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INEQUALITIES FOR ELECTRIC AND ELASTIC POLARIZATION TENSORS WITH APPLICATIONS TO RANDOM COMPOSITES

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ABSTRACT

NEW BOUNDS on the elastic and electric polarization tensors are found for grains of arbitrary shape or connectivity. For a grain shape specified by the characteristic function $\chi(x)$, the bounds are given explicitly in terms of the geometric function $|\hat{\chi}(k)|^2$. For electric polarizations one of the bounds may be interpreted as the polarization of a homogeneous ellipsoidal inclusion with axes determined by $|\hat{\chi}(k)|^2$. The other bound corresponds to a convex sum of polarization tensors for plate-like inclusions. Here the plate normals and weights are specified by $|\hat{\chi}(k)|^2$. These bounds are used to predict the range of effective transport properties for hierarchical random suspensions and aggregates that realize the Effective Medium Approximation. The inequalities also provide rigorous bounds for the effective properties of dilute statistically anisotropic random suspensions.

1. INTRODUCTION

CONSIDER AN inclusion of a given conductivity σ_2 immersed in an infinite medium of a different conductivity σ_1. We subject the system to a homogeneous electric field ζ. The average electric polarization induced inside the inclusion is given by the linear relation

$$\text{average electric polarization} = (\sigma_2-\sigma_1)P_2\zeta. \tag{1.1}$$

Here the tensor $(\sigma_2-\sigma_1)P_2$ is commonly known as the polarization tensor of the inclusion. The tensor P_2 depends upon the shape of the inclusion and transforms the electric field at infinity into the average field inside the inclusion. Similarly, consider an inclusion of elasticity $\mathbb{C}_2$ immersed in an infinite medium of elasticity $\mathbb{C}_1$, subjected to a homogeneous strain ε. The average elastic polarization induced inside the inclusion is given by

$$\text{average elastic polarization} = (\mathbb{C}_2-\mathbb{C}_1)\mathbf{T}_2\varepsilon. \tag{1.2}$$

The elastic polarization tensor is defined by $(\mathbb{C}_2-\mathbb{C}_1)\mathbf{T}_2$ and the tensor $\mathbf{T}_2$ is referred to as the Wu strain tensor, see WU (1966). This tensor transforms the elastic field at infinity into the average field inside the inclusion.

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For ellipsoidal inclusions explicit formulas for the electric polarization are known (see for example STRATTON, 1941). Explicit formulas for the elastic polarization tensor of ellipsoidal inclusions can be found in the work of ESHELBY (1957).

In this analysis we do not restrict ourselves to cases where the inclusion is simply connected or even connected. To fix ideas we label material 2 as the inclusion material and material 1 as the matrix. The generalized inclusions treated here can be viewed as an archipelago of finite diameter immersed in an unbroken sea of material 1. The archipelago is made up of grains of material 2 separated by material 1, see Fig. 1. The only constraint placed on the generalized inclusions is that their diameter be finite.

We consider a generalized inclusion of conductivity σ_2 immersed in a matrix of conductivity σ_1. The geometry of the inclusion is described mathematically by the characteristic function χ_2 of the inclusion, i.e. $\chi_2 = 1$ in material 2, $\chi_2 = 0$ outside. Partial information on the geometry of the inclusion is contained in the function $|\hat{\chi}_2(k)|^2$. Here

$$\hat{\chi}_2(k) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ik \cdot x} \chi_2(x) \, dx$$

is the Fourier transform of $\chi_2(x)$ at wave vector k. When the inclusion is more conducting than the matrix, i.e. $\sigma_2 > \sigma_1$, we show that the electric polarization tensor

