Saturation of the even-order bounds on effective elastic moduli by finite-rank laminates

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Sufficient conditions on the phase geometry for the reduction of even-order bounds to bounds of second order are derived. It is found that finite-rank laminates satisfy the sufficient conditions asymptotically as the length scale of the laminar microstructure goes to zero. This result is used to show that the effective elastic tensors of finite-rank laminates saturate the even-order bounds in the fine-scale limit.

I. INTRODUCTION

There has been much recent progress in the development of bounds on the effective moduli of two-phase composites. These bounds are written in terms of the known statistical properties of the composite (see Brown,[1] Beran,[2] Bergman,[3] Milton,[4] and Torquato[5]). The statistical properties of composites are usually given in terms of structural parameters that depend upon n-point correlation functions. There now exists means of computing the structural paramters and correlation functions for nontrivial models.[6,7]

We consider the even-order bounds on the effective elastic moduli of a two-phase composite originally derived by Kröner.[8] These bounds incorporate structural parameters that depend upon n-point correlation functions up to order $2N$. In this paper we obtain sufficient conditions on the phase geometry for the reduction of bounds of order $2N$ to bounds of order 2 (see Theorem 2.1).

For a two-phase composite we denote the ratio between the fine structure and the macroscopic dimensions of the composite by ϵ. The $\epsilon = 0$ limit of a sequence of composites is referred to as the fine-scale limit of the sequence. We remark that the volume fraction of the inhomogeneities need not go to zero in this limit.

We consider a special class of effective materials called finite-rank laminates.[9–11] These effective materials come from the theory of homogenization of elliptic operators.[12,13] They correspond not to a particular phase geometry but to the fine-scale limit of a family of microstructures. We expand the effective elasticity tensor of a laminate in powers of the anisotropy[8] to find that in the fine-scale limit it satisfies the conditions sufficient for the reduction of even-order bounds to bounds of second order (see Theorem 4.1). Using this fact a simple proof shows that laminates saturate the even-order bounds in the fine-scale limit (see Theorems 4.2 and 4.3).

Important previous work has been done establishing the optimality of even-order bounds in the context of conductivity. More generally, the effective conductivity tensor of a mixture of two isotropic phases has well-known analytic properties as a function of its component conductivities.[14] Milton[14] has shown in two dimensions that every matrix-valued rational function compatible with these analytic properties is the effective conductivity function of some finite-rank laminar composite. The saturation of even-order bounds in two dimensions follows from Milton's results.

II. EVEN-ORDER BOUNDS AND THEIR REDUCTION TO BOUNDS OF SECOND ORDER

We consider an elastic composite material made from two isotropic elastic phases described by Lame shear moduli μ_i, $i = 1,2$ and bulk moduli κ_i, $i = 1,2$. The phases are assumed to be well ordered, i.e., $\mu_1 < \mu_2$, $\kappa_1 < \kappa_2$. The composite is treated as a periodic material with unit period cell Q in R^3. This hypothesis is general provided that the length scale of the inhomogeneities is much smaller than unity.[15] Furthermore, we suppose that the composite is statistically homogeneous, enabling us to write ensemble averages as volume averages. We remark that although the phases are isotropic, the resulting composite can be anisotropic.

The phase geometry of the composite is given exactly by the characteristic functions of materials 1 and 2

$$\chi_2 = \begin{cases} 1 & \text{if } x \text{ is in material 2} \\ 0 & \text{otherwise}, \end{cases}$$
$$\chi_1 = 1 - \chi_2, \tag{2.1}$$

where

$$\int_Q \chi_1 \, dy = \theta_1, \quad \int_Q \chi_2 \, dy = \theta_2.$$

The elasticity tensor of the composite is defined by the piecewise-constant tensor

$$\mathsf{C}(x) = (1 - \chi_2)\mathsf{C} + \chi_2 \mathsf{C}_2. \tag{2.2}$$

Here

$$\mathsf{C}_i = 2\mu_i \mathsf{I} + (\kappa_i - \tfrac{2}{3}\mu_i) I \otimes I, \quad i = 1,2,$$

where I is the identity on 3×3 symmetric matrices and I is the 3×3 identity matrix.

We suppose that the average strain in a unit period cell of the composite is given by the symmetric 3×3 matrix ϵ. The local strain is given by $e(u) + \epsilon$, where u is the Q-periodic displacement field and

$$e_{ij}(u) = \frac{u_{i,j} + u_{j,i}}{2}. \tag{2.3}$$

The local strain solves

$$\operatorname{div}\{\mathsf{C}(x)[e(u) + \epsilon]\} = 0 \text{ in } Q, \tag{2.4}$$

and the effective elastic tensor C^e is defined by

$$\mathsf{C}^e \epsilon : \epsilon = \langle \mathsf{C}(x)[e(u) + \epsilon] : e(u) + \epsilon \rangle. \tag{2.5}$$

Here and throughout the paper $\langle \ \rangle$ denotes integration over the unit period cell Q.

We introduce the space $\mathscr{E}$ of Q-periodic mean-zero strain fields. The space $\mathscr{E}$ can be written as the sum of two orthogonal subspaces $\mathscr{E}_h$ and $\mathscr{E}_s$, i.e., $\mathscr{E} = \mathscr{E}_h \oplus \mathscr{E}_s$.[16] Here $\mathscr{E}_h$ is the set of strain fields in $\mathscr{E}$ that are derived from a Q-periodic displacement field v that itself is the gradient of some scalar potential, and $\mathscr{E}_s$ is the set of trace-free strain fields. We define the operators $\Gamma_{1,h}$ and $\Gamma_{1,s}$ to be the projections onto $\mathscr{E}_h$ and $\mathscr{E}_s$, respectively.

