

matrices. The bounds given in **[15]** imply that the set $P_\theta\{\alpha, \beta\}$ lies within the set of all symmetric linear transforms on $T_{N\times N}$ with eigenvalues satisfying (2.0).

The following bounds lead to a more precise characterisation of the set $P_\theta\{\alpha, \beta\}$ for $N>2$ and in the case of two dimensions ($N=2$) they completely 3
describe $P_\theta\{\alpha, \beta\}$.

THEOREM 2.1. *The set $P_\theta\{\alpha, \beta\}$ is contained in the set of all symmetric linear transformations on $T_{N\times N}$ such that their $N(N+1)/2-1$ eigenvalues lie in the convex region R_θ^N given by*

$$\sum_{i=1}^{m}\frac{1}{\lambda_i-2\alpha}\leqq(N-1)\left(\frac{1}{h_\theta-2\alpha}+\frac{N/2}{m_\theta-2\alpha}\right), \tag{2.1}$$

$$\sum_{i=1}^{m}\frac{1}{2\beta-\lambda_i}\leqq(N-1)\left(\frac{1}{2\beta-h_\theta}+\frac{N/2}{2\beta-m_\theta}\right), \tag{2.2}$$

where $m=N(N+1)/2-1$ and N is the dimension of the space $\mathbb{R}^N$.

Remark 2.2. The convexity of the region R_θ^N given by (2.1) and (2.2) follows from the convexity of the functions $\Psi(\lambda)=(\lambda-2\alpha)^{-1}$ for $\lambda>2\alpha$ and $\phi(\lambda)=(2\beta-\lambda)^{-1}$ for $\lambda<2\beta$ (see **[13, 21]**).

THEOREM 2.3. *In two dimensions the set $P_\theta\{\alpha, \beta\}$ is precisely the set of all symmetric linear transformations on $T_{2\times 2}$ which have their* (*two*) *eigenvalues inside the convex region R_θ^2 given by* (2.1) *and* (2.2) *for $N=2$.* (See Figure 1.)

Remark 2.4. For $N=2$ the region R_θ^2 is contained inside the region given by the bounds in **[15]**. However, in three dimensions there exist choices of the parameters α, β, θ for which R_θ^3 is no longer contained inside the bounds given in **[15]**.

This observation motivates the following remark.

Remark 2.5. For $N>2$ the set $P_\theta\{\alpha, \beta\}$ is a subset of all symmetric linear

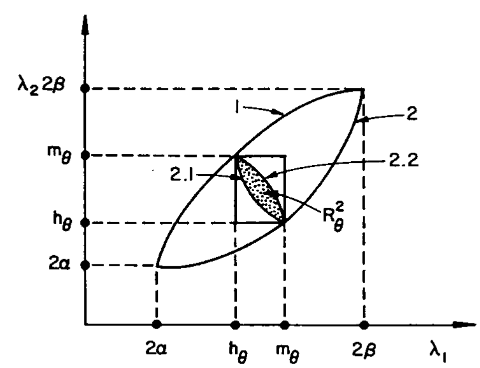

Figure 1. The shaded region R_θ^2 bounded by curves (2.1) and (2.2) is the set of eigenvalues of effective tensors in $P_\theta\{\alpha, \beta\}$ for two dimensions. The square region is given by the bounds of **[15]**. The region lying between (1) and (2) is the set of eigenvalues of the effective tensor for all periodic mixtures of α and β materials.

mappings on $T_{N\times N}$ with eigenvalues lying in the intersection of the region R_θ^N given by (2.1), (2.2) and the region defined by the bounds (2.0).

Remark 2.6. Theorems 2.1 and 2.3 can be generalised to hold for all limits of H-converging sequences as given by Theorem 1.0. This generalisation is technical. The interested reader may refer to **[5]**.

3. Derivation of the eigenvalue bounds for $N \geqq 2$-dimensional homogenised incompressible elasticity

To illustrate our method we derive the lower eigenvalue bound for N-dimensional periodically homogenised incompressible elasticity (inequality (2.1) with $N \geqq 2$). The following inequality on the effective elasticity is obtained from the variational principle (1.2).

THEOREM 3.0. *If A^0 is the limit of a PC sequence of tensors A^ε, then A^0 satisfies*

$$((A^0 - 2\alpha I)\xi, \xi) - 2(1-\theta)(\mu, \xi) \geqq -(\beta-\alpha)^{-1}\frac{(1-\theta)}{2}|\mu|^2 + (F_\beta \mu, \mu), \quad (3.0)$$

where μ and ξ are two matrices in $T_{N\times N}$, $(\cdot, \cdot)$ is the inner product on $T_{N\times N}$ (i.e. $(\xi, \mu) = \operatorname{tr}(\xi\mu)$), and F_β is a symmetric linear mapping on $T_{N\times N}$. For $k \in Z^N$ we denote the Fourier coefficients of the characteristic function $\chi_\beta(x)$ by $\hat{\chi}_\beta(k)$. Then F_β reads as