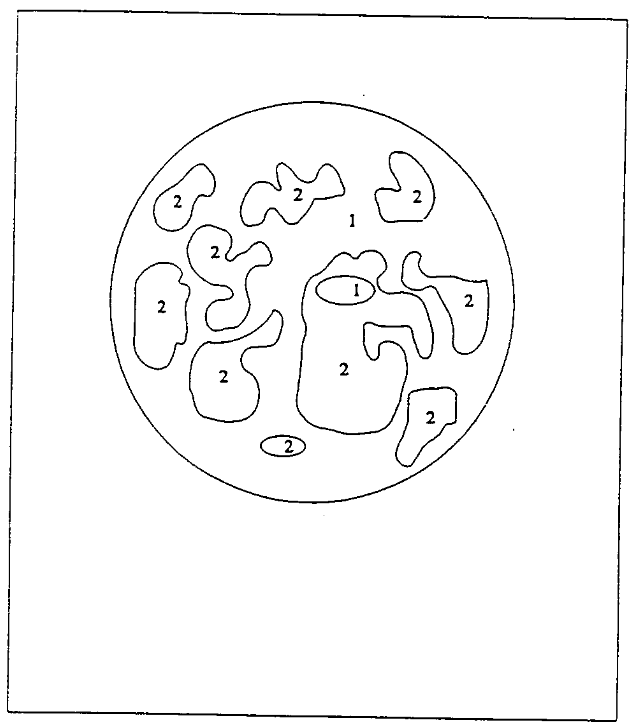

FIG. 1. A generalized inclusion is viewed as an archipelago of material 2 immersed in an unbroken sea of material 1.

is bounded below by the polarization tensor of a homogeneous ellipsoidal inclusion with axes explicitly determined by the geometric function $|\hat{\chi}_2(k)|^2$, see Section 3. The upper bound for the polarization tensor is given by a convex combination of polarization tensors associated with plate-like inclusions of conductivity σ_2. The plate normals and weights appearing in the convex sum are also determined by the geometry of the inclusions through the function $|\hat{\chi}_2(k)|^2$, see Section 3. When the inclusion is less conducting the bounds switch, and the upper bound becomes the polarization of a homogeneous ellipsoidal inclusion and the lower bound corresponds to the convex sum of plate-like polarization tensors.

For the case of a generalized elastic inclusion embedded in a matrix of a second elasticity we obtain upper and lower bounds on the Wu strain tensor of the inclusion. These bounds are also given in terms of the geometric function $|\hat{\chi}_2(k)|^2$. To fix ideas we consider a generalized inclusion of elasticity $\mathbb{C}_2$ embedded in a matrix of $\mathbb{C}_1$ and we suppose that $\mathbb{C}_2 > \mathbb{C}_1$. For this case we are able to show that the upper bound on the elastic polarization is a convex sum of polarization tensors associated with plate-like inclusions. The weights and plate normals appearing in the sum are determined by $|\hat{\chi}_2(k)|^2$, see Section 4. The lower bound on the polarization is given by (2.48). Due to the higher-order tensorial nature of the elastic problem it is not as easy to associate the lower bound on the polarization tensor with a polarization tensor of a homogeneous ellipsoidal inclusion as in the case of conductivity. It is conjectured that one can do so and progress is being made in this direction. Nevertheless, it is possible to derive a lower bound on the polarization tensor that is insensitive to the geometry of the generalized inclusion, see (5.11). It is easy to see that this lower bound corresponds to the elastic polarization of a spherical inclusion. We show, as one would expect, that the geometry independent lower bound lies below all geometry dependent lower bounds given by (2.48). This easily follows from the convexity property of the lower bound with respect to the geometric function $|\hat{\chi}_2(k)|^2$, see (4.6). For the case $\mathbb{C}_2 < \mathbb{C}_1$ the bounds switch and the upper bound is given by (2.48) whereas the lower bound is given by (2.49).

The bounds on the elastic and electric polarization tensors stated here follow from the low volume asymptotics of the Hashin–Shtrikman second-order bounds, see Section 2. The interpretation of various bounds as polarization tensors of homogeneous ellipsoidal inclusions follows from topological degree methods, see Section 3, and the Appendix.

We apply these results to modeling the effective transport properties for random composites. We consider the Symmetric Effective Medium Approximation (EMA) also known as the Coherent Potential Approximation (CPA), see LANDAUER (1978). These approximations are well known in the physics and continuum mechanics literature as they tend to do well in approximating the overall properties of random composites (see LANDAUER, 1978; CHRISTENSEN, 1990). The EMA was originally developed in the electrostatics context by BRUGGEMAN (1935). The application of EMA to elasticity was made by KRÖNER (1958).