It is well known from perturbation theory[8] that the fluctuating part of the local strain satisfies the following integral equations:

$$e(u) = -\mathsf{T}^1[\chi_2\delta\mathsf{C}(e(u)+\epsilon)] \tag{2.6}$$

$$e(u) = \mathsf{T}^2[\chi_1\delta\mathsf{C}(e(u)+\epsilon)], \tag{2.7}$$

where $\delta\mathsf{C} = \mathsf{C}_2 - \mathsf{C}_1$ and

$$\mathsf{T}^i = \frac{\frac{3}{2}}{3\kappa_i + 4\mu_i}\Gamma_{1,h} + \frac{1}{2\mu_i}\Gamma_{1,s}, \quad i=1,2. \tag{2.8}$$

In Fourier space the operators T^i are local. Indeed, given a square-integrable Q-periodic matrix field $\sigma = \Sigma_\kappa \hat{\sigma}(\kappa)e^{2\pi i\kappa\cdot x}$, T^1 and T^2 are defined by

$$\mathsf{T}^i\sigma = \sum_{\kappa\neq 0}\hat{\mathsf{T}}^i(\hat{\kappa})\hat{\sigma}(\kappa), \quad i=1,2, \tag{2.9}$$

where $\hat{\kappa} = \kappa/|\kappa|$, and

$$\hat{\mathsf{T}}^i(\hat{\kappa}) = \frac{\frac{3}{2}}{3\kappa_i + 4\mu_i}\hat{\Gamma}_{1,h}(\hat{\kappa}) + \frac{1}{2\mu_i}\hat{\Gamma}_{1,s}(\hat{\kappa}). \tag{2.10}$$

Here

$$\hat{\Gamma}_{1,h}(\hat{\kappa})\hat{\sigma}(\kappa) = 2(\hat{\sigma}(\kappa)\hat{\kappa},\hat{\kappa})\hat{\kappa}\otimes\hat{\kappa} \tag{2.11}$$

and

$$\hat{\Gamma}_{1,s}(\hat{\kappa})\hat{\sigma}(\kappa) = [\hat{\sigma}(\kappa)\hat{\kappa}]\otimes\hat{\kappa} + \hat{\kappa}\otimes[\hat{\sigma}(\kappa)\hat{\kappa}] - 2(\hat{\sigma}(\kappa)\hat{\kappa},\hat{\kappa})\hat{\kappa}\otimes\hat{\kappa}. \tag{2.12}$$

For any Q-periodic square-integrable *polarization* tensor P the classical Hashin–Shtrikman bounds[16–18] on the effective elasticity tensor C^e are given by

$$(\mathsf{C}^e - \mathsf{C}_1)\epsilon:\epsilon \geqslant \sup_P\{2\epsilon:\langle\chi_2 P\rangle - \langle\chi_2 P:(\delta\mathsf{C}^{-1}+\mathsf{T}^1)\chi_2 P\rangle\} \tag{2.13}$$

and

$$(\mathsf{C}_2 - \mathsf{C}^e)\,\epsilon:\epsilon \geqslant \sup_P\{2\epsilon:\langle(1-\chi_2)P\rangle - \langle(1-\chi_2)P:(\delta\mathsf{C}^{-1}-\mathsf{T}^2)\times(1-\chi_2)P\rangle\}. \tag{2.14}$$

One has equality in (2.13) for the choice

$$P = P^* = \chi_2(\delta\mathsf{C})[e(u)+\epsilon]. \tag{2.15}$$

Here $e(u)+\epsilon$ is the actual strain field in the composite, i.e., $e(u)$ solves Eq. (2.4). Similarly equality in (2.14) holds for the choice

$$P = P^* = \chi_1(\delta C)[e(u)+\epsilon]. \tag{2.16}$$

There is great freedom in the choice of polarization tensors appearing in the Hashin–Shtrikman bounds. The fundamental idea is to choose a class of polarization tensors that yield bounds in terms of known statistical information about the composite.[2,8,18,19]

For a sufficiently small perturbation $\delta\mathsf{C}$ we can approximate the optimal polarization P^* given in Eq. (2.15) by expanding the strain field $e(u)+\epsilon$ in a perturbation series taking only the first few terms. Indeed, from Eq. (2.6) we have

$$e(u)+\epsilon = \epsilon + \sum_{i=1}^{\infty}(-1)^i(\mathsf{T}^1\chi_2\delta\mathsf{C})^i\epsilon, \tag{2.17}$$

and the optimal polarization tensor is approximated by

$$P = \chi_2\delta\mathsf{C}\left(\epsilon + \sum_{i=1}^{N-1}(-1)^i(\mathsf{T}^1\chi_2\delta\mathsf{C})^i\epsilon\right). \tag{2.18}$$

This choice of P in Eq. (2.13) provides a lower bound on the effective elastic tensor in terms of n-point correlation functions up to order $2N$. The n-point correlation function is the probability that n points lie for example in material 2. Similar remarks can be made about the upper Hashin–Shtrikman bound.

Motivated by Eq. (2.18) we consider the finite-dimensional class $\underline{\mathscr{C}}$ of polarization tensors given by

$$\underline{\mathscr{C}} = \left\{P \,\middle|\, P = \chi_2\delta C\left(\sum_{i=0}^{N-1}(-1)^i(\mathsf{T}^1\chi_2\delta\mathsf{C})^i\eta^{(i)}\right)\right\}, \tag{2.19}$$

where $\eta^0,\eta^{(1)},\ldots,\eta^{(N-1)}$ are arbitrary symmetric 3×3 matrices. We write out Eq. (2.13) in terms of the polarizations given by Eq. (2.19) to obtain an even-order lower bound of order $2N$:

$$(\mathsf{C}^e - \mathsf{C}_1)\epsilon:\epsilon \geqslant \sup_{P\in\underline{\mathscr{C}}}\{2\epsilon:\langle\chi_2 P\rangle - \langle\chi_2 P:(\delta\mathsf{C}^{-1}+\mathsf{T}^1)\chi_2 P\rangle\} = \sup_{\{\eta^{(0)},\eta^{(1)},\ldots,\eta^{(N-1)}\}}\mathsf{L}^{2N}(\eta^{(0)},\eta^{(1)},\ldots,\eta^{(N-1)}), \tag{2.20}$$

where

$$\mathsf{L}^{2N}(\eta^{(0)},\ldots,\eta^{(N-1)}) = 2\sum_{i=0}^{N-1}\langle\delta\mathsf{C}\chi_2(\mathsf{T}^1\chi_2\delta\mathsf{C})^i\eta^{(i)}\rangle:\epsilon - \left(\sum_{\substack{i=0\\j=0}}^{N-1}\langle\delta\mathsf{C}\chi_2(\mathsf{T}^1\chi_2\delta\mathsf{C})^i\eta^{(i)}:\chi_2(\mathsf{T}^1\chi_2\delta\mathsf{C})^j\eta^{(j)}\rangle + \sum_{\substack{i=0\\j=0}}^{N-1}\langle\delta\mathsf{C}\chi_2(\mathsf{T}^1\chi_2\delta\mathsf{C})^i\eta^{(i)}:\mathsf{T}^1\delta\mathsf{C}\chi_2(\mathsf{T}^1\chi_2\delta\mathsf{C})^j\eta^{(j)}\rangle\right). \tag{2.21}$$

