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Configurations of nonlinear materials with electric fields that minimize L^p norms

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Abstract

A methodology is given for the construction of configurations of multi-phase nonlinear dielectric materials with electric fields that have the smallest L^p norm among all configurations subject to a resource constraint. Examples are given for configurations of two isotropic linear and nonlinear dielectrics with electric fields minimizing the L^2 norm and L^4 norm respectively.

Key words: Nonlinear dielectric materials. Configurations.

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1 Introduction

Consider a bounded simply connected domain Ω in the plane and prescribe an electric potential U_0 on the boundary $\partial\Omega$. The domain contains N nonlinear dielectric materials and the DC electric potential inside the composite is denoted by ϕ with $\phi = U_0$ on the boundary. In the i^{th} material the energy density is given by $(\gamma_i/p)|\nabla\phi|^p$. Here γ_i is the nonlinear susceptibility of the i^{th} material and p is any positive number greater than unity. For $p = 2$ the material is a linear dielectric. A configuration of these materials in Ω is described by the piecewise constant susceptibility given by $\gamma(\mathbf{x}) = \sum_{i=1}^{N} \chi_i(\mathbf{x})\gamma_i$, where the indicator function $\chi_i(\mathbf{x})$ equals one in the i^{th} phase and zero outside. The potential is a solution of

$$\operatorname{div}\left(\gamma(\mathbf{x})|\nabla\phi|^{p-2}\nabla\phi\right) = 0. \tag{1}$$

By a solution of (1) we mean that ϕ is continuous on $\Omega \cup \partial\Omega$, p-harmonic inside each phase, i.e., $\operatorname{div}\left(|\nabla\phi|^{p-2}\nabla\phi\right) = 0$ in each phase, and on interfaces between materials i and j

$$\mathbf{n}\cdot\gamma_i|\nabla\phi|^{p-2}\nabla\phi_i = \mathbf{n}\cdot\gamma_j|\nabla\phi|^{p-2}\nabla\phi_j. \tag{2}$$

Here $\mathbf{n}$ is the unit normal pointing into phase j and the subscripts indicate the side of the interface that a quantity is evaluated on. The electric field inside the composite is given by $\mathbf{E} = -\nabla\phi$.

We address the problem of finding a configuration of N nonlinear materials that minimizes the L^p norm of the electric field $\int_\Omega |\mathbf{E}|^p d\mathbf{x}$. Here a minimizing configuration is sought over the class of configurations satisfying the resource constraints $\int_\Omega \chi_i d\mathbf{x} = \beta_i$, $i = 1, ..., N$, with $\sum_{i=1}^{N} \beta_i = area(\Omega)$. We consider the class of problems for which p can take any value in $1 < p < \infty$.

A lower bound on the p norm of the electric field is given by Dirichlet's principle which states that among all potentials $\psi = U_0$ on $\partial\Omega$ such that $\int_\Omega |\psi|^p d\mathbf{x} < \infty$ and $\int_\Omega |\nabla\psi|^p d\mathbf{x} < \infty$ the p norm of the electric field $\mathbf{E} = -\nabla\psi$ is bounded below by

$$\int_\Omega |\mathbf{E}_u|^p d\mathbf{x} \leq \int_\Omega |\mathbf{E}|^p d\mathbf{x}. \tag{3}$$

where $\mathbf{E}_u = -\nabla u$ with $u = U_0$ on $\partial\Omega$ and u is a solution of the p-Laplace equation

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0. \tag{4}$$

Based upon this observation a methodology is given for the construction of optimal configurations. It is shown here that an optimal configuration is made by placing each nonlinear material inside subdomains of Ω with boundaries given by the level lines of the q-harmonic function v that is conjugate to u. Here $1/p + 1/q = 1$. It is shown that the electric field for these configurations is precisely $\mathbf{E}_u$. This is rigorously demonstrated for a large class of boundary data. Examples are given for $p = 2$ and $p = 4$.

