Journal of Applied Physics Vol. 96 (2004) pp. 2821–2827.

Optimal lower bounds on the electric field concentration in composite media.

Robert Lipton
Department of Mathematics, Louisiana State University,
Baton Rouge, LA 70803.

May, 2004

Abstract. Composites made from two linear isotropic dielectric materials are considered. It is assumed that only the volume fraction and two point correlation function of each dielectric material are known. Lower bounds on all r^{th} moments of the electric field intensity inside each phase are obtained for $r \geq 2$. A lower bound on the maximum field intensity inside the composite is also obtained. The bounds are given in terms of the one and two point statistics of the microgeometry. All of these bounds are shown to be the best possible as they are attained by the electric field associated with a suitably constructed space filling confocal ellipsoid assemblage. The bounds provide a new opportunity for the assessment of local field behavior in terms of a statistical description of the microstructure.

1 Introduction

The study of failure initiation in dielectric composites requires one to assess the magnitude of the local electric field arising from macroscopic potential gradients. Macroscopic quantities sensitive to the local field behavior include higher order moments of the electric field inside the composite. In this work we focus on two–phase dielectric composites and develop optimal lower bounds on the higher moments of the electric field that depend on statistics of the microgeometry gathered from image analysis [1].

The composite is contained inside a cube Q and no constraints are placed upon the arrangement of the two materials inside Q. The subsets of Q occupied by materials one and two are denoted by Q_1 and Q_2 respectively. The indicator function of material one is denoted by χ_1 and takes the value one inside Q_1 and zero outside. The indicator function of material two is given by χ_2 and $\chi_2 = 1 - \chi_1$. It is supposed that Q is the period cell for an infinite periodic medium. The one point and two point correlation functions are given by

$$S_1^1 = \frac{1}{|Q|}\int_Q \chi_1(\mathbf{x})\, d\mathbf{x} \quad \text{and} \quad S_1^2(\mathbf{t}) = \frac{1}{|Q|}\int_Q \chi_1(\mathbf{x})\chi_1(\mathbf{x}+\mathbf{t})\, d\mathbf{x}, \tag{1.1}$$

where $\mathbf{t}$ is any vector and $|Q|$ is the volume of Q. The one-point correlation S_1^1 gives the volume fraction of material one. The two-point correlation $S_1^2(\mathbf{t})$ gives the probability that a rod of length and orientation specified by $\mathbf{t}$ has both ends in material one when its translated and dropped inside the periodic medium. Image processing techniques have recently been developed in [1] to determine the one-point, two-point and three-point correlation functions from images of composite microstructure.

The electric and displacement fields $\mathbf{E}(\mathbf{x})$ and $\mathbf{D}(\mathbf{x})$ inside the two-phase dielectric satisfy $\mathbf{E}(\mathbf{x}) = \nabla\phi(\mathbf{x})$ and $\mathbf{D}(\mathbf{x}) = \epsilon(\mathbf{x})\mathbf{E}(\mathbf{x})$. Here $-\phi$ is the electric potential and the dielectric constant $\epsilon(\mathbf{x})$ takes the two values ϵ_1 and ϵ_2, with $\epsilon_1 > \epsilon_2$, and

$$\begin{aligned} \Delta\phi &= 0, \text{ in phase } 1, \\ \Delta\phi &= 0, \text{ in phase } 2. \end{aligned} \tag{1.2}$$

It is assumed that there is perfect contact between the dielectrics so the electric potential and normal component of the displacement are continuous across the two phase interface, i.e.,

$$\begin{aligned} \phi_1 &= \phi_2, \\ \mathbf{D}_1 \cdot \mathbf{n} &= \mathbf{D}_2 \cdot \mathbf{n}. \end{aligned} \tag{1.3}$$

Here $\mathbf{n}$ is the unit normal to the interface pointing into material 2 and the subscripts indicate the side of the interface that the fields are evaluated on. For a prescribed constant electric field $\overline{\mathbf{E}}$ the average electric field $< \mathbf{E} >$ satisfies $< \mathbf{E} >= \overline{\mathbf{E}}$ and $\phi(\mathbf{x}) - \overline{\mathbf{E}} \cdot \mathbf{x}$ is periodic on Q. The effective dielectric tensor is defined by

$$< \mathbf{D} >= \mathcal{E}^e \overline{\mathbf{E}}. \tag{1.4}$$

In this work we consider the moments of the electric field intensity inside each phase given by

$$< \chi_1 |\mathbf{E}(\mathbf{x})|^r >^{1/r} \quad \text{and} \quad < \chi_2 |\mathbf{E}(\mathbf{x})|^r >^{1/r} \tag{1.5}$$

for $2 \le r < \infty$. Here $< \cdot >$ indicates the volume average of a quantity over the cube Q. We also consider the L^∞ norms given by

$$
\begin{aligned}
\|\mathbf{E}(\mathbf{x})\|_{L^\infty(Q_1)} &= \lim_{r\to\infty} <\chi_1|\mathbf{E}(\mathbf{x})|^r>^{1/r} \\
\|\mathbf{E}(\mathbf{x})\|_{L^\infty(Q_2)} &= \lim_{r\to\infty} <\chi_2|\mathbf{E}(\mathbf{x})|^r>^{1/r} \\
\|\mathbf{E}(\mathbf{x})\|_{L^\infty(Q)} &= \lim_{r\to\infty} <|\mathbf{E}(\mathbf{x})|^r>^{1/r} .
\end{aligned} \tag{1.6}
$$

In Section 3 we present explicit optimal lower bounds on the moments (1.5) and L^∞ norms (1.6) that are given in terms of S_1^1 and S_1^2. In this work the minimizing configurations are shown to be given by suitably constructed confocal ellipsoid assemblages [2, 3]. These configurations include the Hashin–Shtrikman [4] coated sphere assemblage as a special case. The bounding technique presented in Sections 2 and 3 also applies when loss becomes significant in the dielectrics, ie., for complex values of ϵ_1 and ϵ_2. This issue is taken up in Section 7 where explicit bounds are given for statistically isotropic composites.

The optimal lower bounds for the higher moments of the electric fields can be used to assess the effective higher order response of weakly nonlinear composite media. This is due to the fact that the effective higher order nonlinear response for weakly nonlinear dielectric media can be approximated by suitable higher moments of the linear electric field, see Refs. [5, 6, 7, 8].