Similarly we consider the class $\overline{\mathscr{C}}$ of polarization tensors given by

$$\overline{\mathscr{C}} = \left\{ P \,\middle|\, P = \chi_1 \delta \mathrm{C} \left(\sum_{i=0}^{N-1} (\mathrm{T}^2 \chi_1 \delta \mathrm{C})^i \eta^{(i)} \right) \right\} \tag{2.22}$$

and obtain from Eq. (2.14) an upper even-order bound of order $2N$:

$$(\mathrm{C}_2 - \mathrm{C}^e)\epsilon : \epsilon \geqslant \sup_{\{\eta^{(0)},\ldots,\eta^{(N-1)}\}} \mathrm{U}^{2N}(\eta^{(0)},\ldots,\eta^{(N-1)}) , \tag{2.23}$$

where

$$\mathrm{U}^{2N}(\eta^{(0)},\ldots,\eta^{(N-1)})$$

$$= 2 \sum_{i=0}^{N-1} \langle \delta \mathrm{C} \chi_1 (\mathrm{T}^2 \chi_1 \delta \mathrm{C})^i \eta^{(i)} \rangle : \epsilon$$

$$- \left(\sum_{\substack{i=0\\ j=0}}^{N-1} \langle \delta \mathrm{C} \chi_1 (\mathrm{T}^2 \chi_1 \delta \mathrm{C})^i \eta^{(i)} : \chi_1 (\mathrm{T}^2 \chi_1 \delta \mathrm{C})^j \eta^{(j)} \rangle - \sum_{\substack{i=0\\ j=0}}^{N-1} \langle \delta \mathrm{C} \chi_1 (\mathrm{T}^2 \chi_1 \delta \mathrm{C})^i \eta^{(i)} : \mathrm{T}^2 \delta \mathrm{C} \chi_1 (\mathrm{T}^2 \chi_1 \delta \mathrm{C})^j \eta^{(j)} \rangle \right). \tag{2.24}$$

When the phase function χ_2 is given by a statistically homogeneous random field the upper and lower even-order bounds contain statistical information given in terms of n-point correlation functions up to order $2N$. We remark that although the perturbation expansion (2.17) holds only for small perturbations $\delta \mathrm{C}$ the even-order bounds apply for all values of the perturbation.

We observe that for the following smaller classes of polarization tensors given by

$$\underline{\mathscr{C}}^2 = \{ P \,|\, P = \chi_2 \delta \mathrm{C} \eta^{(0)} \} \tag{2.25}$$

and

$$\overline{\mathscr{C}}^2 = \{ P \,|\, P = \chi_1 \delta \mathrm{C} \eta^{(0)} \} \tag{2.26}$$

that the even-order bounds (2.20) and (2.24) reduce to the well-known second-order bounds[18–20] given by

$$(\mathrm{C}^e - \mathrm{C}_1)\epsilon : \epsilon \geqslant \sup_{\eta^{(0)}} \mathrm{L}^2(\eta^{(0)}) ,$$

$$\mathrm{L}^2(\eta^{(0)}) = 2\theta_2 \eta^{(0)} : \epsilon - \eta^{(0)} : (\theta_2 \delta \mathrm{C}^{-1} + \langle \chi_2 \mathrm{T}^1 \chi_2 \rangle) \eta^{(0)} \tag{2.27}$$

and

$$(\mathrm{C}_2 - \mathrm{C}^e)\epsilon : \epsilon \geqslant \sup_{\eta^{(0)}} \mathrm{U}^2(\eta^{(0)}) ,$$

$$\mathrm{U}^2(\eta^0) = 2\theta_1 \eta^{(0)} : \epsilon - \eta^0 : (\theta_1 \delta \mathrm{C}^{-1} - \langle \chi_1 \mathrm{T}^2 \chi_1 \rangle) \eta^{(0)} . \tag{2.28}$$

We now present sufficient conditions on the phase geometry such that the even-order bounds algebraically reduce to second-order bounds.

Theorem 2.1. Given the phase functions $\chi_1, \chi_2 = 1 - \chi_1$, then the even-order lower bound of order $2N$ given by Eq. (2.20) reduces algebraically to the second-order bound given by Eq. (2.27) if

$$\langle \chi_2 (\mathrm{T}^1 \chi_2 \delta \mathrm{C})^i \rangle = \frac{(\langle \chi_2 \mathrm{T}^1 \chi_2 \delta \mathrm{C} \rangle)^i}{\theta_2^{i-1}}, \quad i = 1,\ldots,2N-1 \tag{2.29}$$

and the even-order upper bound of order $2N$ given by Eq. (2.24) reduces algebraically to the second-order bound given by Eq. (2.28) if

$$\langle \chi_1 (\mathrm{T}^2 \chi_1 \delta \mathrm{C})^i \rangle = \frac{(\langle \chi_1 \mathrm{T}^2 \chi_1 \delta \mathrm{C} \rangle)^i}{\theta_1^{i-1}}, \quad i = 1,\ldots,2N-1 . \tag{2.30}$$

Proof. We prove the theorem for the even-order lower bound, noting that the proof for the upper bound is identical. The theorem is proved using two lemmas.