$$(F_\beta \mu, \theta) = \frac{1}{\alpha}\sum_{k\neq 0} |\hat{\chi}_\beta(k)|^2 \left\{\left(\mu\frac{k}{|k|}, \frac{k}{|k|}\right)^2 - \left(\mu\frac{k}{|k|}, \mu\frac{k}{|k|}\right)\right\} \quad (3.1)$$

and

$$F_\beta \mu = \frac{1}{\alpha}\sum_{k\neq 0} |\hat{\chi}_\beta(k)|^2 \left\{\left(\mu\frac{k}{|k|}, \frac{k}{|k|}\right)\frac{k}{|k|}\cdot\frac{k}{|k|} - \left(\mu\frac{k}{|k|}\right)\cdot\frac{k}{|k|}\right\}, \quad (3.2)$$

where $|k| = (\sum_{i=1}^N k_i^2)^{1/2}$ and for a, b in $\mathbb{R}^N a \cdot b = \dfrac{a\otimes b + b\otimes a}{2}$.

The proof of Theorem 3.0 is along the lines of **[8]** and can also be found in **[6, 8]**:

Proof. As in **[6, 8]** the Hashin–Shtrikman variational principle **[4]** is derived from (1.2). Let γ be a positive real number less than α. We add and subtract $2\gamma(e_{ij}(\phi) + \xi_{ij})^2$ on the right-hand side of (1.2) to obtain

$$(A^0\xi, \xi) = \inf_\phi \int_Q \{2(a(y)-\gamma)\,|\xi + e(\phi)|^2 + 2\gamma\,|e(\phi) + \xi|^2\}\,dy. \quad (3.3)$$

Let σ be a square integrable Q-periodic $N\times N$ symmetric trace-free tensor field, then

$$(2a(y) - 2\gamma)\,|\xi + e(\phi)|^2 \geqq 2(e(\phi) + \xi, \sigma) - (2a(y) - 2\gamma)^{-1}|\sigma|^2 \quad (3.4)$$

almost everywhere in Q. Substituting (3.4) into (3.3) and integrating by parts gives

$$(A^0\xi, \xi) \geqq \int_Q \{2(\sigma, \xi) - (2a - 2\gamma)^{-1} |\sigma|^2 + 2\gamma |\xi|^2\} \, dy$$

$$+ \inf_{\phi} \int_Q \{2(\sigma, e(\phi)) + 2\gamma |e(\phi)|^2\} \, dy.$$

Fixing σ the best choice of ϕ solves

$$\left.\begin{array}{r} \gamma\Delta\phi^* + \nabla P^* = -\operatorname{div} \sigma \\ \nabla \cdot \phi^* = 0 \end{array}\right\} \text{ in } Q, \tag{3.5}$$

$$\phi^* \text{ is } Q\text{-periodic.}$$

Substituting ϕ^* given by (3.5) for ϕ and rearranging terms gives the Hashin-Shtrikman variational principle

$$((A^0 - 2\gamma)\xi, \xi) \geqq \int_Q \{2(\sigma, \xi) - (2a - 2\gamma)^{-1} |\sigma|^2 + (\sigma, e(\phi^*))\} \, dy. \tag{3.6}$$

We make the choice $\sigma = \chi_\beta \mu$ and pass to the limit as γ tends to α in (3.6) to obtain

$$((A^0 - 2\gamma)\xi, \xi) \geqq 2(1 - \theta)(\mu, \xi) - (\beta - \alpha)^{-1} \frac{(1 - \theta)}{2} |\mu|^2 + \int_Q (\chi_\beta \mu, e(\phi^*)) \, dy. \tag{3.7}$$

To complete the derivation of (3.0) we show that

$$\int_Q (\chi_\beta \mu, e(\phi^*)) \, dy = (F_\beta \mu, \mu).$$

Since ϕ^*, $\sigma = \chi_\beta \mu$, and P^* are Q-periodic, they have a Fourier series expansion

$$\phi^*(y) = \sum_{k \in Z^N} \hat{\phi}^*(k) e^{2\pi i k.y},$$

$$\chi_\beta(y)\mu = \mu\left(\sum_{k \in Z^N} \hat{\chi}_\beta(k) e^{2\pi i k.y}\right),$$

and

$$P^*(y) = \sum_{k \in Z^N} \hat{P}^*(k) e^{2\pi i k.y}.$$

The problem (3.5) determines ϕ^* up to an additive constant so we may take

$$\hat{\phi}^*(0) = 0.$$

From equation (3.5) we deduce for each $k \neq 0$ that

$$\hat{P}^*(k) = -\left(\mu \frac{k}{|k|}, \frac{k}{|k|}\right) \hat{\chi}_\beta(k)$$

and

$$\hat{\phi}^*(k) = \frac{i}{2\pi\alpha} \left(\hat{\chi}_\beta(k) \, \mu \frac{k}{|k|} + \hat{P}^*(k) \frac{k}{|k|}\right).$$

It follows that for $k \neq 0$

$$\hat{e}(\phi^*)(k) = -\frac{1}{\alpha}\hat{\chi}_\beta(k)\left\{\left(\mu\frac{k}{|k|}\right)\cdot\frac{k}{|k|} - \left(\mu\frac{k}{|k|}, \frac{k}{|k|}\right)\frac{k}{|k|}\cdot\frac{k}{|k|}\right\}, \tag{3.8}$$

Hence, from (3.1), (3.8) and the Plancherel equality it is evident that

$$\int_Q (\chi_\beta\mu, e(\phi^*))\,dy = \sum_{k\neq 0}(\hat{\chi}_\beta(k)\,\mu, \hat{e}(\phi^*)(k)) = (F_\beta\mu, \mu),$$

which together with (3.7) completes the proof of Theorem 3.0.