Recently, it has been shown that for every EMA one can construct a composite for which the EMA gives the actual effective transport properties. This was observed by MILTON (1985) for the EMA in the context of multiphase conducting composites. At about the same time NORRIS (1985) introduced a generalized differential effective medium scheme to show the realizability of the EMA for multiphase isotropic elastic composites. More recently, AVELLANEDA (1987) applied the mathematically precise notion of G-convergence to prove the realizability of the EMA for multiphase anisotropic elastic composites, and establish power law convergence for the differential scheme.

In view of the previous remarks we see that EMA, while providing estimates for random aggregates and suspensions are also exact effective properties for a class of hierarchical random models. In this article we characterize the range of effective transport properties for these hierarchical models. The well-known bounds on effective properties provided by the Hashin–Shtrikman variational principle (HASHIN and SHTRIKMAN, 1963), the translation method (TARTAR, 1985; LURIE and CHERKAEV, 1984), and the analytic continuation method (BERGMAN, 1978; MILTON, 1981; GOLDEN and PAPANICOLAOU, 1983) are too wide to predict accurately the range of transport properties for these models. Here we apply techniques based upon differential inequalities to understand the range of effective properties for the class of hierarchical micromechanical models that realize the EMA, see Sections 5 and 6.

In this treatment we consider the EMA associated with isotropic two phase elastic and conducting mixtures. Bounds on the polarization and Wu tensors are used to obtain inequalities on the effective properties for an EMA associated with a hierarchical mixture of two elastic or conducting grains.

For EMAs associated with two phase conductors and prescribed grain shape we show that there exist extremal EMAs associated with ellipsoidal and plate-like grains that bound the EMA above and below, see Section 5. In fact it is possible to characterize the set of all effective conductivities for the hierarchical models, see Section 6. We show that this set can be realized by just considering hierarchical models of spheroidal inclusions. For EMAs associated with a hierarchical mixture of two elastic grains it is shown that the EMA is bounded above and below by extremal EMAs associated with spherical and plate-like grains, see Theorem 5.A. We note that these bounds are the absolute lower and upper bounds for all EMAs as one considers all grain shapes.

A second application of the bounds on electric and elastic polarizations is to the study of dilute anisotropic random suspensions. We apply our results to find bounds on the effective properties of dilute anisotropic random suspensions valid to second order in the volume fraction, see Section 7. For the special case of isotropic random suspensions we show that Maxwell's result (MAXWELL, 1873) for suspensions of conducting spheres provides optimal bounds on the effective properties of isotropic dilute suspensions of particles of arbitrary shape.

2. ESTIMATES FOR POLARIZATION AND WU STRAIN TENSORS

We consider electrically conducting the elastic composites made from two isotropic phases. The electrical conductivities for phases 1 and 2 are given by σ_i, $i = 1, 2$. The elastic properties of the two phases are described by Lamé shear moduli μ_i, $i = 1, 2$ and bulk moduli κ_i, $i = 1, 2$. The elasticities are assumed to be well ordered, i.e. $\mu_1 < \mu_2$, $\kappa_1 < \kappa_2$, and without loss of generality we take $\sigma_1 < \sigma_2$. The composite is treated as a periodic material with a square period cell Q_R of edge length R. This hypothesis is general provided the length scale of the inhomogeneities is much smaller than unity, see GOLDEN and PAPANICOLAOU (1983). The phase geometry is given exactly by the characteristic functions of materials 1 and 2.

$$\chi_2^R = \begin{cases} 1 & \text{if } x \text{ is in material 2} \\ 0 & \text{otherwise for } x \in Q_R, \end{cases} \tag{2.1}$$

$$\chi_1^R = 1 - \chi_2^R,$$

where

$$\frac{1}{|Q_R|}\int_{Q_R} \chi_1^R \, \mathrm{d}x = \theta_1, \quad \frac{1}{|Q_R|}\int_{Q_R} \chi_2^R \, \mathrm{d}x = \theta_2.$$