Lemma 2.2. Given χ_2, the even-order lower bound (2.20) reduces algebraically to the second-order lower bound (2.27) when

$$\langle \chi_2 P : (\delta \mathrm{C}^{-1} + \mathrm{T}^1) \chi_2 P \rangle$$

$$= \langle \chi_2 P \rangle : \left(\frac{\delta \mathrm{C}^{-1}}{\theta_2} + \frac{\langle \chi_2 \mathrm{T}^1 \chi_2 \rangle}{\theta_2^2} \right) \langle \chi_2 P \rangle \tag{2.31}$$

for all P in $\underline{\mathscr{C}}$.

Proof of Lemma 2.2. If Eq. (2.31) holds then the lower even order bound given by (2.20) is written

$$(\mathrm{C}^e - \mathrm{C}_1)\epsilon : \epsilon \geqslant \sup_{P \in \underline{\mathscr{C}}} \Bigg[2\epsilon : \langle \chi_2 P \rangle - \langle \chi_2 P \rangle : \left(\frac{\delta \mathrm{C}^{-1}}{\theta_2} + \frac{\langle \chi_2 T^1 \chi_2 \rangle}{\theta_2^2} \right) \langle \chi_2 P \rangle \Bigg] . \tag{2.32}$$

As P ranges over all elements in $\underline{\mathscr{C}}$ it is evident that $\langle \chi_2 P \rangle$ ranges over all symmetric 3×3 matrices. Therefore it follows that Eq. (2.32) reduces to the second-order bound (2.27) by replacing $\langle \chi_2 P \rangle$ by $\theta_2 M$ in Eq. (2.32), where M is any symmetric 3×3 matrix. ■

Lemma 2.3. Given χ_2, the equality

$$\langle \chi_2 P : (\delta \mathrm{C}^{-1} + \mathrm{T}^1) \chi_2 P \rangle$$

$$= \langle \chi_2 P \rangle : \left(\frac{\delta \mathrm{C}^{-1}}{\theta_2} + \frac{\langle \chi_2 \mathrm{T}^1 \chi_2 \rangle}{\theta_2^2} \right) \langle \chi_2 P \rangle \tag{2.33}$$

holds for P in $\underline{\mathscr{C}}$ if and only if

$$\langle \chi_2 (\mathsf{T}^1 \chi_2 \delta \mathsf{C})^i \rangle = \frac{(\langle \chi_2 \mathsf{T}^1 \chi_2 \delta \mathsf{C} \rangle)^i}{\theta_2^{i-1}}, \quad i = 1,...,2N-1. \tag{2.34}$$

Proof of Lemma 2.3. We first suppose that Eq. (2.33) holds and deduce Eq. (2.34). We observe from Eq. (2.8) that T^1 is a linear combination of the two projection operators $\Gamma_{1,h}$ and $\Gamma_{1,s}$, therefore T^1 is a self-adjoint operator on the space of Q-periodic square-integrable 3×3 matrix fields with respect to the $[L^2(Q)]^{3\times 3}$ inner product. From this fact we deduce the following useful identities:

$$\langle \chi_2 \delta \mathsf{C} (\mathsf{T}^1 \chi_2 \delta \mathsf{C})^i \eta^{(i)} : \chi_2 (\mathsf{T}^1 \chi_2 \delta \mathsf{C})^j \eta^{(j)} \rangle = \langle \chi_2 \delta \mathsf{C} (\mathsf{T}^1 \chi_2 \delta \mathsf{C})^{i+j} \rangle \eta^{(i)} : \eta^{(j)}, \quad i,j = 1,...,N-1 \tag{2.35}$$

$$\langle \chi_2 \delta \mathsf{C} (\mathsf{T}^1 \chi_2 \delta \mathsf{C})^i \eta^{(i)} : \mathsf{T}^1 \chi_2 \delta \mathsf{C} (\mathsf{T}^1 \chi_2 \delta \mathsf{C})^j \eta^{(j)} \rangle = \langle \chi_2 \delta \mathsf{C} (\mathsf{T}^1 \chi_2 \delta \mathsf{C})^{i+j} \rangle \eta^{(i)} : \eta^{(j)}, \quad i,j = 1,...,N-1 \tag{2.36}$$

and

$$\langle \chi_2 \delta \mathsf{C} (\mathsf{T} \chi_2 \delta \mathsf{C})^j \rangle^T = \langle \chi_2 \delta \mathsf{C} (\mathsf{T} \chi_2 \delta \mathsf{C})^j \rangle, \quad j = 1,...,N-1. \tag{2.37}$$

Expanding P in Eq. (2.33), using the identities (2.35)–(2.37), and noting that Eq. (2.33) holds for all choices of $\eta^{(0)}, \eta^{(1)},...,\eta^{(N-1)}$ yields

$$\langle \chi_2 \delta \mathsf{C} (\mathsf{T}^1 \chi_2 \delta \mathsf{C})^{i+j} \rangle = \frac{1}{\theta_2} \langle \chi_2 \delta \mathsf{C} (\mathsf{T}^1 \chi_2 \delta \mathsf{C})^j \rangle \langle \chi_2 (\mathsf{T}^1 \chi_2 \delta \mathsf{C})^i \rangle \tag{2.38}$$

for $i,j = 0,1,...,N-1$ and

$$\langle \chi_2 \delta \mathsf{C} (\mathsf{T}^1 \chi_2 \delta \mathsf{C})^{i+j+1} \rangle = \langle \chi_2 \delta \mathsf{C} (\mathsf{T}^1 \chi_2 \delta \mathsf{C})^j \rangle \frac{\langle \chi_2 \mathsf{T}^1 \chi_2 \rangle}{\theta_2^2} \langle \chi_2 \delta \mathsf{C} (\mathsf{T}^1 \chi_2 \delta \mathsf{C})^i \rangle \tag{2.39}$$

for $i,j = 0,1,...,N-1$.

A simple computation shows that Eqs. (2.38) and (2.39) are equivalent to Eq. (2.34). Therefore (2.33) implies (2.34). To show the necessity of condition (2.34) we note that the conditions (2.38) and (2.39) imply (2.33). Thus we see that (2.34) implies (2.33) since (2.34) is equivalent to the conditions (2.38) and (2.39).

Theorem 2.1 now follows directly from Lemmas 2.2 and 2.3.