Related earlier work addresses optimal design problems for the case $p = 2$ in the presence of a charge density in Ω. There the goal is to minimize the L^2 norm of the difference of the gradient of state and a target electric field. The problem was proposed in [1] and minimizing sequences of locally layered configurations were characterized for a G_δ dense set of targets. It was shown in [1] that for this class of targets only one scale of oscillation would develop in minimizing sequences of configurations. In [2] and [3] minimizing sequences made from locally layered materials with a single scale of oscillation were rigorously identified for all target fields. However this result doesn't give the full story as that analysis does not rule out the appearance of several scales of oscillation in minimizing sequences. This question is answered in [4] (for 2 and 3 dimensional problems) and in [5] (2 dimensional problems) where the explicit fully relaxed problem formulation is given. They show that there is only one scale of oscillation for minimizing sequences of locally layered materials for all choice of targets.

2 L^p minimizing configurations

We show how to choose a configuration of N nonlinear dielectric materials so that u is also a solution of

$$\mathrm{div}\,(\gamma(\mathbf{x})|\nabla u|^{p-2}\nabla u) = 0. \tag{5}$$

It is clear that the electric field for this type of configuration is $\mathbf{E}_u$. For this reason we call such configurations L^p minimizing configurations.

In what follows it is supposed that the unit tangent $\mathbf{t}$ and normal $\mathbf{n}$ vectors are defined almost everywhere on $\partial\Omega$ and that the tangential derivative $\partial_t U_0$ exists on $\partial\Omega$. The p-harmonic conjugate [6] to u is denoted by v and is the q-harmonic function inside Ω with $1/q + 1/p = 1$ and $\partial_n v = \partial_t U_0$ on $\partial\Omega$, where $\partial_n v$ is the normal derivative of v on $\partial\Omega$. Here v satisfies the p-Cauchy–Riemann equations [6] $v_x = -|\nabla u|^{p-2}u_y$ and $v_y = |\nabla u|^{p-2}u_x$. It is evident that the stream lines of u are the equipotential lines of v. Both u and v have locally Hölder continuous gradients and the zeros of ∇u and ∇v are isolated.

See [6] and [7] for a complete discussion and references to the literature. Let $\mathbf{n}$ denote the unit normal to an equipotential line of v then it is evident that

$$\mathbf{n} \cdot |\nabla u|^{p-2} \nabla u = 0 \tag{6}$$

on equipotential lines of v.

Now we show how to construct configurations of N isotropic materials for which u is also a solution of (5). For $p \neq 2$ we assume that there are no critical points of u in Ω, i.e. $\nabla u \neq 0$ in Ω. Then it is known [8] that u is real analytic in Ω. For $p = 2$ we make no such assumption and note that u is real analytic in Ω anyway. Thus for $p \neq 2$ it follows from the p-Cauchy–Riemann equations that v has no critical values. For $p = 2$ Sard's theorem [9] together with the Cauchy–Riemann equations shows that the set of critical values of v has measure zero. Let V_0^+ and V_0^- be the maximum and minimum values of v. Pick numbers $t_1, \ldots, t_{N+1}$ such that $t_2, \ldots, t_N$ are not critical values of v and $t_1 = V_0^- < t_2 < t_3 < \cdots < t_N < t_{N+1} = V_0^+$. The open sets of points in Ω where $t_i < v < t_{i+1}$ are denoted by $\{t_i < v < t_{i+1}\}$. The boundaries $\{v = t_i\}$ are smooth curves.

Construction of L^p minimizing configurations.
Let $\tilde{\chi}_i$ denote the indicator function of the sets $\{t_i < v < t_{i+1}\}$ and put $\tilde{\gamma}(\mathbf{x}) = \sum_{i=1}^N \tilde{\chi}_i(\mathbf{x})\gamma_i$ then $\mathrm{div}\,(\tilde{\gamma}(\mathbf{x})|\nabla u|^{p-2}\nabla u) = 0$.

Proof. Its clear that u is p-harmonic in each phase and continuous across interfaces. Since material interfaces correspond to the equipotential lines of v it is evident that the transmission condition (2) follows immediately from (6).

It is noted that there is considerable leeway in the choice of the numbers $t_2, ..., t_N$ so there is no unique L^p minimizing configuration.