Theorem 3.0 is used to obtain an inequality between symmetric mappings on $T_{N\times N}$. By minimising the left-hand side of (3.0) over all ξ for μ fixed, we obtain the minimiser

$$\xi = (1-\theta)(A^0 - 2\alpha I)^{-1}\mu,$$

and substitution into (3.0) yields

$$((A^0 - 2\alpha I)^{-1}\mu, \mu) \leqq \frac{(\beta-\alpha)^{-1}}{2(1-\theta)}|\mu|^2 - \frac{(F_\beta\mu, \mu)}{(1-\theta)^2}.$$

This gives the following relation between symmetric mappings on $T_{N\times N}$

$$(A^0 - 2\alpha I)^{-1} \leqq \frac{(\beta-\alpha)^{-1}}{2(1-\theta)}I - \frac{1}{(1-\theta)^2}F_\beta. \tag{3.9}$$

Remark 3.1. We stress that the above arguments and formulae hold for any dimension $N \geqq 2$.

The lower eigenvalue bound is derived by computing the trace of the right-hand side in (3.9). Indeed, the traces of the symmetric mappings I and F_β are computed using the orthonormal basis of $T_{N\times N}$ given by

$$\delta^i \quad i = 1, \ldots, N-1,$$

$$e^{rs} \quad \begin{array}{l} r = 1, \ldots, N-1, \\ N \geqq s > r \end{array}$$

where

$$\delta^i = (i^{-1}+1)^{-\frac{1}{2}}\left\{\sum_{k=1}^{i} i^{-1}e_k \otimes e_k - e_{i+1}\otimes e_{i+1}\right\},$$

$$e^{rs} = (e_r \otimes e_s + e_s \otimes e_r)/\sqrt{2}, \tag{3.10}$$

and

$$e_r \in \mathbb{R}^N,\ e_r = \Big(\underset{r}{\xrightarrow{0,\ldots,0,1}}, 0, \ldots, 0\Big).$$

Since there are $N(N+1)/2-1$ basis elements in $T_{N\times N}$, the trace of I is $N(N+1)/2-1$. The trace of F_β is given by the following lemma.

LEMMA 3.2. *The trace of* F_β *is* $-\dfrac{\theta(1-\theta)(N-1)}{2\alpha}$.

Proof. The following identity is used to compute the trace:

$$\operatorname{tr}(F_\beta)=\sum_{i=1}^{N-1}(F_\beta\delta^i,\delta^i)+\sum_{\substack{r=1\\ r<s\leqq N}}^{N-1}(F_\beta e^{rs},e^{rs}). \tag{3.11}$$

Use of (3.2) and (3.10) in (3.11) yields

$$\operatorname{tr}(F_\beta)=\sum_{k\neq 0}\frac{|\hat{\chi}_\beta(k)|^2}{\alpha}\left\{S_1^N\left(\frac{k}{|k|}\right)+S_2^N\left(\frac{k}{|k|}\right)\right\}, \tag{3.12}$$

where

$$S_1^N\left(\frac{k}{|k|}\right)=-\sum_{\substack{r=1\\ r<s\leqq N}}^{N-1}\frac{k_r^2+k_s^2}{2|k|^2}$$

and

$$S_2^N=A^N\left(\frac{k}{|k|}\right)+B^N\left(\frac{k}{|k|}\right),$$

where

$$A^N\left(\frac{k}{|k|}\right)=\sum_{\substack{r=1\\ r<s\leqq N}}^{N-1}2\frac{k_r^2k_s^2}{|k|^4},$$

$$B^N\left(\frac{k}{|k|}\right)=\sum_{i=1}^{N-1}\frac{i}{i+1}\left(\frac{1}{i}\left(\sum_{m=1}^{i}\frac{k_m^2}{|k|^2}\right)-\frac{k_{i+1}^2}{|k|^2}\right)^2-\sum_{i=1}^{N-1}\frac{i}{i+1}\left(\frac{1}{i^2}\left(\sum_{m=1}^{i}\frac{k_m^2}{|k|^2}\right)+\frac{k_{i+1}^2}{|k|^2}\right).$$

We prove that for all k in Z^N

$$S_1^N\left(\frac{k}{|k|}\right)=-\frac{N-1}{2} \tag{3.13}$$

and

$$S_2^N\left(\frac{k}{|k|}\right)=0. \tag{3.14}$$

Since $k/|k|$ is a unit vector it is sufficient to establish (3.13) and (3.14) for all unit vectors n in $\mathbb{R}^N$.