III. EFFECTIVE ELASTICITY TENSORS OF FINITE-RANK LAMINATES

A finite-rank laminate is defined iteratively. To fix ideas we show how to construct a rank-2 laminate. One starts with a core of material 2 and layers it with a coating of material 1 in layers of thickness ϵ^2 perpendicular to a specified direction n_1. One then takes this finely layered material and again layers with a coating of material 1 in layers of thickness ϵ perpendicular to a second direction n_2. The $\epsilon = 0$ limit of this microgeometry is called a rank-2 laminate. Conversely, one could start with a core of material 1 and layer it with a coating of material 2 and so on. Laminates of higher rank are constructed in the same way.

Laminates made starting with the weaker material 1 as core and reinforcing with layers of the stronger material 2 are referred to as strong laminates. The weak laminates correspond to using material 2 as core and layering with material 1.

The characteristic function of material 2 for the weak rank-2 laminate described in the introduction can be written as

$$\chi_2^\epsilon (x) = \chi^1 \left(\frac{x}{\epsilon}\right) \chi^2 \left(\frac{x}{\epsilon^2}\right) \tag{3.1}$$

for $\epsilon > 0$. Here $\chi^1(x/\epsilon)$ is the characteristic function of the layers of width ϵ containing material 2 and $\chi^2(x/\epsilon^2)$ is the characteristic function of material 2 inside those layers. Analogously for finite ϵ the characteristic function of material 2 for a weak rank-j laminate can be written as

$$\chi_2^\epsilon (x) = \chi^1 \left(\frac{x}{\epsilon}\right) \chi^2 \left(\frac{x}{\epsilon^2}\right) \cdots \chi^j \left(\frac{x}{\epsilon^j}\right). \tag{3.2}$$

Here $\chi^1(x/\epsilon)$ is the characteristic function of the layers of width ϵ containing material 2, $\chi^2(x/\epsilon^2)$ is the characteristic function of the layers of width ϵ^2 containing material 2 and so on. Finally, $\chi^j(x/\epsilon^j)$ is the characteristic function of material 2 inside the layers of width ϵ^{j-1}.

For future reference let $\langle \chi^i \rangle$ denote the average of the characteristic function χ^i over the unit period cell. It follows that the total volume fraction of material 2 is given by

$$\prod_{i=1}^{j} \langle \chi^i \rangle = \theta_2 \tag{3.3}$$

and

$$\theta_2 \leqslant \prod_{i=2}^{j} \langle \chi^i \rangle \leqslant \cdots \leqslant \prod_{i=\kappa}^{j} \langle \chi^i \rangle \leqslant \prod_{i=\kappa+1}^{j} \langle \chi^i \rangle \leqslant \cdots \leqslant \langle \chi^j \rangle \leqslant 1. \tag{3.4}$$

Explicit formulas have been developed for tensors describing the effective properties of finite-rank laminates.[9–11] For prescribed volume fractions θ_1 and θ_2 of materials 1 and 2 the effective elasticity tensors $\underline{\mathsf{C}}$ and $\overline{\mathsf{C}}$ of rank-j laminates with layer directions given by the unit vectors $n^1, n^2,...,n^j$ are

$$\theta_2 (\underline{\mathsf{C}} - \mathsf{C}_1)^{-1} = \delta \mathsf{C}^{-1} + \theta_1 \mathsf{T}^{*1} \tag{3.5}$$

for weak laminates and

$$\theta_1 (\mathsf{C}_2 - \overline{\mathsf{C}})^{-1} = \delta \mathsf{C}^{-1} - \theta_2 \mathsf{T}^{*2} \tag{3.6}$$

for strong laminates.

Here

$$T^{*s} = \sum_{i=1}^{j} \rho_i \hat{\mathsf{T}}^s (n^i), \quad s = 1,2, \quad 0 \leqslant \rho_i \leqslant 1, \quad \sum_{i=1}^{j} \rho_i = 1 \tag{3.7}$$

and for any unit vector v, $\hat{\mathsf{T}}^s(v)$ is given by formula (2.10). The quantities $\theta_1 \rho_i$ and $\theta_2 \rho_i$ appearing in $\theta_1 \mathsf{T}^{*1}$ and $\theta_2 \mathsf{T}^{*2}$ are the relative proportions of layer material introduced in the ith lamination. Formulas (3.5) and (3.6) were developed by Francfort and Murat.[9]

We now indicate how the effective tensor $\underline{C}$ of a weak finite-rank laminate can be written in terms of the sequence of phase geometries $\{\chi_2^\epsilon\}$ given by Eq. (3.2). We consider the constant tensor

$$\langle \chi_2^\epsilon \mathsf{T}^1 \chi_2^\epsilon \rangle , \tag{3.8}$$

where χ_2^ϵ is given by Eq. (3.2) and T^1 is given by Eq. (2.8). Expanding the integrand of Eq. (3.8) in a Fourier series yields

$$\langle \chi_2^\epsilon \mathsf{T}^1 \chi_2^\epsilon \rangle = \sum_{\kappa \neq 0} |\hat{\chi}_2^\epsilon(\kappa)|^2 \hat{\mathsf{T}}^1(\hat{\kappa}) , \tag{3.9}$$

where $\hat{\mathsf{T}}^1(\hat{\kappa})$ is given by (2.10) and

$$\hat{\chi}_2^\epsilon(\kappa) = \langle e^{2\pi i \kappa \cdot x} \chi_2^\epsilon(x) \rangle . \tag{3.10}$$

Upon taking the limit as ϵ goes to zero in Eq. (3.9), one finds after an appropriate iteration of Eq. (15) in the paper by Avellaneda and Milton[20] that

$$\lim_{\epsilon \to 0} \langle \chi_2^\epsilon \mathsf{T}^1 \chi_2^\epsilon \rangle = \theta_1 \theta_2 \sum_{i=1}^{j} \rho_i \hat{\mathsf{T}}^1(n^i) , \tag{3.11}$$

where

$$\rho_i = \frac{\Pi_{m=i+1}^{j} \langle \chi^m \rangle}{\theta_1} - \frac{\Pi_{m=i}^{j} \langle \chi^m \rangle}{\theta_1} \tag{3.12}$$

for $i = 1,\ldots, j-1$ and

$$\rho_j = \frac{1 - \langle \chi^j \rangle}{\theta_1} . \tag{3.13}$$

It follows from Eqs. (3.4), (3.12), and (3.13) that

$$\rho_i \geqslant 0 , \quad \sum_{i=1}^{j} \rho_i = 1 . \tag{3.14}$$

We remark that as we vary over all laminar microstructures $\{\chi_2^\epsilon\}$ the sequence $\rho_1, \rho_2, \ldots, \rho_j$ can be made to be any set of positive weights satisfying Eq. (3.14).[21]