3 Application to optimal design

In this section the design problem of finding a configuration of N nonlinear dielectrics that minimizes the L^p norm of the electric field $\int_\Omega |\mathbf{E}|^p d\mathbf{x}$ is addressed. In what follows we suppose that $\partial_t U_0$ exists. For boundary data U_0 such that u has no critical points in Ω we have the following result.

Optimal design result. Consider the q-harmonic function v conjugate to u. One can find numbers $t_2 < t_3 < \cdots < t_N$ such that the characteristic functions $\tilde{\chi}_i$ of the sets $\{t_i < v < t_{i+1}\}$ satisfy the constraints $\beta_i = \int_\Omega \tilde{\chi}_i d\mathbf{x}$ and $\sum_{i=1}^N \beta_i = area(\Omega)$. Then a configuration of nonlinear dielectric materials that supports the electric field with minimum L^p norm among all admissible

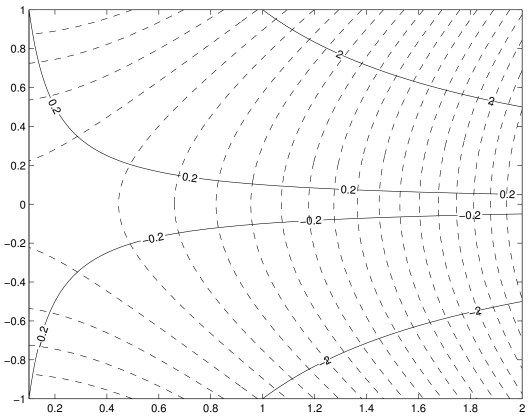

Fig. 1. The level lines of $x^2 - y^2$ are given by the dashed curves. The solid curves are the level lines of $2xy$ and form the interfaces between different dielectrics.

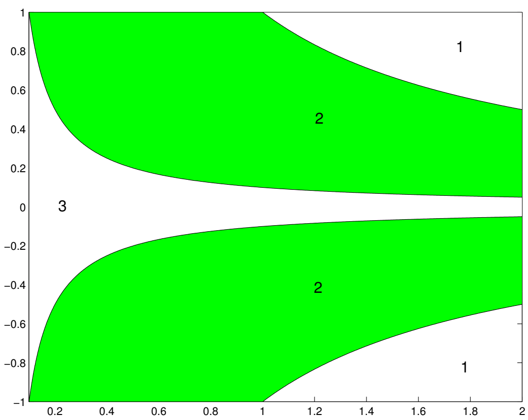

Fig. 2. An optimal configuration for minimizing the L^2 norm of the electric field.

configurations is constructed by placing the material with susceptibility γ_i in the set $\{t_i < v < t_{i+1}\}$ for $i = 1, ..N$. Moreover the electric field for this design is precisely the one that minimizes Dirichlet's principle (3).

Proof. Introduce the distribution function $\lambda_v(t) = |\{v > t\}|$ where $\{v > t\}$ is the set of points in Ω where $v > t$ and $|\{v > t\}|$ is the Lebesgue measure of $\{v > t\}$. Since there are no critical points of u in Ω one sees that there are no critical points of v and so the function $\lambda_v(t)$ is continuous and strictly decreasing on the interval $[V_0^-, V_0^+]$. Appealing to the intermediate value theorem we find $t_2 < t_3 < \cdot\cdot < t_N$ such that $\lambda_v(t_i) = \beta_i + \beta_{i+1} + \cdots + \beta_N$ for $i = 2, ...N$. For this choice the measure of the sets $\{t_i < v < t_{i+1}\}$ is $\lambda_v(t_i) - \lambda_v(t_{i+1}) = \beta_i$. It is clear that the characteristic functions $\tilde{\chi}_i$ of the sets $\{t_i < v < t_{i+1}\}$ satisfy the constraints $\beta_i = \int_\Omega \tilde{\chi}_i d\mathbf{x}$. We now show that the susceptibility $\tilde{\gamma}(\mathbf{x}) = \sum_{i=1}^N \tilde{\chi}_i(\mathbf{x})\gamma_i$ describes an optimal configuration of nonlinear dielectric materials. Indeed, from the previous section we see that $\tilde{\gamma}(\mathbf{x})$ satisfies div $(\tilde{\gamma}(\mathbf{x})|\nabla u|^{p-2}\nabla u) = 0$ and optimality follows from Dirichlet's principle.