We prove (3.13) by induction. For any unit vector n in $\mathbb{R}^2$ we have

$$S_1^2(n)=-\frac{1}{2}.$$

We suppose for any unit vector n in $\mathbb{R}^N$ that $S_1^N(n)=-(N-1)/2$ and consider the sum $S_1^{N+1}(b)$ where b is a unit vector in $\mathbb{R}^{N+1}$. Then $S_1^{N+1}(b)$ is written

$$S_1^{N+1}(b)=-\sum_{\substack{s=2\\ 1\leqq r<s}}^{N-1}\frac{b_r^2+b_s^2}{2}-\sum_{r=1}^{N-1}\frac{b_{N+1}^2+b_N^2+b_r^2}{2}-\sum_{i=1}^{N+1}\frac{b_i^2}{2}. \tag{3.15}$$

We define the unit vector v in $\mathbb{R}^N$ by

$$v_i^2=b_i^2 \quad i=1,\ldots,N-1 \tag{3.16}$$

and

$$v_N^2 = b_N^2 + b_{N+1}^2.$$

Substitution of (3.16) in the right-hand side of (3.15) and application of the induction hypothesis gives

$$S_1^{N+1}(b) = -\frac{N-1}{2} - \frac{1}{2},$$

thus (3.13) is proved. We prove (3.14) by showing that $B^N(n) = -A^N(n)$ for any unit vector n in $\mathbb{R}^N$. The proof is by induction. For $N=2$ the identity is trivial. Suppose for any unit vector n in $\mathbb{R}^N$ that $B^N(n) = -A^N(n)$ and consider $B^{N+1}(b)$ where b is a unit vector in $\mathbb{R}^{N+1}$. We make use of the identity $|b|=1$ and write $B^{N+1}(b)$ as

$$\begin{aligned} B^{N+1}(b) = \sum_{i=1}^{N-2} \frac{i}{1+i} \left\{ \left(\frac{1}{i} \left(\sum_{m=1}^{i} b_m^2 \right) - b_{i+1}^2 \right)^2 - \frac{1}{i^2} \left(\sum_{m=1}^{i} b_m^2 \right) - b_{i+1}^2 \right\} \\ + \frac{(1 - b_{N+1}^2 - b_N^2)}{N(N-1)} (-b_{N+1}^2 - b_N^2) - \frac{2b_N^2(1 - b_{N+1}^2 - b_N^2)}{N} \\ - \frac{(N-1)b_N^2(1-b_N^2)}{N} - \frac{(N+1)b_{N+1}^2(1-b_{N+1}^2)}{N}. \end{aligned} \tag{3.17}$$

We write $2\,(b_{N+1}^2 b_N^2)$ as

$$2\left(\frac{1}{N} + \frac{N-1}{N}\right) b_{N+1}^2 b_N^2 \tag{3.18}$$

and add and subtract (3.18) to the right-hand side of (3.17) to obtain

$$\begin{aligned} B^{N+1}(b) = \sum_{i=1}^{N-2} \frac{i}{i+1} \left\{ \left(\frac{1}{i} \left(\sum_{m=1}^{i} b_m^2 \right) - b_{i+1}^2 \right)^2 - \frac{1}{i^2} \left(\sum_{m=1}^{i} b_m^2 \right) - b_{i+1}^2 \right\} \\ + \frac{(1 - b_{N+1}^2 - b_N^2)}{N(N-1)} (-b_{N+1}^2 - b_N^2) - \frac{2(b_{N+1}^2 + b_N^2)(1 - b_{N+1}^2 - b_N^2)}{N} \\ - \frac{(N-1)(b_{N+1}^2 + b_N^2)(1 - b_{N+1}^2 - b_N^2)}{N} - 2b_{N+1}^2 b_N^2. \end{aligned} \tag{3.19}$$

We define the unit vector v in $\mathbb{R}^N$ by

$$v_i^2 = b_i^2 \quad i = 1, \ldots, N-1 \tag{3.20}$$

and

$$v_N^2 = b_N^2 + b_{N+1}^2.$$

Substitution of (3.20) in the right-hand side of (3.19) and application of the induction hypothesis yields

$$B^{N+1}(b) = -A^N(v) - 2b_{N+1}^2 b_N^2. \tag{3.21}$$

Substitution of b for v in (3.21) yields

$$B^{N+1}(b) = -A^{N+1}(b);$$

thus (3.14) is proved. Finally, Plancherel's equality gives

$$\sum_{k\neq 0} |\hat{\chi}_\beta(k)|^2 = \int_Q \{\chi_\beta(y) - (1-\theta)\}^2\, dy$$
$$= \theta(1-\theta),$$

therefore from (3.12), (3.13), and (3.14) we obtain

$$\operatorname{tr}(F_\beta) = -\frac{\theta(1-\theta)(N-1)}{2\alpha}.$$

Taking the trace of (3.9) yields

$$\operatorname{tr}\{(A^0 - 2\alpha I)^{-1}\} \leqq \frac{(\beta-\alpha)^{-1}}{2(1-\theta)}\left(N\frac{(N+1)}{2} - 1\right) + \frac{\theta(N-1)}{2\alpha(1-\theta)}. \tag{3.22}$$

A straightforward calculation shows that (3.22) is identical to (2.1). The upper bound (2.2) is computed using similar methods.