The conductivity and elasticity tensors of the composite are defined by the piecewise constant tensors

$$\sigma(x) = (\chi_1^R \sigma_1 + \chi_2^R \sigma_2) I \tag{2.2}$$

and

$$\mathbb{C}(x) = \chi_1^R \mathbb{C}_1 + \chi_2^R \mathbb{C}_2. \tag{2.3}$$

respectively. Here

$$\mathbb{C}_i = 2\mu_i \mathbf{I} + (k_i - \tfrac{2}{3}\mu_i) I \otimes I, \quad i = 1, 2, \tag{2.4}$$

where $\mathbf{I}$ is the identity on 3×3 symmetric matrices and I is the 3×3 identity matrix.

We suppose that the average electric field in a period cell of the composite is given by the 3 vector ζ. The local field is given by $\nabla\varphi + \zeta$, where φ is the Q_R-periodic potential field. The local electric field solves

$$\operatorname{div}(\sigma(x)[\nabla\varphi + \zeta]) = 0 \quad \text{in} \quad Q_R \tag{2.5}$$

and the effective conductivity tensor is defined by

$$\sigma^e \zeta \cdot \zeta = \frac{1}{|Q_R|}\int_{Q_R} \sigma(x)|\nabla\varphi + \zeta|^2 \, \mathrm{d}x. \tag{2.6}$$

Similarly, suppose the average strain in a period cell is given by the 3×3 symmetric matrix ε. The local strain is given by $e(u) + \varepsilon$, where u is the Q_R-periodic displacement field and

$$e_{ij}(u) = \frac{u_{i,j} + u_{j,i}}{2}. \tag{2.7}$$

The local strain solves

$$\operatorname{div}(\mathbb{C}(x)[e(u) + \varepsilon]) = 0 \quad \text{in} \quad Q_R, \tag{2.8}$$

and the effective elastic tensor $\mathbb{C}^e$ is defined by

$$\mathbb{C}^e\varepsilon:\varepsilon = \frac{1}{|Q_R|}\int_{Q_R}\mathbb{C}(x)(e(u)+\varepsilon):(e(u)+\varepsilon)\,\mathrm{d}x. \tag{2.9}$$

We introduce the spaces $\mathscr{E}^R$ and $\mathscr{S}^R$ of Q_R-periodic square integrable mean zero gradient fields and strain fields respectively. The space $\mathscr{S}^R$ of strain fields can be written as the sum of two orthogonal subspaces $\mathscr{S}_h^R$ and $\mathscr{S}_s^R$, i.e. $\mathscr{S}^R = \mathscr{S}_h^R \oplus \mathscr{S}_s^R$ (cf. MILTON and KOHN, 1988). Here $\mathscr{S}_h^R$ is the set of strain fields in $\mathscr{S}^R$ that are derived from a Q_R-periodic strain field v that itself is the gradient of some scalar potential, and $\mathscr{S}_s^R$ is the set of trace free strain fields. We define the operators Γ_R, $\Gamma_{h,R}$, $\Gamma_{s,R}$ to be the projections onto $\mathscr{E}^R$, $\mathscr{S}_h^R$ and $\mathscr{S}_s^R$ respectively. We define the following operators for conductivity

$$\Lambda_R^i = \frac{1}{\sigma^i}\Gamma_R, \quad i = 1,2, \tag{2.10}$$

and elasticity

$$T_R^i = \frac{3/2}{3k_i+4\mu_i}\Gamma_{h,R}+\frac{1}{2\mu_i}\Gamma_{s,R}, \quad i = 1,2. \tag{2.11}$$