Previous investigations have provided upper and lower bounds on the second moments of the electric field in composite media, see Refs. [9, 10, 11, 12, 13, 14]. Higher order moments of the electric field have been calculated numerically for two dimensional dispersions of disk, needle, and square shaped inclusions [15] as well as the density of states for the Hashin coated cylinder assemblage, see [16]. For multi-phase nonlinear power law dielectric composites optimal lower bounds on the moments of the electric field are found when the degree of the moment matches the power of the nonlinearity [17]. For completeness we list recent work done in the context of two-phase linear elasticity. Here optimal inclusion shapes are sought that minimize the maximum eigenvalue of the local stress for a given applied stress. The work presented in Ref. [18] provides an optimal lower bound on the supremum of the maximum principle stress for a single simply connected stiff inclusion in an infinite matrix subject to a remote stress at infinity. The optimal shapes are given by ellipsoids. The work presented in Ref. [19] provides an optimal lower bound on the supremum of the maximum principle stress for two-dimensional periodic composites consisting of a single simply connected stiff inclusion in the period cell. The bound is given in terms of the area fraction of the included phase and for an explicit range of prescribed average stress the optimal inclusions are given by Vigdergauz shapes [20].

2 Lower bounds on the electric field intensity in anisotropic composites and sufficient conditions for optimality.

In this section we establish lower bounds on the L^∞ norm of the electric field inside each material. Sufficient conditions on the electric field are identified that guarantee that lower bound is attained. These conditions are used to establish the optimality of the bounds presented in Section 3.

For $0<\theta_1<1$ we suppose that the volume fraction of material one S_1^1 is fixed and given by θ_1. The volume fraction of material two is given by $\theta_2 = 1-\theta_1$. For any vector field $\mathbf{F}(\mathbf{x})$ defined on Q one has

$$
<\chi_2(\mathbf{x})|\mathbf{E}(\mathbf{x})-\mathbf{F}(\mathbf{x})|^2> \;\geq\; 0. \tag{2.1}
$$

Setting $\mathbf{F}$ equal to a constant vector $\overline{\mathbf{F}}$ one obtains

$$
<\chi_2(\mathbf{x})|\mathbf{E}(\mathbf{x})|^2> \;\geq 2\overline{\mathbf{F}}\cdot <\chi_2(\mathbf{x})\mathbf{E}(\mathbf{x})> -\theta_2|\overline{\mathbf{F}}|^2. \tag{2.2}
$$

Optimizing over $\overline{\mathbf{F}}$ gives

$$< \chi_2(\mathbf{x})|\mathbf{E}(\mathbf{x})|^2 > \ \geq \frac{1}{\theta_2}| < \chi_2(\mathbf{x})\mathbf{E}(\mathbf{x}) > |^2. \tag{2.3}$$

Expanding (1.4) one obtains

$$\mathcal{E}^e\overline{\mathbf{E}} =< ((\epsilon_1 + \chi_2(\epsilon_2 - \epsilon_1))\, \mathbf{E}(\mathbf{x}) > . \tag{2.4}$$

Recalling that $< \mathbf{E}(\mathbf{x}) >= \overline{\mathbf{E}}$ one easily deduces the identity given by

$$(\epsilon_2 - \epsilon_1)^{-1} (\mathcal{E}^e - \epsilon_1 I)\, \overline{\mathbf{E}} =< \chi_2(\mathbf{x})\mathbf{E}(\mathbf{x}) > . \tag{2.5}$$

From (2.3) one obtains

$$< \chi_2(\mathbf{x})|\mathbf{E}(\mathbf{x})|^2 > \ \geq \frac{1}{\theta_2(\epsilon_2 - \epsilon_1)^2}|\, (\mathcal{E}^e - \epsilon_1 I)\, \overline{\mathbf{E}}\, |^2. \tag{2.6}$$

For p and q such that $p \geq 1$ and $1/p + 1/q = 1$, an elementary estimate gives

$$|Q|^{1/q}\theta_2^{1/q}(\int_Q \chi_2(\mathbf{x})|\mathbf{E}(\mathbf{x})|^{2p}d\mathbf{x})^{1/p} \geq \ |Q|\times < \chi_2(\mathbf{x})|\mathbf{E}(\mathbf{x})|^2 > \tag{2.7}$$

and it follows that

$$< \chi_2(\mathbf{x})|\mathbf{E}(\mathbf{x})|^r >^{1/r} \geq \frac{\theta_2^{1/r}}{\theta_2|\epsilon_2 - \epsilon_1|}|\, (\mathcal{E}^e - \epsilon_1 I)\, \overline{\mathbf{E}}\, |, \tag{2.8}$$

for $2 \leq r \leq \infty$. From (2.5) one easily sees that the lower bound given by (2.8) is optimal when the electric field is constant inside material two.

Similar arguments give the lower bound

$$< \chi_1(\mathbf{x})|\mathbf{E}(\mathbf{x})|^2 > \ \geq \frac{1}{\theta_1(\epsilon_1 - \epsilon_2)^2}|\, (\mathcal{E}^e - \epsilon_2 I)\, \overline{\mathbf{E}}\, |^2. \tag{2.9}$$

and it follows that

$$< \chi_1(\mathbf{x})|\mathbf{E}(\mathbf{x})|^r >^{1/r} \geq \frac{\theta_1^{1/r}}{\theta_1|\epsilon_1 - \epsilon_2|}|\, (\mathcal{E}^e - \epsilon_2 I)\, \overline{\mathbf{E}}\, |, \tag{2.10}$$

for $2 \leq r \leq \infty$. Here equality holds in (2.10) when the electric field is constant inside phase one.

3 Optimal lower bounds on the moments of the electric field.

Optimal lower bounds on the moments and L^∞ norms of the electric field are presented. The bounds are given in terms of the volume fraction of material one and the eigenvalues of a tensor of geometric parameters that depends explicitly on the two point correlation function. The tensor of geometric parameters $\mathcal{M}(S_1^1, S_1^2)$ is now well known and is given by [2, 21]

$$\mathcal{M}(S_1^1, S_1^2) = \frac{1}{S_1^1(1 - S_1^1)} \sum_{\mathbf{k}\neq 0} \hat{S_1^2}(\mathbf{k}) \frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2} \tag{3.1}$$

where $\mathbf{k}$ is a vector on the integer lattice, $\mathbf{k}\otimes\mathbf{k}$ is the rank-1 matrix with entries $\mathbf{k}_i\mathbf{k}_j$ and $\hat{S_1^2}(\mathbf{k})$ are the Fourier coefficients of $S_1^2(\mathbf{t})$ computed over the cube Q. Here $trace\{\mathcal{M}(S_1^1, S_1^2)\} = 1$ and $\mathcal{M}(S_1^1, S_1^2)$ is positive semidefinite. The eigenvalues of $\mathcal{M}(S_1^1, S_1^2)$ are written in ascending order and are denoted by $\lambda_1(S_1^1, S_1^2), \lambda_2(S_1^1, S_1^2), \lambda_3(S_1^1, S_1^2)$. It is evident that one can introduce the one and two point correlation functions for material two denoted by S_2^1 and $S_2^2(\mathbf{t})$. For future reference we point out that it is well known and easy to see that the associated tensor of geometric parameters $\mathcal{M}(S_2^1, S_2^2)$ is identical to $\mathcal{M}(S_1^1, S_1^2)$.