4. Optimality of the eigenvalue bounds for two-dimensional homogenised incompressible elasticity

The optimality proof of the bounds (2.1) and (2.2) for two dimensions proceeds in two steps: It is first shown that the upper and lower eigenvalue bounds are achieved by H-converging sequences of rank-2 laminar mixtures with volume fractions θ and $(1-\theta)$. These extremal rank-2 mixtures are then used to construct H-convergent sequences of rank-3 laminar mixtures that achieve all points inside the region R^2_θ. Once this is done optimality is proved by the following remarks.

Remark 4.0. Let U_2 be the space of all symmetric transforms on $T_{2\times 2}$. If we can exhibit one tensor in $P_\theta\{\alpha, \beta\}$ with eigenvalues λ_1, λ_2 in R^2_θ then all tensors in U_2 with eigenvalues λ_1 and λ_2 are elements of $P_\theta\{\alpha, \beta\}$. To demonstrate this remark we note that it is easily shown for the set U_2 that if tensors D and C in U^2 have the same eigenvalues then there exists an orthogonal matrix q such that

$$C_{ijkl} = D_{mnop} q_{mi} q_{nj} q_{ok} q_{pl}. \tag{4.0}$$

Remark 4.1. Given a sequence of periodic mixtures with microstructure $\chi^1_\alpha(x)$ and effective elasticity C^1 it is possible to construct a sequence of mixtures with microstructure χ^2_α and effective elasticity C^2 such that

$$C^2_{ijkl} = C^1_{mnop} q_{mi} q_{nj} q_{ok} q_{pl}$$

by letting

$$\chi^2_\alpha(y) = \chi^1_\alpha(qy), \quad \text{where} \quad qq^T = 1.$$

This remark follows by changing coordinates $\tilde{y} = qy$ in (1.2). We demonstrate Remark 4.0. Suppose that for λ_1, λ_2 in R^2_θ, a sequence of periodic mixtures with volume fractions θ, $(1-\theta)$ is found such that the effective tensor has eigenvalues

λ_1, λ_2. It is evident from (4.0) and Remark 4.1 that every tensor in U_2 with eigenvalues λ_1, λ_2 can be constructed by rotating and reflecting the microstructure through an orthogonal change of coordinates $\bar{y} = qy$.

Remark 4.2. All points in R^2_θ are achieved by sequences of rank $\leqq 3$ laminar mixtures. Although these mixtures are not periodic their effective tensors can be approximated arbitrarily well by those associated with periodic mixtures having volume fractions θ, $1-\theta$.

We start by giving a general formula for the effective elasticity of an H-convergent sequence of laminar mixtures of two anisotropic incompressible elastic materials. Our formula is a modified version of [**3**, Theorem 4.1] which applies to the compressible case.

Consider two incompressible but not necessarily isotropic materials with elasticities A^1 and A^2. It is assumed that A^1, A^2 are symmetric mappings on $T_{N\times N}$ and $2\alpha \leqq A^i \leqq 2\beta$, $i = 1,2$.

THEOREM 4.3. *Let A^1 and A^2 be as above; consider a layered composite in which A^1 is present with volume fraction p and A^2 with volume fraction $1-p$. Let $e \in \mathbb{R}^N$ be the unit vector perpendicular to the layers and let χ be a periodic step function on $\mathbb{R}$ with period 1 such that*

$$p = \int_0^1 \chi(t)\, dt.$$

Then the sequence of tensors

$$A^\varepsilon(x) = A^1\chi\left(\frac{x\cdot e}{\varepsilon}\right) + A^2\left(1 - \chi\left(\frac{x\cdot e}{\varepsilon}\right)\right) \tag{4.1}$$

H-converges to A^0 given by

$$A^0\mu = A^1\mu = A^2\mu \textit{ for } \mu \textit{ in } \operatorname{Ker}(A^2 - A^1), \tag{4.2}$$

$$(A^0 - A^1)^{-1}\mu = \frac{(A^2-A^1)^{-1}}{1-p}\mu + \frac{p}{1-p}\{e\cdot q(e)\{\mu e - (\mu e, e)e\}\} \tag{4.3}$$

for μ in $\operatorname{Ker}(A^2-A^1)^\perp$, *where the symmetric linear mapping $q(e)$ is defined by*

$$q(e)^{-1}\{a\} = \{A^1(e\cdot a)\}e - (\{A^1(e\cdot a)\}e, e)e \tag{4.4}$$

for all a in $\mathbb{R}^N$ perpendicular to e.