It follows from KOHN and MILTON (1986) that the upper and lower Hashin–Shtrikman second-order bounds for the effective electrical conductivity are given by

$$\sigma^l \leqslant \sigma^e \leqslant \sigma^u \tag{2.12}$$

where

$$\sigma^l = \sigma_1+\theta_2(\sigma_2-\sigma_1)\left[I+(1-\theta_2)\frac{1}{|Q_R|}\left(\left(\int_{Q_R}(\chi_2^R\Lambda_2^1\chi_2^R\,\mathrm{d}x)(\sigma_2-\sigma_1)\right)\right]^{-1} \tag{2.13}$$

and

$$\sigma^u = \sigma_1+\theta_2(\sigma_2-\sigma_1)\left[I-\theta_2\frac{1}{|Q_R|}\left(\int_{Q_R}\chi_1^R\Lambda_R^2\chi_1^R\,\mathrm{d}x\right)(\sigma_2-\sigma_1)\right]^{-1}. \tag{2.14}$$

The upper and lower bounds for the effective elasticity are given by

$$\mathbb{C}^l \leqslant \mathbb{C}^e \leqslant \mathbb{C}^u \tag{2.15}$$

where

$$\mathbb{C}^l = \mathbb{C}_1+\theta_2(\mathbb{C}_2-\mathbb{C}_1)\left[\mathbf{I}+(1-\theta_2)\frac{1}{|Q_R|}\left(\left(\int_{Q_R}(\chi_2^R T_R^1\chi_2^R\,\mathrm{d}x)(\mathbb{C}_2-\mathbb{C}_1)\right)\right]^{-1} \tag{2.16}$$

and

$$\mathbb{C}^u = \mathbb{C}_1+\theta_2(\mathbb{C}_2-\mathbb{C}_1)\left[\mathbf{I}-\theta_2\frac{1}{|Q_R|}\left(\left(\int_{Q_R}(\chi_1^R T_R^2\chi_1^R\,\mathrm{d}x)(\mathbb{C}_2-\mathbb{C}_1)\right)\right]^{-1}. \tag{2.17}$$

Here the resulting composites may be anisotropic so that the inequalities (2.12) and (2.15) hold in the sense of quadratic forms.

Bounds on the Wu strain tensor and polarization tensor are obtained by expanding the effective properties and Hashin–Shtrikman bounds in low volume fraction expansions. It is easily seen that the low volume fraction limit may be realized by taking the period cell Q_R to be infinite. We consider a suspension of inclusions E_2 of material 2 within a matrix of material 1. Passing to the $R = \infty$ limit in (2.6) gives the low volume fraction expansion for the effective conductivity

$$\sigma^e = \sigma_1 + \theta_2(\sigma_2 - \sigma_1)P_2(\sigma_1, \sigma_2) + O(\theta_2^2). \tag{2.18}$$

Here $(\sigma_2 - \sigma_1)P_2$ is the polarization tensor of the inclusion E_2 and

$$P_2(\sigma_1, \sigma_2) = \frac{1}{|E_2|}\int_{E_2} (\nabla \varphi^\infty + \mathbf{I})\, dx, \tag{2.19}$$

where the potentials $\varphi^\infty = (\varphi_i^\infty, \varphi_2^\infty, \varphi_3^\infty)$, $i = 1, 2, 3$ solve

$$\operatorname{div}(\sigma(x)(\nabla \varphi_i^\infty + e^i)) = 0, \tag{2.20}$$

$$\int_{\mathbb{R}^3} |\nabla \varphi_i^\infty|^2 < \infty, \tag{2.21}$$

and

$$\sigma(x) = \chi_1^\infty \sigma_1 + \chi_2^\infty \sigma_2. \tag{2.22}$$

Here $\chi_2^\infty = 1$ for x in E_2 and zero outside $(\chi_1^\infty = 1 - \chi_2^\infty)$. The vectors e^i, $i = 1, 2, 3$ are unit vectors along the three coordinate axes (cf. AVELLANEDA, 1987). The polarization is seen to depend upon the shape of the inclusion E_2, and upon the conductivities σ_1, σ_2. Analogously one has the low volume fraction expansion for the effective elasticity given by

$$\mathbb{C}^e = \mathbb{C}_1 + \theta_2(\mathbb{C}_2 - \mathbb{C}_1)\mathbf{T}_2(\mathbb{C}_1, \mathbb{C}_2) + O(\theta_2^2). \tag{2.23}$$