We introduce the set $\mathcal{K}^+$ of all vectors $\mathbf{d} = (d_1, d_2, d_3)$ such that $0 \leq d_1 \leq d_2 \leq d_3$ and $\sum_i d_i = 1$. The class of microgeometries (configurations of the two materials) for which $S_1^1 = \theta_1$ and $\lambda_i(S_1^1, S_1^2) = d_i$, $i = 1, 2, 3$ is denoted by $\mathcal{R}(\theta_1, \mathbf{d})$. In what follows we provide optimal lower bounds on the moments and L^∞ norms of the electric field for microstructures in the class $\mathcal{R}(\theta_1, \mathbf{d})$. From a mathematical perspective this problem is an optimization problem, i.e., among all configurations in $\mathcal{R}(\theta_1, \mathbf{d})$ we seek a configuration of two dielectrics that minimize the moments and L^∞ norm. It is shown here that the extremal microgeometries that attain the bounds are given by the confocal ellipsoid assemblages.

The construction of a confocal ellipsoid assemblage with core of material two and coating of material one is described as follows. One considers the cube containing a space filling assemblage of ellipsoids. Here all ellipsoids are contained inside Q and have the same shape and orientation of axes and differ only in their size. Inside each ellipsoid one places a smaller confocal ellipsoid filled with material two and the surrounding shell is filled with material one. We call these coated ellipsoids. The part of Q not covered by the coated ellipsoids has zero volume (measure). The volume fractions of materials one and two are the same for each coated ellipsoid and are given by the proportions θ_1 and θ_2 respectively, see Figure 1. A confocal ellipsoid assemblage with core of material one and coating of material two is constructed in an identical way.

For future reference we list the well known properties [2, 3] of the local electric field inside the confocal ellipsoid assemblage that are useful for the subsequent analysis. To fix ideas we consider an assemblage with a core of material two and coating of material one. We select a prototypical coated ellipsoid from the assemblage. One recalls that electric field in the composite is given by $\mathbf{E}(\mathbf{x}) = \nabla\phi$. Here ϕ is continuous inside the coated ellipsoid, harmonic in the core phase and coating phase, and satisfies the transmission conditions (1.3) on the core-coating interface. The fields inside the coated ellipsoid exhibit several distinguishing features. The first and foremost is that $\phi = \overline{\mathbf{E}} \cdot \mathbf{x}$ on the external boundary of the coated ellipsoid. This implies that on the external boundary that

$$\tau \cdot \mathbf{E}(\mathbf{x}) = \tau \cdot \overline{\mathbf{E}} \tag{3.2}$$

for every vector τ tangent to the external boundary at $\mathbf{x}$. Secondly on the external boundary one has the following flux condition given by

$$\mathbf{n} \cdot \epsilon_1 \mathbf{E}(\mathbf{x}) = \mathbf{n} \cdot \overline{\mathcal{E}}^e \overline{\mathbf{E}}, \tag{3.3}$$

where $\mathbf{n}$ is the exterior unit normal and $\overline{\mathcal{E}}^e$ is the effective dielectric constant of the confocal ellipsoid assemblage. Last the electric field inside the core material two is constant and given by

$$\mathbf{E}(\mathbf{x}) = \frac{1}{\theta_2(\epsilon_2 - \epsilon_1)} \left(\overline{\mathcal{E}}^e - \epsilon_1 I\right) \overline{\mathbf{E}}. \tag{3.4}$$

The confocal ellipsoid assemblage consists of translated and rescaled versions of the prototypical coated ellipsoid. The electric field $\tilde{\mathbf{E}}(\mathbf{x})$ in a rescaled and translated coated ellipsoid with scale factor $t > 0$ is related to the electric field $\mathbf{E}(\mathbf{x})$ in the prototype by $\tilde{\mathbf{E}}(\mathbf{x}) = \mathbf{E}(t^{-1}\mathbf{x})$ and (3.2), (3.3) and (3.4) are satisfied for every rescaled and translated confocal ellipsoid. Thus the electric field in material two is given by (3.4) and the lower bound (2.8) is attained. Interchanging core and coating materials one sees that the field inside phase one is constant for a confocal ellipsoid assemblage of core material one and coating material two and is given by

$$\mathbf{E}(\mathbf{x}) = \frac{1}{\theta_1(\epsilon_1 - \epsilon_2)} \left(\underline{\mathcal{E}}^e - \epsilon_2 I\right) \overline{\mathbf{E}}. \tag{3.5}$$

and the lower bound (2.10) is attained.

For $\mathbf{d}$ and θ_1 fixed we define

$$L^1 = \frac{\epsilon_2}{\epsilon_2 + (\epsilon_1 - \epsilon_2)(1 - \theta_1)d_3} \tag{3.6}$$

and

$$L^2 = \frac{\epsilon_1}{\epsilon_1 - (\epsilon_1 - \epsilon_2)\theta_1 d_1}, \tag{3.7}$$

here $L^1 \leq 1 \leq L^2$.

The optimal lower bounds on the moments of the electric field are given in the following.

Optimal lower bound on the moments of the electric field intensity in material one.

For every θ_1 and $\mathbf{d}$ in $\mathcal{K}^+$, the electric field $\mathbf{E}(\mathbf{x})$ associated with any micro geometry in $\mathcal{R}(\theta_1, \mathbf{d})$ satisfies

$$\theta_1^{1/r} L^1 |\overline{\mathbf{E}}| \leq \; < \chi_1 |\mathbf{E}(\mathbf{x})|^r >^{1/r}, \text{ for } 2 \leq r < \infty. \tag{3.8}$$

Moreover for every θ_1 and $\mathbf{d}$ in $\mathcal{K}^+$ there exists a confocal ellipsoid assemblage with core of material one in $\mathcal{R}(\theta_1, \mathbf{d})$ for which the minor axis of the ellipsoids are aligned with $\overline{\mathbf{E}}$ and (3.8) holds with equality for every r in $2 \leq r < \infty$.