It now follows from Eqs. (3.5) and (3.11) that the effective elastic tensor of a weak finite-rank laminate characterized by the sequence of phase functions $\{\chi_2^\epsilon\}$ is given by

$$\theta_2(\underline{\mathsf{C}} - \mathsf{C}_1)^{-1} = \delta\mathsf{C}^{-1} + \lim_{\epsilon \to 0} \frac{\langle \chi_2^\epsilon \mathsf{T}^1 \chi_2^\epsilon \rangle}{\theta_2} , \tag{3.15}$$

where $\langle \chi_2^\epsilon \rangle = \theta_2$ for all $\epsilon > 0$.

Similarly, the effective elastic tensor of a strong finite rank laminate characterized by the sequence of phase functions $\{\chi_1^\epsilon\}$ is given by

$$\theta_1(\mathsf{C}_2 - \overline{\mathsf{C}})^{-1} = \delta\mathsf{C}^{-1} - \lim_{\epsilon \to 0} \frac{\langle \chi_1^\epsilon \mathsf{T}^2 \chi_1^\epsilon \rangle}{\theta_1} , \tag{3.16}$$

where $\langle \chi_1^\epsilon \rangle = \theta_1$ for all $\epsilon > 0$.

IV. SATURATION OF EVEN-ORDER BOUNDS BY LAMINATES

In this section we consider the lower even-order bounds (2.20) and (2.21) for a family of laminar phase geometries $\{\chi_2^\epsilon\}$ given by Eq. (3.2) for $\epsilon > 0$. We shall pass to the $\epsilon = 0$ limit in these bounds to find that the bounds are saturated by the effective elasticity tensors of weak finite-rank laminates. A similar conclusion holds for the even-order upper bounds.

We denote by $\underline{\mathsf{C}}^\epsilon$ the effective elasticity tensor of a weak finite-rank laminar phase geometry χ_2^ϵ given by Eq. (3.2) and write the corresponding even-order lower bounds:

$$(\underline{\mathsf{C}}^\epsilon - \mathsf{C}_1)\, \epsilon : \epsilon \geqslant \sup_{\eta^{(0)},\ldots,\eta^{(N-1)}} \Bigg[2 \sum_{i=0}^{N-1} \langle \delta\mathsf{C} \chi_2^\epsilon (\mathsf{T}^1 \chi_2^\epsilon \delta\mathsf{C})^i \eta^{(i)} \rangle : \epsilon - \Bigg(\sum_{\substack{i=0\\ j=0}}^{N-1} \langle \delta\mathsf{C} \chi_2^\epsilon (\mathsf{T}^1 \chi_2^\epsilon \delta\mathsf{C})^i \eta^{(i)} : \chi_2^\epsilon (\mathsf{T}^1 \chi_2^\epsilon \delta\mathsf{C})^j \eta^{(j)} \rangle$$
$$+ \sum_{\substack{i=0\\ j=0}}^{N-1} \langle \delta\mathsf{C} \chi_2^\epsilon (\mathsf{T}^1 \chi_2^\epsilon \delta\mathsf{C})^i \eta^{(i)} : \mathsf{T}^1 \delta\mathsf{C} \chi_2^\epsilon (\mathsf{T}^1 \chi_2^\epsilon \delta\mathsf{C})^j \eta^{(j)} \rangle \Bigg) \Bigg] . \tag{4.1}$$

Applying the identities (2.36) and (2.37), passing to the $\epsilon = 0$ limit in Eq. (4.1), and noting that $\lim_{\epsilon \to 0} \underline{\mathsf{C}}^\epsilon = \underline{\mathsf{C}}$ yields

$$(\underline{\mathsf{C}} - \mathsf{C}_1)\, \epsilon : \epsilon \geqslant \sup_{\eta^{(0)},\ldots,\eta^{(N-1)}} \mathsf{L}(\eta^{(0)},\ldots,\eta^{(N-1)}) .$$

Here L is defined by

$$\mathsf{L}(\eta^{(0)},\ldots,\eta^{(N-1)}) = 2 \sum_{i=0}^{N-1} \lim_{\epsilon \to 0} \langle \delta\mathsf{C} \chi_2^\epsilon (\mathsf{T}^1 \chi_2^\epsilon \delta\mathsf{C})^i \rangle \eta^{(i)} : \epsilon$$
$$- \Bigg(\sum_{\substack{i=0\\ j=0}}^{N-1} \lim_{\epsilon \to 0} \langle \delta\mathsf{C} \chi_2^\epsilon (\mathsf{T}^1 \chi_2^\epsilon \delta\mathsf{C})^{i+j} \rangle \eta^{(i)} : \eta^{(j)} + \sum_{\substack{i=0\\ j=0}}^{N-1} \lim_{\epsilon \to 0} \langle \delta\mathsf{C} \chi_2^\epsilon (\mathsf{T}^1 \chi_2^\epsilon \delta\mathsf{C})^{i+j+1} \rangle \eta^{(i)} : \eta^{(j)} \Bigg) . \tag{4.2}$$

From perturbation theory we have the following identities for finite-rank laminates:

Theorem 4.1. Given the sequence of laminar phase geometries $\{\chi_2^\epsilon\}$ described by Eq. (3.2), one has

$$\lim_{\epsilon \to 0} \langle \chi_2^\epsilon (\mathsf{T}^1 \chi_2^\epsilon \mathsf{C})^i \rangle = \frac{(\lim_{\epsilon \to 0} \langle \chi_2^\epsilon \mathsf{T}^1 \chi_2^\epsilon \delta\mathsf{C} \rangle)^i}{\theta_2^{i-1}} \tag{4.3}$$

for $i = 1,2,\ldots,2N-1$. (The proof of Theorem 4.1 is given at the end of this section.)