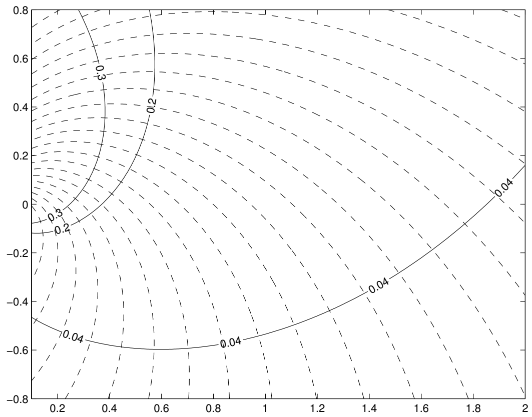

Fig. 3. The dashed curves are the level lines of u described by (7). The solid curves are the level lines of the conjugate function v given by (8). These form the interfaces between different dielectrics.

4 Methodology and Examples

The methodology for finding the optimal configuration is easily summarized. For given boundary data U_0 compute the q-harmonic solution v for which $\partial_n v = \partial_\tau U_0$. Then place each nonlinear material inside subdomains with boundaries given by the equipotentials of v. This method is illustrated in the following examples.

For the first example the design domain is the rectangle $0.1 < x < 2$, $-1 < y < 1$. The problem is to find an arrangement of three linear dielectric materials that minimizes the L^2 norm of the electric field for a prescribed boundary potential U_0. Here the resource constraints on each dielectric are specified as follows: 16.15% of the rectangle is occupied by material 1, 68.08% is occupied by material 2 and 15.77% is occupied by material 3. The boundary data U_0 is given by the trace of the harmonic function $x^2 - y^2$ on the boundary of the rectangle. It is clear from the methodology that an optimal design is obtained by placing material inside the level lines of the function $2xy$, see Fig. 1. The optimal design is given in Fig. 2. It is important to note that this design is optimal for any choice of the dielectric constants provided that they are all positive.

The second design problem is to find a configuration of four nonlinear dielectric materials with susceptibilities γ_i, $i = 1, .., 4$ and local energy densities $(\gamma_i/4)|\nabla\phi|^4$, that minimizes the L^4 norm of the electric field for a prescribed boundary potential U_0. Here the design domain is the rectangle $0.1 < x < 2$, $-0.8 < y < 0.8$. The boundary data U_0 is given by the trace of the 4-harmonic function u chosen from 'the quasi-radial zoo' (see [6], [7] and [10]). Here u is given by

$$u = r^{(1/3)} \exp(\theta/3), \tag{7}$$

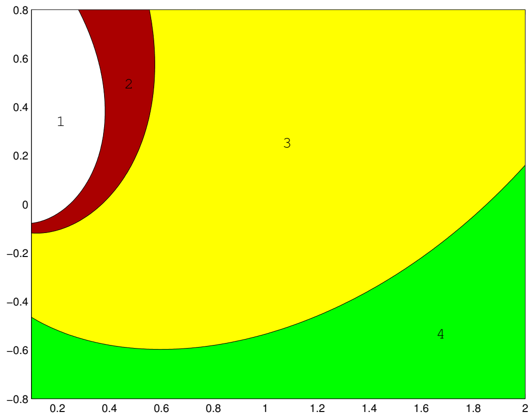

Fig. 4. An optimal configuration for minimizing the L^4 norm of the electric field.

where $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan y/x$. An optimal design is obtained by placing material inside the level lines of the 4/3-harmonic function v conjugate to u given by

$$v = (2/27)r^{-1}\exp(\theta), \tag{8}$$

see Fig. 3. The optimal design is given in Fig. 4. This design supports an electric field that is a minimizer for Dirichlet's principle (3) for $p = 4$ with U_0 prescribed as above.

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