Optimal lower bound on the L^∞ norm of the electric field intensity in material one.

For every θ_1 and $\mathbf{d}$ in $\mathcal{K}^+$, the electric field $\mathbf{E}(\mathbf{x})$ associated with any micro geometry in $\mathcal{R}(\theta_1, \mathbf{d})$ satisfies

$$L^1 |\overline{\mathbf{E}}| \leq \|\mathbf{E}(\mathbf{x})\|_{L^\infty(Q_1)}. \tag{3.9}$$

Moreover for every θ_1 and $\mathbf{d}$ in $\mathcal{K}^+$ there exists a confocal ellipsoid assemblage with core of material one in $\mathcal{R}(\theta_1, \mathbf{d})$ for which the minor axis of the ellipsoids are aligned with $\overline{\mathbf{E}}$ and (3.9) holds with equality.

Optimal lower bound on the moments of the electric field intensity in material two.

For every θ_1 and $\mathbf{d}$ in $\mathcal{K}^+$, the electric field $\mathbf{E}(\mathbf{x})$ associated with any micro geometry in $\mathcal{R}(\theta_1, \mathbf{d})$ satisfies

$$(1 - \theta_1)^{1/r} L^2 |\overline{\mathbf{E}}| \leq \; < \chi_2 |\mathbf{E}(\mathbf{x})|^r >^{1/r}, \text{ for } 2 \leq r < \infty. \tag{3.10}$$

Moreover for every θ_1 and $\mathbf{d}$ in $\mathcal{K}^+$ there exists a confocal ellipsoid assemblage with core of material two in $\mathcal{R}(\theta_1, \mathbf{d})$ for which the major axis of the ellipsoids are aligned with $\overline{\mathbf{E}}$ and (3.10) holds with equality for every r in $2 \leq r < \infty$.

Optimal lower bound on the L^∞ norm of the electric field intensity in material two.

For every θ_1 and $\mathbf{d}$ in $\mathcal{K}^+$, the electric field $\mathbf{E}(\mathbf{x})$ associated with any micro geometry in $\mathcal{R}(\theta_1, \mathbf{d})$ satisfies

$$L^2|\overline{\mathbf{E}}| \quad \leq \quad \|\mathbf{E}(\mathbf{x})\|_{L^\infty(Q_2)}. \tag{3.11}$$

Moreover for every θ_1 and $\mathbf{d}$ in $\mathcal{K}^+$ there exists a confocal ellipsoid assemblage with core of material two in $\mathcal{R}(\theta_1, \mathbf{d})$ for which the major axis of the ellipsoids are aligned with $\overline{\mathbf{E}}$ and (3.11) holds with equality.

Optimal lower bound on the L^∞ norm of the electric field intensity.

For every θ_1 and $\mathbf{d}$ in $\mathcal{K}^+$, the electric field $\mathbf{E}(\mathbf{x})$ associated with any micro geometry in $\mathcal{R}(\theta_1, \mathbf{d})$ satisfies

$$L^2|\overline{\mathbf{E}}| \quad \leq \quad \|\mathbf{E}(\mathbf{x})\|_{L^\infty(Q)}. \tag{3.12}$$

Moreover for every θ_1 and $\mathbf{d}$ in $\mathcal{K}^+$ there exists a confocal ellipsoid assemblage with core of material two in $\mathcal{R}(\theta_1, \mathbf{d})$ for which the major axis of the ellipsoids are aligned with $\overline{\mathbf{E}}$ and (3.12) holds with equality.

When the composite is statistically isotropic $\mathbf{d} = (1/3, 1/3, 1/3)$ and

$$L^2 = 3\epsilon_1/(3\epsilon_1 - (\epsilon_1 - \epsilon_2)\theta_1)). \tag{3.13}$$

For this case one has the following optimal lower bound on the L^∞ norm of the electric field intensity inside the composite given by

Optimal lower bound on the L^∞ norm of the electric field intensity for statistically isotropic composites

Consider all microgeometries in $\mathcal{R}(\theta_1, \mathbf{d})$ for the case $\mathbf{d} = (1/3, 1/3, 1/3)$. For a prescribed average electric field $\overline{\mathbf{E}}$ the lower bound on the L^∞ norm of the electric field concentration is given by

$$(3\epsilon_1/(3\epsilon_1 - (\epsilon_1 - \epsilon_2)\theta_1)))|\overline{\mathbf{E}}| \leq \|\mathbf{E}(\mathbf{x})\|_{L^\infty(Q)}. \tag{3.14}$$

The lower bound is attained by the electric field inside the Hashin Shtrikman concentric coated sphere assemblage with a core of material two for every choice of applied field $\overline{\mathbf{E}}$.

4 Derivation of the lower bounds

We recall the energy bounds [21] given by

$$\mathcal{U}^-\eta\cdot\eta \leq \quad \mathcal{E}^e\eta\cdot\eta \quad \leq \mathcal{U}^+\eta\cdot\eta, \tag{4.1}$$

for every constant vector η. Where

$$\mathcal{U}^+ \quad = \quad \epsilon_1 I - (1 - S_1^1)\epsilon_1[\epsilon_1(\epsilon_1 - \epsilon_2)^{-1}I - S_1^1\mathcal{M}(S_1^1, S_1^2)]^{-1} \tag{4.2}$$
$$\mathcal{U}^- \quad = \quad \epsilon_2 I + S_1^1\epsilon_2[\epsilon_2(\epsilon_1 - \epsilon_2)^{-1}I + (1 - S_1^1)\mathcal{M}(S_1^1, S_1^2)]^{-1} \tag{4.3}$$

We write the eigenvalues of $\mathcal{E}^e$ in ascending order $\epsilon_1^e, \epsilon_2^e, \epsilon_3^e$. For any micro geometry in $\mathcal{R}(\theta_1, \mathbf{d})$ it follows easily from (4.1) that

$$\underline{\lambda}_i \quad \leq \epsilon_i^e \leq \quad \overline{\lambda}_i, \text{ for } i = 1, 2, 3, \tag{4.4}$$

where

$$
\begin{aligned}
&\underline{\lambda}_1 = \epsilon_2 + \theta_1\epsilon_2(\epsilon_2(\epsilon_1-\epsilon_2)^{-1} + \theta_2 d_3)^{-1}, && \overline{\lambda}_1 = \epsilon_1 - \theta_2\epsilon_1(\epsilon_1(\epsilon_1-\epsilon_2)^{-1} - \theta_1 d_3)^{-1}\\
&\underline{\lambda}_2 = \epsilon_2 + \theta_1\epsilon_2(\epsilon_2(\epsilon_1-\epsilon_2)^{-1} + \theta_2 d_2)^{-1}, && \overline{\lambda}_2 = \epsilon_1 - \theta_2\epsilon_1(\epsilon_1(\epsilon_1-\epsilon_2)^{-1} - \theta_1 d_2)^{-1}\\
&\underline{\lambda}_3 = \epsilon_2 + \theta_1\epsilon_2(\epsilon_2(\epsilon_1-\epsilon_2)^{-1} + \theta_2 d_1)^{-1}, && \overline{\lambda}_3 = \epsilon_1 - \theta_2\epsilon_1(\epsilon_1(\epsilon_1-\epsilon_2)^{-1} - \theta_1 d_1)^{-1}.
\end{aligned} \tag{4.5}
$$