Remark 4.4. For a unit vector e, $q(e)$ is a well-defined symmetric map on the space V^N of all vectors in $\mathbb{R}^N$ orthogonal to e. This follows from the fact that A^1 is a symmetric map on $T_{N\times N}$, $A^1 > 2\alpha$ and $e\cdot a$ is in $T_{N\times N}$ for a in V^N. Indeed,

$$(q(e)^{-1}\{a\}, a) = (\{A^1(e\cdot a)\}e, a) \geqq \alpha\,|a|^2 \quad \text{for} \quad a \text{ in } V^N, \tag{4.5}$$

thus $q(e)^{-1}$ is invertible on V^N.

If we suppose that one material is isotropic, i.e. $A^1 = 2\lambda I$, we obtain the following corollary to Theorem 4.3:

COROLLARY 4.5. *Given that* $A^1 = 2\lambda^1 I$ *in Theorem* 4.3, *then* A^0 *is given by*

$$A^0\mu = A^2\mu = 2\lambda^1\mu \text{ for } \mu \text{ in } \operatorname{Ker}(A^2 - 2\lambda^1 I), \tag{4.6}$$

$$(A^0 - 2\lambda^1 I)^{-1} = \frac{(A^2 - 2\lambda^1 I)^{-1}}{1-p}\mu + \frac{p}{\lambda^1(1-p)}\{(\mu e)\cdot e - (\mu e, e)e \cdot e\} \tag{4.7}$$

for μ *in* $\operatorname{Ker}(A^2 - 2\lambda^1 I)^{\perp}$.

Proof. We compute $q^{-1}(e)$ for $A^1 = 2\lambda^1 I$. From (4.4) we obtain

$$q(e)^{-1}\{a\} = \lambda^1 a.$$

Therefore

$$q(e)\{a\} = a/\lambda^1 \tag{4.8}$$

and the corollary follows from (4.3) and (4.8).

Remark 4.6. Corollary 4.5 can be used iteratively as in **[3]** to obtain the effective tensors for multiply-layered materials (see equations (4.25), (4.27)).

Remark 4.7. Equation (4.7) can be obtained directly from **[3**, equation (4.6)**]** by letting the bulk modulus K^1 tend to $+\infty$ in that formula.

We outline the proof of Theorem 4.3. The proof is based on the proof of **[3**, Theorem 4.1**]**.

Proof of Theorem 4.3. For any choice of layer direction e in S^{N-1} there exists a cubic lattice in which the layer structure is periodic, therefore one may work within the context of periodic homogenisation. From **[14]** the H-limit of the sequence (4.1) is given by

$$A^0\xi = \int_Q a(y)(e(\phi^{\xi}) + \xi)\, dy, \tag{4.9}$$

where

$$a(y) = A^1\chi(y\cdot e) + A^2(1 - \chi(y \cdot e)),$$

and ϕ^{ξ} solves the "cell" problem

$$\left.\begin{aligned} \partial_{y_j}(a(y)[e_{ij}(\phi^{\xi}) + \xi_{ij}]) + \partial_{y_i}P^{\xi} = 0 \\ \operatorname{div}\phi^{\xi} = 0 \end{aligned}\right\} \quad \text{on } \mathbb{R}^N, \tag{4.10}$$

$$\phi^{\xi} \text{ is } Q\text{-periodic, } \xi \text{ is in } T_{N\times N}.$$

The solution of (4.10) is linear and the piecewise constant strain takes values in $T_{N\times N}$ such that

$$e(\phi^{\xi}) + \xi = \chi(y\cdot e)\xi_{A1} + (1 - \chi(y\cdot e))\xi_{A2}.$$

The pressure p^{ξ} is piecewise constant and is given by

$$p^{\xi} = \chi(y\cdot e)P_{A1} + (1-\chi(y\cdot e))P_{A2}.$$

Thus from (4.9) we have

$$A^0\xi = pA^1\xi_{A1} + (1-p)A^2\xi_{A2}. \tag{4.11}$$

Integration of $e(\phi^{\xi}) + \xi$ over Q gives

$$\xi = p\xi_{A1} + (1-p)\xi_{A2}. \tag{4.12}$$

Lastly, we have

$$(A^2\xi_{A2} - A^1\xi_{A1})e = \Gamma e, \quad \Gamma = P_{A1} - P_{A2} \tag{4.13}$$

and

$$\xi_{A1} = \xi_{A2} + e \cdot a \quad \text{for} \quad a \text{ in } \mathbb{R}^N, \tag{4.14}$$

such that

$$(e, a) = 0. \tag{4.15}$$

Equation (4.13) follows from the continuity of the traction at the layer interface; (4.14) is the consistency condition for the existence of a deformation with specified piecewise constant strain; and (4.15) follows from the condition that ξ_{A1} and ξ_{A2} be trace-free.