Here $\mathbf{T}_2$ is the Wu strain tensor of the inclusion E_2 and

$$\mathbf{T}_2 = \frac{1}{|E_2|}\int_{E_2} (e(\mathbf{u}^\infty) + \mathbf{I})\, dx, \tag{2.24}$$

where the displacement fields $\mathbf{u}^\infty = u^{\infty ij}$, $i, j = 1, 2, 3$ solve

$$\operatorname{div}(\mathbb{C}(x)(e(u^\infty)^{ij} + e^i \cdot e^j)) = 0, \tag{2.25}$$

$$\int_{\mathbb{R}^3} |e(u^{\infty ij})|^2\, dx < \infty, \tag{2.26}$$

and

$$\mathbb{C}(x) = \chi_1^\infty \mathbb{C}_1 + \chi_2^\infty \mathbb{C}_2. \tag{2.27}$$

The matrices $e^i \cdot e^j$ are defined by $(e^i \otimes e^j + e^j \otimes e^i)/2$. The polarization and Wu tensors map the homogeneous electric and strain fields at infinity to the average electric and strain fields inside the inclusion E_2. If one considers suspensions of inclusions E_1

within a matrix of material 2 the associated low volume fraction expansions are given by

$$\sigma^e = \sigma_2 + \theta_1(\sigma_1 - \sigma_2)P_1(\sigma_1, \sigma_2) + O(\theta_1^2) \tag{2.28}$$

and

$$\mathbb{C}^e = \mathbb{C}_2 + \theta_1(\mathbb{C}_1 - \mathbb{C}_2)\mathbf{T}_1(\mathbb{C}_1, \mathbb{C}_2) + O(\theta_1^2). \tag{2.29}$$

Here $(\sigma_1 - \sigma_2)P_1$ and $\mathbf{T}_1$ are the polarization and Wu strain tensors associated with the inclusion E_1.

To fix ideas we show how to obtain bounds on the Wu strain tensor $\mathbf{T}_2$. For sufficiently small values of inclusion volume fraction θ_2 we may expand the Hashin–Shtrikman upper bound (2.17) to obtain

$$\mathbb{C}^u = \mathbb{C}_1 + \theta_2(\mathbb{C}_2 - \mathbb{C}_1)\left[\mathbf{I} - \alpha_R \int_{Q_R} \chi_1^R T_R^2 \chi_1^R \, dx \, (\mathbb{C}_2 - \mathbb{C}_1)\right] + O(\theta_2^2). \tag{2.30}$$

Here R is sufficiently large and $\alpha_R = |Q_R|/(|E_2|\,|E_1|)$ with

$$|E_2| = \int_{Q_R} \chi_2 \, dx, \quad |E_1| = \int_{Q_R} \chi_1 \, dx.$$

A straightforward calculation shows that

$$\int_{Q_R} \chi_1^R T_R^2 \chi_1^R \, dx = \int_{Q_R} \chi_2^R T_R^2 \chi_2^R \, dx. \tag{2.31}$$

Now as $\lim_{R \to \infty} (|E_1|/|Q_R|) = 1$ one finds [using methods found in AVELLANEDA (1988), LIPTON (1990)] that

$$\lim_{R \to \infty} \left(\alpha_R \int_{Q_R} \chi_2^R T_R^2 \chi_2^R \, dx\right) = \frac{1}{|E_2|} \int_{\mathbb{R}^3} \chi_2^\infty T_\infty^2 \chi_2^\infty \, dx, \tag{2.32}$$

where χ_2^∞ is the shape function of the grain ; indeed,

$$\chi_2^\infty = 1 \text{ in } E_2, \quad \chi_2^\infty = 0 \text{ outside.} \tag{2.33}$$