It follows immediately from (4.4) and (4.5) that

$$
\begin{aligned}
L^1|\overline{\mathbf{E}}| &\leq \frac{1}{\theta_1|\epsilon_1-\epsilon_2|}|(\mathcal{E}^e - \epsilon_2 I)\overline{\mathbf{E}}|,\\
L^2|\overline{\mathbf{E}}| &\leq \frac{1}{\theta_2|\epsilon_1-\epsilon_2|}|(\mathcal{E}^e - \epsilon_1 I)\overline{\mathbf{E}}|,
\end{aligned} \tag{4.6}
$$

and the lower bounds (3.9) and (3.11) now follow immediately from (2.8), (2.10) and (4.6).

To obtain bounds on the moments note that (2.6), (2.9), (4.6) together with Hölders inequality implies that

$$
\begin{aligned}
|Q|\theta_1(L^1)^2|\overline{\mathbf{E}}|^2 &\leq \textstyle\int_Q \chi_1|\mathbf{E}(\mathbf{x})|^2\,d\mathbf{x} \leq \; (|Q|\theta_1)^{1/q}(\displaystyle\int_Q \chi_1|\mathbf{E}(\mathbf{x})|^{2p}\,d\mathbf{x})^{1/p},\\
|Q|\theta_2(L^2)^2|\overline{\mathbf{E}}|^2 &\leq \textstyle\int_Q \chi_2|\mathbf{E}(\mathbf{x})|^2\,d\mathbf{x} \leq \; (|Q|\theta_2)^{1/q}(\displaystyle\int_Q \chi_2|\mathbf{E}(\mathbf{x})|^{2p}\,d\mathbf{x})^{1/p},
\end{aligned} \tag{4.7}
$$

where $p \geq 1$, $1/p + 1/q = 1$ and the lower bounds (3.8) and (3.10) follow immediately.

The lower bound given by (3.12) follows immediately from (3.11) and

$$
L^2|\overline{\mathbf{E}}| \leq \|\mathbf{E}(\mathbf{x})\|_{L^\infty(Q_2)} \leq \|\mathbf{E}(\mathbf{x})\|_{L^\infty(Q)}. \tag{4.8}
$$

5 Optimality

In this section the lower bounds are shown to be attained within the class of microstructures given by the confocal ellipsoid assemblages. We write the formulas for the effective dielectric tensors for the confocal ellipsoid assemblage in a form that is suitable for our purposes. The formulas differ from the original formulas given in [2, 3] and are derived in the following section. The effective dielectric tensor for the confocal ellipsoid assemblage with core material 2 and coating material 1 is given by

$$
\epsilon_1 I - \overline{\mathcal{E}}^e \;=\; \theta_2\epsilon_1[\epsilon_1(\epsilon_1-\epsilon_2)^{-1}I - \theta_1\mathcal{M}(S_1^1, S_1^2)]^{-1} \tag{5.1}
$$

and the eigenvalues are given by $\overline{\lambda}_i$, for $i = 1, 2, 3$ defined in (4.5). Eigenvectors corresponding to $\overline{\lambda}_3$ include vectors parallel to the major axis of the coated ellipsoids. Eigenvectors corresponding to $\overline{\lambda}_1$ include vectors parallel to the minor axis of the coated ellipsoids.

The effective dielectric tensor with core material 1 and coating material 2 is given by

$$
\underline{\mathcal{E}}^e - \epsilon_2 I \;=\; \theta_1\epsilon_2[\epsilon_2(\epsilon_1-\epsilon_2)^{-1}I + \theta_2\mathcal{M}(S_1^1, S_1^2)]^{-1}. \tag{5.2}
$$

and the eigenvalues are given by $\underline{\lambda}_i$, for $i = 1, 2, 3$ defined in (4.5). Here eigenvectors corresponding to $\underline{\lambda}_3$ include vectors parallel to the major axis of the coated ellipsoids. Eigenvectors corresponding to $\underline{\lambda}_1$ include vectors parallel to the minor axis of the coated ellipsoids.

We now state the following

Attainability property

For any choice of θ_1 and any symmetric tensor $\mathcal{H}$ with eigenvalues in $\mathcal{K}^+$ there exists a confocal ellipsoid assemblage with core of material one and coating of material two such that $S_1^1 = \theta_1$,

$$\mathcal{M}(S_1^1, S_1^2) = \mathcal{H}, \tag{5.3}$$

and there exists a confocal ellipsoid assemblage with core of material two and coating of material one such that $S_1^1 = \theta_1$ and

$$\mathcal{M}(S_2^1, S_2^2) = \mathcal{H}. \tag{5.4}$$

The attainability property is established in the following section.

We now establish the optimality of the lower bound (3.8). The characteristic function of material one in the confocal ellipsoid assemblage with core of material one and coating of material two is denoted by χ_1^{el}. The volume fraction of material one prescribed to be θ_1, i.e., $<\chi_1^{el}>=\theta_1$. The electric field inside material one the of the confocal ellipsoid assemblage is constant and from (3.5) one recalls that the electric field in material one is given by

$$\mathbf{E}(\mathbf{x}) = \frac{1}{\theta_1(\epsilon_1 - \epsilon_2)}(\underline{\mathcal{E}}^e - \epsilon_2 I)\overline{\mathbf{E}}. \tag{5.5}$$

For a prescribed vector $\mathbf{d}$ in $\mathcal{K}^+$ it follows from (5.3) that we can choose the confocal ellipsoid assemblage such that the eigenvalues of $\mathcal{M}(S_1^1, S_1^2)$ correspond to $\mathbf{d}$ and

$$\mathcal{M}(S_1^1, S_1^2)\overline{\mathbf{E}} = d_3\overline{\mathbf{E}}. \tag{5.6}$$