It follows immediately from Theorems 4.1 and 2.1 that the even-order bounds in the $\epsilon = 0$ limit reduce to bounds of second-order for laminar phase geometries. Indeed, we have

$$\sup_{\eta^{(0)},\ldots,\eta^{(N-1)}} \mathsf{L}(\eta^{(0)},\ldots,\eta^{(N-1)})$$

$$= \sup_{\eta^{(0)}} \{2\theta_2 \eta^{(0)} : \epsilon - \eta^{(0)} : (\theta_2 \delta C^{-1} + \lim_{\epsilon \to 0} \langle \chi_2^\epsilon \mathsf{T}^1 \chi_2^\epsilon \rangle) \eta^{(0)}\} . \quad (4.4)$$

Moreover, a simple proof shows the following.

Theorem 4.2. For a given sequence of laminar phase geometries $\{\chi_2^\epsilon\}$ described by Eq. (3.2) the associated effective elasticity tensor $\underline{\mathsf{C}}$ saturates the even-order lower bound in the $\epsilon = 0$ limit, i.e.,

$$(\underline{\mathsf{C}} - \mathsf{C}_1)\, \epsilon : \epsilon = \sup_{\eta^{(0)},\ldots,\eta^{(N-1)}} \mathsf{L}(\eta^{(1)},\ldots,\eta^{(N-1)}) . \quad (4.5)$$

Proof:

$$(\underline{\mathsf{C}} - \mathsf{C}_1)\, \epsilon : \epsilon \geqslant \sup_{\eta^{(0)},\ldots,\eta^{(N-1)}} \mathsf{L}(\eta^{(0)},\ldots,\eta^{(N-1)})$$

$$= \sup_{\eta^{(0)}} \{2\theta_2 \eta^{(0)} : \epsilon - \eta^{(0)} : (\theta_2 \delta \mathsf{C}^{-1} + \lim_{\epsilon \to 0} \langle \chi_2^\epsilon \mathsf{T}^1 \chi_2^\epsilon \rangle) \eta^{(0)}\}$$

$$= \sup_{\eta^{(0)}} \{2\theta_2 \eta^{(0)} : \epsilon - \theta_2^2 \eta^{(0)} : (\underline{\mathsf{C}} - \mathsf{C}_1)^{-1} \eta^{(0)}\}$$

$$= (\underline{\mathsf{C}} - \mathsf{C}_1)\, \epsilon : \epsilon . \quad (4.6)$$

Here the second to the last equality follows from formula (3.15) for the effective tensor of a finite rank laminate. ▩

A similar argument show the following.

Theorem 4.3. For a given sequence of laminar phase geometries $\{\chi_2^\epsilon\}$, the associated effective elasticity tensor $\overline{\mathsf{C}}$ of a strong laminate saturates the even-order upper bound in the $\epsilon = 0$ limit.

We conclude this section with a proof of Theorem 4.1. To prove Theorem 4.1 we shall expand $\underline{\mathsf{C}}^\epsilon$ as a power series in the anisotropy $\delta\mathsf{C}$ and pass to the $\epsilon = 0$ limit to find a power-series expansion for $\underline{\mathsf{C}}$ having terms of the form

$$\lim_{\epsilon \to 0} \langle \delta \mathsf{C} \chi_2^\epsilon (\mathsf{T}^1 \chi_2^\epsilon \delta \mathsf{C})^i \rangle . \quad (4.7)$$

Alternately, we use the laminate formula (3.15) to find an equivalent series expansion in $\delta\mathsf{C}$ for the effective tensor $\underline{\mathsf{C}}$ with terms given by

$$\delta \mathsf{C} \frac{(\lim_{\epsilon \to 0} \langle \chi_2^\epsilon \mathsf{T}^1 \chi_2^\epsilon \delta \mathsf{C} \rangle)^i}{\theta_2^{i-1}} . \quad (4.8)$$

Theorem 4.1 then follows from the uniqueness of power series.

Proof of Theorem 4.1. We introduce the following.

Lemma 4.4. The effective tensor $\underline{\mathsf{C}}$ of a finite-rank weak laminate characterized by the sequence $\{\chi_2^\epsilon\}$ can be expanded for sufficiently small δC in the power series

$$\underline{\mathsf{C}} = C_1 + \sum_{i=0}^{\infty} \lim_{\epsilon \to 0} [\delta \mathsf{C} (-1)^i \langle \chi_2^\epsilon (\mathsf{T}^1 \chi_2^\epsilon \delta \mathsf{C})^i \rangle] . \quad (4.9)$$

Proof. Given a sequence of finite-rank laminar geometries $\{\chi_2^\epsilon\}$ the associated effective tensor for $\epsilon > 0$ is given by

$$\underline{\mathsf{C}}^\epsilon\, \epsilon : \epsilon = \langle \mathsf{C}^\epsilon(x) [e(u^\epsilon) + \epsilon] : e(u^\epsilon) + \epsilon \rangle , \quad (4.10)$$

where $\mathsf{C}^\epsilon = (1 - \chi_2^\epsilon) \mathsf{C}_1 + \chi_2^\epsilon \mathsf{C}_2$, $\langle e(u^\epsilon) + \epsilon \rangle = \epsilon$, and $e(u^\epsilon)$ solves

$$e(u^\epsilon) = -\mathsf{T}^1 \{\delta \mathsf{C} \chi_2^\epsilon [e(u^\epsilon) + \epsilon]\} . \quad (4.11)$$

From Eq. (4.11) we see that

$$(I + \mathsf{T}^1 \delta \mathsf{C} \chi_2^\epsilon) [e(u^\epsilon) + \epsilon] = \epsilon , \quad (4.12)$$

thus

$$e(u^\epsilon) + \epsilon = (I + \mathsf{T}^1 \delta \mathsf{C} \chi_2^\epsilon)^{-1} \epsilon . \quad (4.13)$$