From (4.13) and (4.14) we obtain

$$(A^2 - A^1)\xi_{A2}e = \{A^1(e \cdot a)\}e + \Gamma e. \tag{4.16}$$

Now define μ by

$$\mu = (A^2 - A^1)\xi_{A2} \tag{4.17}$$

and multiply (4.16) by e to obtain

$$\Gamma = (\mu e, e) - (\{A^1(e \cdot a)\}e, e).$$

Therefore from (4.16) we obtain

$$\mu e - (\mu e, e)e = \{A^1(e \cdot a)\}e - (\{A^1(e \cdot a)\}e, e)e = q^{-1}(e)\{a\},$$

hence

$$a = q(e)\{\mu e - (\mu e, e)e\}, \tag{4.18}$$

where $\mu e - (\mu e, e)\, e$ is in V^N. We see from (4.14), (4.17) and (4.18) that ξ_{A1} and a are uniquely determined functions of ξ_{A2} where ξ_{A2} is an arbitrary element of $T_{N\times N}$.

From (4.11) and (4.12) we obtain

$$(1-p)(A^2 - A^1)\xi_{A2} = (A^0 - A^1)(p\xi_{A1} + (1-p)\xi_{A2}). \tag{4.19}$$

Use of equations (4.14), (4.17) and (4.18) in (4.19) gives

$$(A^0 - A^1)[\xi_{A2} + p\, e \cdot q(e)\{\mu e - (\mu e, e)e\}] = (1-p)\mu. \tag{4.20}$$

We obtain (4.2) and (4.3) by examining the two cases: ξ_{A2} lies in Ker $(A^2 - A^1)$ or ξ_{A2} lies in Ker $(A^2 - A^1)^{\perp}$. If ξ_{A2} belongs to Ker $(A^2 - A^1)$ then $\mu = 0$ and from (4.17) and (4.20) we have

$$A^0\xi_{A2} = A^1\xi_{A2} = A^2\xi_{A2}$$

which proves (4.2). If ξ_{A2} belongs to Ker $(A^2 - A^1)^{\perp}$, we let $(A^2 - A^1)^{-1}$ denote the inverse of $A^2 - A^1$ restricted to Ker $(A^2 - A^1)^{\perp}$. Then μ given by (4.17) is in

$\mathrm{Ker}\,(A^2-A^1)^{\perp}$ and (4.20) is written

$$(A^0-A^1)S^e\mu=\mu, \tag{4.21}$$

where

$$S^e\mu=\frac{(A^2-A^1)^{-1}}{1-p}\mu+\frac{p}{1-p}e\cdot q(e)\{\mu e-(\mu e,e)e\}. \tag{4.22}$$

In view of equations (4.21) and (4.22), equation (4.3) is proved if we can show that $(S^e)^{-1}$ exists for μ in $\mathrm{Ker}\,(A^2-A^1)^{\perp}$. Indeed, we show that $\mathrm{Ker}\,(S^e)=0$ for μ in $\mathrm{Ker}\,(A^2-A^1)^{\perp}$. From (4.4), (4.5) and as $A_2>0$ we write

$$\begin{aligned}(S^e\mu,\,&(A_2-A_1)[e\cdot q(e)\{\mu e-(\mu e,e)e\}])\\&\geqq(\{\mu e-(\mu e,e)e\},\,q(e)\{\mu e-(\mu e,e)e\})\geqq 0.\end{aligned} \tag{4.23}$$

If μ is in $\mathrm{Ker}\,(S^e)$ then from (4.23)

$$\mu e-(\mu e,e)e=0$$

as $q(e)$ is positive definite from (4.5). Then

$$q(e)\{\mu e-(\mu e,e)e\}=0$$

and (4.22) yields

$$(A^2-A^1)^{-1}\mu=0, \tag{4.24}$$

but as $\mu\in\mathrm{Ker}\,(A^2-A^1)^{\perp}$ we conclude from (4.24) that $\mu=0$.

In the following, we show how to attain all points in the set R^2_θ using rank $\leqq 3$ laminar mixtures. We choose $\lambda_1=\alpha$ and $A^2=2\beta I$ in formula (4.7) and use (4.7) iteratively to derive the following formula for the rank-2 iterated laminar mixture with effective tensor A^0 given by

$$\begin{aligned}(A^0-2\alpha I)^{-1}\mu=\frac{(\beta-\alpha)^{-1}}{2(1-\theta)}\mu+\frac{1}{\alpha}\frac{1}{(1-\theta)}\{&\rho_1[(\mu a)\cdot a-a\cdot a(\mu a,a)]\\&+(\theta-\rho_1)[(\mu b)\cdot b-b\cdot b(\mu b,b)]\},\end{aligned} \tag{4.25}$$

where $a=(a_1,a_2)$, $b=(b_1,b_2)$ are unit vectors orthogonal to layer directions, ρ_1 is the volume fraction of α-material in the first iterate, and θ is the total volume fraction of α-material in the mixture. We use (4.25) to construct a rank-2 iterated laminate with eigenvalues on the lower boundary of R^2_θ (equation (2.1) for $N=2$). Upon choosing layer directions $a=(1,0)$ and $b=(1/\sqrt{2},1/\sqrt{2})$ in (4.25) the matrix of A^0 relative to the basis in $T_{2\times 2}$