For this case it follows from earlier remarks that the minor axes of the ellipsoids are aligned with the applied field $\overline{\mathbf{E}}$ and from (5.2), (5.5) and (5.6) it follows that the electric field in material one is given by

$$\mathbf{E}(\mathbf{x}) = \tfrac{1}{\theta_1(\epsilon_1-\epsilon_2)}(\underline{\lambda}_1 - \epsilon_2 I)\overline{\mathbf{E}} \quad = L^1\overline{\mathbf{E}}. \tag{5.7}$$

Substitution gives

$$<\chi_1^{el}|\mathbf{E}(\mathbf{x})|^r>^{1/r}=\theta_1^{1/r}L^1|\overline{\mathbf{E}}| \tag{5.8}$$

and the optimality of (3.8) is established. The optimality of (3.9) follows immediately from (5.7). The optimality of the bounds (3.10) and (3.11) follow from identical considerations using confocal ellipsoid assemblages with core material two and coating material one. For this case the major axes of the ellipsoids are aligned with $\overline{\mathbf{E}}$ and $\mathbf{E}(\mathbf{x}) = L^2\overline{\mathbf{E}}$ in the core.

Last we establish the optimality of (3.12). To do this we examine the electric field inside the confocal ellipsoid assemblage with core of material two for which all ellipsoids have major axes aligned with the applied field $\overline{\mathbf{E}}$. Consider any coated ellipsoid in the assemblage and note that $|\mathbf{E}(\mathbf{x})| = L^2|\overline{\mathbf{E}}|$ in the core. In what follows we will show that the electric field intensity in the coating is bounded above by $L^2|\overline{\mathbf{E}}|$. Since the coated ellipsoids cover Q (up to a set of measure zero) it then follows that

$$L^2|\overline{\mathbf{E}}| \quad = \quad \|\mathbf{E}(\mathbf{x})\|_{L^\infty(Q)} \tag{5.9}$$

for this confocal ellipsoid assemblage.

Now we show that the electric field intensity is bounded above by $L^2|\overline{\mathbf{E}}|$ in the coating. To do this recall that $|\mathbf{E}(\mathbf{x})| = |\nabla\phi(\mathbf{x})|$ in the coating phase and that ϕ is harmonic there. Thus $|\mathbf{E}(\mathbf{x})|$ is subharmonic in the coating and from the maximum principle it necessarily takes its maximum values either on the interface between the core and coating or on the external boundary of the coated ellipsoid. We denote tangent vectors to the core–coating interface by τ and the unit normal to the interface is denoted by $\mathbf{n}$. The trace of the electric field on the core side of the interface is denoted by $\mathbf{E}_{|_2}$ and the trace of the electric field on the coating side of the interface is denoted by $\mathbf{E}_{|_1}$. Continuity of ϕ across the interface gives

$$\mathbf{E}_{|_1} \cdot \tau = \mathbf{E}_{|_2} \cdot \tau \tag{5.10}$$

and continuity of the normal component of displacement gives

$$\epsilon_1 \mathbf{E}_{|_1} \cdot \mathbf{n} = \epsilon_2 \mathbf{E}_{|_2} \cdot \mathbf{n}. \tag{5.11}$$

For points on the interface where $\mathbf{E}_{|_1} \cdot \mathbf{n} = 0$ it is clear that $|\mathbf{E}_{|_1}| = |\mathbf{E}_{|_2}|$. For all other points on the interface

$$|\mathbf{E}_{|_1} \cdot \mathbf{n}| = \frac{\epsilon_2}{\epsilon_1} |\mathbf{E}_{|_2} \cdot \mathbf{n}| \leq |\mathbf{E}_{|_2} \cdot \mathbf{n}|, \tag{5.12}$$

since $\epsilon_1 > \epsilon_2$. It now follows from (5.10) and (5.12) that

$$|\mathbf{E}_{|_1}| \leq |\mathbf{E}_{|_2}| = L^2 |\overline{\mathbf{E}}| \tag{5.13}$$

on the core–coating interface. The trace of the electric field on the coating side of the external interface is denoted by $\mathbf{E}_{|_{ext}}$. On the external boundary of the coated ellipsoid one recalls (3.2) and

$$\mathbf{E}_{|_{ext}} \cdot \tau = \overline{\mathbf{E}} \cdot \tau \tag{5.14}$$

where τ is any tangent vector to the external boundary. From (3.3) we have

$$\epsilon_1 \mathbf{E}_{|_{ext}} \cdot \mathbf{n} = \overline{\mathcal{E}}^e \overline{\mathbf{E}} \cdot \mathbf{n} = \overline{\lambda}_3 \overline{\mathbf{E}} \cdot \mathbf{n}. \tag{5.15}$$

Here $\mathbf{n}$ is the outer unit normal to the external boundary and from (4.5)

$$\overline{\lambda}_3 < \epsilon_1. \tag{5.16}$$

Using (5.16) we argue as before to deduce that on the outer boundary $|\mathbf{E}_{|_{ext}}| \leq |\overline{\mathbf{E}}|$. We apply the maximum principle and note that $1 \leq L^2$ to deduce that $|\mathbf{E}(\mathbf{x})| \leq L^2 |\overline{\mathbf{E}}|$ in the coating and optimality follows.

Optimality of (3.14) follows immediately from the same arguments used to establish the optimality of (3.12). This can also be checked by directly calculating the electric field inside the Hashin-Shtrikman coated sphere assemblage.

6 Formulas for effective properties

In this section we establish the formulas (5.1) and (5.2) for the effective properties of confocal ellipsoid assemblages. We equate these formulas with the better known formulas for the effective properties for confocal ellipsoid assemblages given in [2, 3] to establish the attainability property expressed in (5.3) and (5.4).