For δC sufficiently small we expand the right-hand side of Eq. (4.13) in powers of $\delta\mathsf{C}$ and write Eq. (4.10) as

$$\underline{\mathsf{C}}^\epsilon\, \epsilon : \epsilon = \mathsf{C}_1\, \epsilon : \epsilon + \sum_{i=0}^{\infty} \langle (-1)^i \delta \mathsf{C} \chi_2^\epsilon (\mathsf{T}^1 \chi_2^\epsilon \delta C)^i \rangle\, \epsilon : \epsilon . \quad (4.14)$$

This expansion is equivalent to

$$\underline{\mathsf{C}}^\epsilon = \mathsf{C}_1 + \sum_{i=0}^{\infty} \langle (-1)^i \delta \mathsf{C} \chi_2^\epsilon (\mathsf{T}^1 \chi_2^\epsilon \delta \mathsf{C})^i \rangle . \quad (4.15)$$

Passing to the $\epsilon = 0$ limit in Eq. (4.15) yields

$$\underline{\mathsf{C}} = \mathsf{C}_1 + \lim_{\epsilon \to 0} \sum_{i=0}^{\infty} \langle (-1)^i \delta \mathsf{C} \chi_2^\epsilon (\mathsf{T}^1 \chi_2^\epsilon \delta \mathsf{C})^i \rangle . \quad (4.16)$$

To conclude the proof of Lemma 4.4 we show that

$$\lim_{\epsilon \to 0} \sum_{i=0}^{\infty} \langle (-1)^i \delta \mathsf{C} \chi_2^\epsilon (\mathsf{T}^1 \chi_2^\epsilon \delta \mathsf{C})^i \rangle = \sum_{i=0}^{\infty} \lim_{\epsilon \to 0} \langle (-1)^i \delta \mathsf{C} \chi_2^\epsilon (\mathsf{T}^1 \chi_2^\epsilon \delta \mathsf{C})^i \rangle . \quad (4.17)$$

We prove Eq. (4.17) by introducing the family of tensor-valued analytic functions $\{R_\epsilon(z)\}$, $\epsilon > 0$ where $R_\epsilon(z)$ is defined by

$$R_\epsilon(z) = \mathsf{C}_1 + \sum_{i=0}^{\infty} \langle (-1)^i \delta \mathsf{C} \chi_2^\epsilon (\mathsf{T}^1 \chi_2^\epsilon \delta \mathsf{C})^i \rangle z^i . \quad (4.18)$$

The set $\{R_\epsilon(z)\}$, $\epsilon > 0$ is a normal family of analytic functions for z in the region $|z| < |\delta \mathsf{C}|^{-1}$. Since $\{R_\epsilon(z)\}$ is a normal family, we see that as ϵ tends to zero, $R_\epsilon(z)$ converges uniformly to the analytic function $R_0(z)$ given by

$$R_0(z) = \mathsf{C}_1 + \sum_{i=0}^{\infty} \mathsf{A}^{(i)} z^i . \quad (4.19)$$

Using the uniform convergence of $R_\epsilon(z)$ we deduce from Cauchy's integral formula that

$$\lim_{\epsilon\to 0} \langle (-1)^n \delta C \chi_2^\epsilon (T^1 \chi_2^\epsilon \delta C)^n \rangle$$

$$= \lim_{\epsilon\to 0} \frac{1}{2\pi i} \int_\gamma \frac{R_\epsilon(\xi)}{\xi^{n+1}} d\xi$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{R_0(\xi)}{\xi^{n+1}} d\xi$$

$$= \frac{1}{n!} \frac{d^n}{dz^n} R_0(z) \bigg|_{z=0} = A^{(n)}. \quad (4.20)$$

Here γ is a simple closed curve containing $z=0$ in the region $|z| < |\delta C|^{-1}$. Therefore for δC sufficiently small, $z=1$ lies inside $|z| < |\delta C|^{-1}$ and Eq. (4.17) follows. ■

Alternately, it follows from the laminate formula (3.15) that

Lemma 4.5. The effective tensor $\underset{\sim}{C}$ of a finite-rank laminate characterized by the sequence $\{\chi_2^\epsilon\}$ can be expanded for sufficiently small δC in the power series

$$\underset{\sim}{C} = C_1 + \sum_{i=0}^{\infty} \delta C (-1)^i \frac{(\lim_{\epsilon\to 0} \langle \chi_2^\epsilon T^1 \chi_2^\epsilon \rangle \delta C)^i}{\theta_2^{i-1}}. \quad (4.21)$$

Proof. From Eq. (3.15) simple algebraic manipulation yields

$$\underset{\sim}{C} = C_1 + \theta_2 \delta C \left(I + \lim_{\epsilon\to 0} \frac{\langle \chi_2^\epsilon T^1 \chi_2^\epsilon \rangle}{\theta_2} \delta C \right)^{-1}. \quad (4.22)$$

For δC sufficiently small we expand the right-hand side of (4.22) in a Neumann series to recover Eq. (4.21). ■

From the uniqueness of power series we obtain from Lemmas 4.4 and 4.5 that

$$\lim_{\epsilon\to 0} \langle \delta C \chi_2^\epsilon (T^1 \chi_2^\epsilon \delta C)^i \rangle = \delta C \frac{\lim_{\epsilon\to 0} \langle \chi_2^\epsilon T^1 \chi_2^\epsilon \rangle \delta C)^i}{\theta_2^{i-1}} \quad (4.23)$$

for δC sufficiently small.

Lastly, we remark that if we replace δC by $\lambda \tilde{\delta C}$ in Eq. (4.23) where λ is a scalar, then it follows from the homogeneity of both sides of Eq. (4.23) that

$$\lim_{\epsilon\to 0} \langle \tilde{\delta C} \chi_2^\epsilon (T^1 \chi_2^\epsilon \tilde{\delta C})^i \rangle = \tilde{\delta C} \frac{(\lim_{\epsilon\to 0} \langle \chi_2^\epsilon T^1 \chi_2^\epsilon \rangle \tilde{\delta C})^i}{\theta_2^{i-1}} \quad (4.24)$$

for the perturbatioin $\tilde{\delta C}$. Therefore we see that Eq. (4.23) holds for perturbations δC of all magnitudes and Theorem 4.1 is proved.

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