$$\delta^1=\begin{pmatrix}1/\sqrt{2} & 0\\ 0 & -1/\sqrt{2}\end{pmatrix},\qquad \delta^2=\begin{pmatrix}0 & 1/\sqrt{2}\\ 1/\sqrt{2} & 0\end{pmatrix}$$

is written

$$\begin{bmatrix}2\alpha+\dfrac{2\alpha(\beta-\alpha)(1-\theta)}{\alpha+(\beta-\alpha)(\theta-\rho_1)} & 0\\ 0 & 2\alpha+\dfrac{2\alpha(\beta-\alpha)(1-\theta)}{\alpha+(\beta-\alpha)\rho_1}\end{bmatrix}. \tag{4.26}$$

We see from (4.26) that as ρ_1 ranges from 0 to θ the eigenvalues of the rank-2 laminates (Fig. 1) sweep out the lower bound ((2.1) for $N=2$).

Remark 4.8. By choosing $\lambda^1=\beta$ and $A^2=2\alpha I$ in (4.7) and iterating, we obtain the formula for the rank-2 iterated laminate

$$(2\beta I-A^0)^{-1}\mu=\frac{(\beta-\alpha)^{-1}}{2\theta}\mu-\frac{1}{\theta\beta}\{(1-\rho_1)[(\mu a)\cdot a-a\cdot a(\mu a,a)] + (\rho_1-\theta)[(\mu b)\cdot b-b\cdot b(\mu b,b)]\}, \quad (4.27)$$

where $1-\rho_1$ is the volume fraction of the β-material in the first iterate and $1-\theta$ is the total volume fraction of β-material in the mixture. Upon choosing layer directions $a=(1,0)$, $b=(1/\sqrt{2},1/\sqrt{2})$ we obtain rank-2 laminates sweeping out the upper bound ((2.2), for $N=2$) of R_θ^2 as $1-\rho_1$ ranges from 0 to $1-\theta$.

Lastly, to obtain effective elasticities with eigenvalues on the interior of R_θ^2 we construct effective tensors by layering a rank-2 laminate A_U^0 having eigenvalues on the upper boundary of R_θ^2 with a rank-2 laminate A_L^0 having eigenvalues on the lower boundary. The effective elasticities A_U^0 and A_L^0 are chosen such that

$$A_U^0=\lambda_U\delta^1\otimes\delta^1+\lambda\delta^2\otimes\delta^2,$$
$$A_L^0=\lambda_L\delta^1\otimes\delta^1+\lambda\delta^2\otimes\delta^2.$$

Consider a layered mixture with layers of A_L^0 and A_U^0 in proportions θ, $1-\theta$ perpendicular to the $(1,0)$ direction. We observe that δ^2 lies in $\operatorname{Ker}(A_U^0-A_L^0)$ and therefore, from Theorem 4.3,

$$A^0\delta^2=A_U^0\delta^2=A_L^0\delta^2=\lambda\delta^2. \quad (4.28)$$

The matrix δ^1 lies in $\operatorname{Ker}(A_U^0-A_L^0)^\perp$ and we compute

$$\delta^1e-(\delta^1e,e)e \quad \text{for} \quad e=(1,0)$$

to obtain

$$\delta^1e-(\delta^1e,e)e=0.$$

Therefore, as

$$q(e)\{\delta^1e-(\delta^1e,e)e\}=0,$$

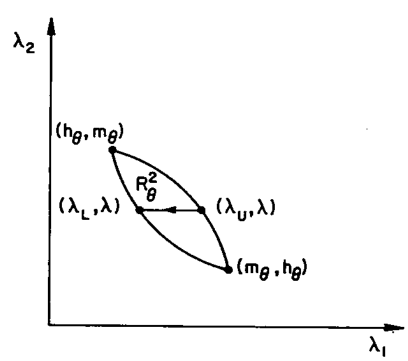

Figure 2

we see from (4.3) that

$$A^0\delta^1 = (\lambda_L\theta + (1-\theta)\lambda_U)\delta^1. \tag{4.29}$$

Equations (4.28) and (4.29) yield

$$A^0 = (\lambda_L\theta + (1-\theta)\lambda_U)\delta^1 \otimes \delta^1 + \lambda\delta^2 \otimes \delta^2.$$

Therefore, as θ ranges from 0 to 1, the eigenvalues of A^0 sweep out all points between (λ_L, λ) and (λ_U, λ). (See Figure 2.)

5. Concluding remarks

The eigenvalue bounds on the effective elasticity in two dimensions resemble the eigenvalue bounds obtained for the two-dimensional conductivity problem established in **[21]** and independently in **[9]**. For both cases extremal mixtures achieving the upper and lower bounds are rank-2 laminates. However, the choices of layer directions are different for each case. In the case of conductivity the two layer directions are perpendicular to each other, while for incompressible elasticity the layer directions differ by 45 degrees.

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