We sketch the ideas behind the derivation of (5.2) noting that (5.1) is established along similar lines. The electric field inside the concentric ellipsoid assemblage with core material one and coating material two admits the decomposition $\mathbf{E}(\mathbf{x}) = \nabla u + \overline{\mathbf{E}}$ where u is Q periodic and is the solution of

$$\text{div}\,\big((\epsilon_1 \chi_1 + \epsilon_2 (1 - \chi_1))(\nabla u + \overline{\mathbf{E}})\big) = 0. \tag{6.1}$$

This is equivalent to

$$\epsilon_2 \Delta u = -\text{div } \left((\epsilon_1 - \epsilon_2)\chi_1(\nabla u + \overline{\mathbf{E}})\right) = 0. \tag{6.2}$$

From (6.2) one easily obtains the identity

$$\mathbf{E}(\mathbf{x}) = -\frac{(\epsilon_1 - \epsilon_2)}{\epsilon_2}\nabla\Delta^{-1}\text{div } (\chi_1 \mathbf{E}(\mathbf{x})) + \overline{\mathbf{E}}. \tag{6.3}$$

Where $w = \Delta^{-1}f$ is the Q periodic solution of $\Delta w = f$. The constant value that $\mathbf{E}(\mathbf{x})$ takes in material one is denoted by $\tilde{\mathbf{E}}$ and its clear that $\chi_1(\mathbf{x})\mathbf{E}(\mathbf{x}) = \chi_1(\mathbf{x})\tilde{\mathbf{E}}$. Multiplying (6.3) by χ_1 and taking averages gives

$$\theta_1 \tilde{\mathbf{E}} = -\frac{(\epsilon_1 - \epsilon_2)}{\epsilon_2} < \chi_1 \nabla\Delta^{-1}\text{div}\, \chi_1 \tilde{\mathbf{E}} > +\theta_1 \overline{\mathbf{E}}. \tag{6.4}$$

It is well known [2] that

$$< \chi_1 \nabla\Delta^{-1}\text{div}\, \chi_1 \tilde{\mathbf{E}} >= \theta_1\theta_2 \mathcal{M}(S_1^1, S_1^2)\tilde{\mathbf{E}}, \tag{6.5}$$

and it follows that

$$\tilde{\mathbf{E}} = \theta_1 \left(\theta_1 I + \frac{(\epsilon_1 - \epsilon_2)}{\epsilon_2}\theta_1\theta_2\mathcal{M}(S_1^1, S_1^2)\right)^{-1} \overline{\mathbf{E}}. \tag{6.6}$$

We recall from (3.5) that

$$\tilde{\mathbf{E}} = \frac{1}{\theta_1(\epsilon_1 - \epsilon_2)}(\underline{\mathcal{E}}^e - \epsilon_2 I)\overline{\mathbf{E}}. \tag{6.7}$$

Equation (5.2) follows on elimination of $\tilde{\mathbf{E}}$ from (6.6) and (6.7).

We establish the attainability result stated in (5.3). From [2, 3] the effective dielectric tensor with core material 1 and coating material 2 is given by

$$\underline{\mathcal{E}}^e - \epsilon_2 I \quad = \quad \theta_1\epsilon_2[\epsilon_2(\epsilon_1 - \epsilon_2)^{-1} I + \theta_2\mathcal{H}]^{-1}, \tag{6.8}$$

where $\mathcal{H}$ is a symmetric positive semidefinite matrix with unit trace. It is shown in [3] that $\mathcal{H}$ ranges over all such matrices as the shape of the ellipsoids are varied while keeping the core volume fraction θ_1 fixed. The attainability property (5.3) follows immediately noting that (5.2) and (6.8) are equal and solving for $\mathcal{M}(S_1^1, S_1^2)$. Identical arguments are used to establish (5.4).

7 Bounds on field concentrations for two phase composites with complex dielectric permittivity

The effects of loss become important when considering optical properties of composite materials. For this case the dielectric constants ϵ_1 and ϵ_2 are complex. A straight forward calculation easily shows that the lower bounds on the field fluctuations given by (2.8) and (2.10) also hold for this case. Moreover (2.8) is optimal when the electric field is constant in material two and (2.10) is optimal when the electric field is constant in material one.

The methodology of Section 3 is applied to obtain lower bounds on the field concentrations in statistically isotropic two-phase dielectric composites when the dielectric constants are complex. To illustrate the method we show how to obtain a lower bound on the field concentration inside material two. For isotropic composites the complex effective dielectric tensor reduces to $\mathcal{E}^e = \epsilon^e I$. Here ϵ^e is the effective complex permittivity. For this case (2.8) becomes

$$< \chi_2(\mathbf{x})|\mathbf{E}(\mathbf{x})|^r >^{1/r} \geq \theta_2^{1/r}\left(\frac{1}{\theta_2}|\frac{\epsilon^e - \epsilon_1}{\epsilon_2 - \epsilon_1}|\right)|\overline{\mathbf{E}}|, \tag{7.9}$$

for $2 \leq r \leq \infty$.

Bounds on ϵ^e that are given in terms of the volume fractions and dielectric constants of the component materials were derived in [22]. These reduce to the Hashin Shtrikman bounds when the component dielectrics are real valued. In this context the bounds are given by curves bounding a region Ω_{θ_2} of the complex plane inside which ϵ^e must lie. The explicit formulas for the boundary are given in terms of ϵ_1, ϵ_2, and θ_2 and can be found in [2]. We set

$$L^2 = \min\left\{ \frac{1}{\theta_2} \left| \frac{z-\epsilon_1}{\epsilon_2-\epsilon_1} \right| ;\, z \text{ in } \Omega_{\theta_2} \right\}. \tag{7.10}$$

The lower bound on field concentrations for statistically isotropic composites in terms of the data ϵ_1, ϵ_2, and θ_2 are given by

$$\theta_2^{1/r} L^2 |\overline{\mathbf{E}}| \leq < \chi_2(\mathbf{x}) |\mathbf{E}(\mathbf{x})|^r >^{1/r}, \tag{7.11}$$

for $2 \leq r \leq \infty$. Here the constant L^2 is computed numerically. To fix ideas for $\epsilon_1 = 20 + i$, $\epsilon_2 = -2 + 3i$, and $\theta_2 = 0.4$ calculation shows that $L^2 = 1.06$. Similar considerations give lower bounds on the field fluctuations in material one. It is pointed out that, for the case of complex dielectric constants , the optimality of the bound (7.11) remains an open question and is the topic of future research.

If more information on the microstructure is available then one has tighter lower bounds on the higher moments of the electric field. This follows from the correspondingly tighter bounds on ϵ^e given in terms of higher order statistical information, see [2]. The tighter bounds restrict ϵ^e to a subset $\mathcal{A}$ of Ω_{θ_2}. It is then clear from (7.9) that minimization of

$$\frac{1}{\theta_2} \left| \frac{z-\epsilon_1}{\epsilon_2-\epsilon_1} \right| \tag{7.12}$$

for z in $\mathcal{A}$ provides a lower bound on $< \chi_2(\mathbf{x}) |\mathbf{E}(\mathbf{x})|^r >^{1/r}$ that is greater than or equal to $\theta_2^{1/r} L^2 |\overline{\mathbf{E}}|$.

8 Discussion: Extreme suppression of field concentrations

One recalls that the L^∞ norm of the electric field intensity inside the confocal ellipsoid assemblage with core material two and major axis aligned with $\overline{\mathbf{E}}$ is given by $\epsilon_1(\epsilon_1 - (\epsilon_1 - \epsilon_2)\theta_1 d_1)^{-1} |\overline{\mathbf{E}}|$. In the limit when the ratio between the major and minor axes of the ellipsoids tend to infinity the geometric parameter d_1 tends to zero and the L^∞ norm of the field intensity is precisely $|\overline{\mathbf{E}}|$. This value agrees with the electric field intensity seen in a layered material with layers parallel to $\overline{\mathbf{E}}$. The largest value of the L^∞ norm of the electric field intensity for this class of assemblages with core two and major axes aligned with $\overline{\mathbf{E}}$ is given by the Hashin-Shtrikman coated sphere assemblage when all axes of the ellipsoids are equal and $d_1 = 1/3$.

It is clear that the L^∞ norm of the electric field intensity inside the confocal ellipsoid assemblage with core material two remains finite even as $\epsilon_2 \to 0$. In this limit it is given by $\frac{1}{1-\theta_1 d_1} |\overline{\mathbf{E}}|$. At first sight this appears counter intuitive as high contrast inclusions can be arbitrarily close together inside the confocal ellipsoid assemblage. However since every coated ellipsoid “sees” the linear Dirichlet boundary conditions given by $\phi = \overline{\mathbf{E}} \cdot \mathbf{x}$ it is clear that the fields inside each coated ellipsoid are not affected by the surrounding inclusions and field concentrations do not occur.

9 Acknowledgments

This research effort is sponsored by NSF through grant DMS-0296064 and by the Air Force Office of Scientific Research, Air Force Materiel Command USAF, under grant number F49620-02-1-

0041. The US Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation thereon. The views and conclusions herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied of the Air Force Office of Scientific Research or the US Government.

References

[1] Berryman, J. G., 1988. *Interpolating and integrating three-point correlation functions on a lattice.* Journal of Computational Physics. **75**, pp. 86–102.

[2] Milton, G.W., 2002. *The Theory of Composites.* Applied and Computational Mathematics, vol. 6. Cambridge University Press, Cambridge.

[3] Tartar, L., 1985. *Estimations fines des coefficients homogénéisés.* In P. Krée (ed.), Ennio de Giorgi Colloquium:Papers presented at a Colloquium Held at the H. Poincaré Institute in November 1983, pp. 168–187. London Pitman Publishing Ltd.

[4] Hashin, Z. and Shtrikman, S., 1962. A variational approach to the theory of the effective magnetic permeability of multiphase materials. J. Appl. Phys. **33**, pp. 3125–3131.

[5] Sarychev, A. K. and Shalaev, V. M., 2000. *Electromagnetic field fluctuations and optical nonlinearities in metal–dielectric composites.* Phys. Rep. **335**, pp. 275–371.

[6] Hui, P. M., 1993. *Higher order nonlinear response in dilute random composites.* J. Appl. Phys. **73**, pp. 4072–4073.

[7] Zhang, G. M., 1996. *Higher order nonlinear response in random resistor networks: numerical studies for arbitrary nonlinearity.* Z. Phys. B. **99**, pp. 599–603.

[8] Gao, L., 2004. *Spectral representation theory for higher-order nonlinear response in random composites.* Phys. Lett. A **322**, pp. 250–259.

[9] Axel, J., 1992. *Bounds for field fluctuations in two-phase materials.* J. Appl. Phys. **72**, pp. 1217–1220.

[10] Bergman, D. J., 1978. *The dielectric constant of a composite material – A problem in classical physics.* Physics Reports **43**, pp. 377–407.

[11] Bobeth, M. and Diener, G., 1986. *Field fluctuations in multicomponent mixtures.* J. Mech. Phys. Solids. **34**, pp. 1–17.

[12] Beran, M. J., 1980. *Field fluctuations in a two-phase random medium.* Journal of Mathematical Physics **21**, pp. 2583–2585.

[13] Grabovsky, Y. 2001. *Optimal design problems for two-phase conducting composites with weakly discontinuous objective functionals.* Advances in Applied Mathematics, **27**, pp. 683-704.

[14] Lipton, R., 2001. *Optimal bounds on electric field fluctuations for random composites – three dimensional problems.* J. Appl. Phys. **89**, pp. 1371–1376.

[15] Cheng, H. and Torquato, S., 1997, *Electric field fluctuations in random dielectric composites.* Physical Review B **56**, pp. 8060–8068.

[16] Cule, D. and Torquato, S., 1998. *Electric field distribution in composite media.* Physical Review B **58**, pp. 11829–11832.

[17] Lipton, R., 2003. *Configurations of nonlinear materials with electric fields that minimize L^p norms.* Physica B **338**, pp. 48–53.

[18] Wheeler, L. T. 1994. *Inhomogeneities of minimum stress concentration.* Anisotropy and Inhomogeneity in Elasticity and Plasticity (ed. Y. C. Angel), Vol. AMD–158. ASME.

[19] Grabovsky, Y. and Kohn, R. V., 1995. *Microstructures minimizing the energy of a two phase elastic composite in two space dimensions. II: The Vigdergauz microstructure.* J. Mech. Phys. Solids. **43**, pp. 949–972.

[20] Vigdergauz, S. B., 1994. *Two-dimensional grained composites of extreme rigidity.* ASME J. Appl. Mech. **61**, pp. 390–394.

[21] Willis, J. R., 1977. *Bounds and self-consistent estimates for the overall properties of anisotropic composites.* J. Mech. Phys. Solids. **25**, pp. 185–202.

[22] Milton, G. W., 1981. *Bounds on the complex permittivity of a two-component composite material.* J. Appl. Phys. **52**, pp. 5286–5293.

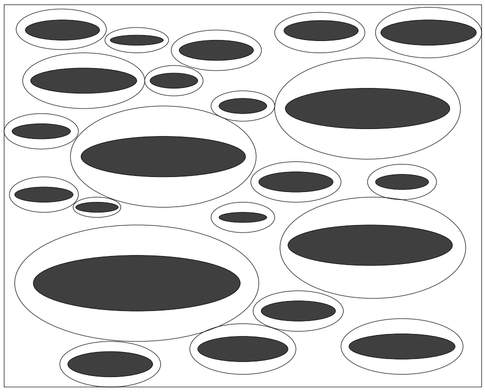

Figure 1: Confocal ellipsoid assemblage