Here the tensor T_∞^i is defined for all τ_{ij} in $L^2(\mathbb{R}^3)^{3\times 3}$ such that for $\hat{k} = k/|k|$,

$$(T_\infty^i \sigma)(x) = \int_{\mathbb{R}^3} e^{-ik\cdot x} \hat{T}_\infty^i(\hat{k})\hat{\tau}(k) \, dk, \tag{2.34}$$

where

$$\hat{T}_\infty^i(\hat{k})\hat{\tau}(k) = \frac{3/2}{3k_i + 4\mu_i} \hat{\Gamma}_{h,\infty}(\hat{k})\hat{\tau}(k) + \frac{1}{2\mu_i} \hat{\Gamma}_{s,\infty}(\hat{k})\hat{\tau}(k), \quad i = 1, 2, \tag{2.35}$$

and

$$\hat{\Gamma}_{h,\infty}(\hat{k})\hat{\tau}(k) = 2(\hat{\tau}(k)\hat{k}, \hat{k})\hat{k} \otimes \hat{k}, \tag{2.36}$$

$$\hat{\Gamma}_{s,\infty}(\hat{k})\hat{\tau}(k) = (\hat{\tau}(k)\hat{k}) \otimes \hat{k} + \hat{k} \otimes (\hat{\tau}(k)\hat{k}) - 2(\hat{\tau}(k)\hat{k}, \hat{k})\hat{k} \otimes \hat{k}. \tag{2.37}$$

From Parseval's identity we obtain

$$\frac{1}{|E_2|}\int_{\mathbb{R}^3} \chi_2^\infty T_\infty^2 \chi_2^\infty \,\mathrm{d}x = \frac{1}{(2\pi)^3|E_2|}\int_{\mathbb{R}^3} |\hat{\chi}_2^\infty(k)|^2 \hat{T}_\infty^2(\hat{k})\,\mathrm{d}k \tag{2.38}$$

and

$$|E_2| = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} |\hat{\chi}_2^\infty(k)|^2\,\mathrm{d}k. \tag{2.39}$$

In view of (2.38) and (2.39) we write

$$\frac{1}{|E_2|}\int_{\mathbb{R}^3} \chi_2^\infty T_\infty^2 \chi_2^\infty \,\mathrm{d}x = \frac{1}{w_2}\int_{\mathbb{R}^3} |\hat{\chi}_2^\infty(k)|^2 \hat{T}_\infty^2(\hat{k})\,\mathrm{d}k, \tag{2.40}$$

where $w_2 = \int_{\mathbb{R}^3} |\hat{\chi}_2^\infty(k)|^2\,\mathrm{d}k$.

The low volume fraction expansion of the Hashin–Shtrikman upper bound is written

$$\mathbb{C}^u = \mathbb{C}_1 + \theta_2(\mathbb{C}_2 - \mathbb{C}_1)\left[\mathbf{I} - \frac{1}{w_2}\left(\int_{\mathbb{R}^3} |\hat{\chi}_2^\infty(k)|^2 \hat{T}_\infty^2(\hat{k})\,\mathrm{d}k\right)(\mathbb{C}_2 - \mathbb{C}_1)\right] + O(\theta_2^2). \tag{2.41}$$

Similar arguments show that the lower bounds for low volume fraction suspensions of identical inclusions of material 2 is given by

$$\mathbb{C}^l = \mathbb{C}_1 + \theta_2(\mathbb{C}_2 - \mathbb{C}_1)\left[\mathbf{I} + \frac{1}{w_2}\left(\int_{\mathbb{R}^3} |\hat{\chi}_2^\infty(k)|^2 \hat{T}_\infty^1(\hat{k})\,\mathrm{d}k\right)(\mathbb{C}_2 - \mathbb{C}_1)\right]^{-1} + O(\theta_2^2). \tag{2.42}$$

We now rewrite the low volume fraction Hashin–Shtrikman upper bound (2.41) in a more symmetric form. Indeed, we note that the tensor

$$I - \frac{1}{w_2}\left(\int_{\mathbb{R}^3} |(\hat{\chi}_2^\infty(k)|^2 \hat{T}_\infty^2(\hat{k})\,\mathrm{d}k\right)(\mathbb{C}_2 - \mathbb{C}_1) = \frac{1}{w_2}\int_{\mathbb{R}^3} (I - \hat{T}_\infty^2(\hat{k})(\mathbb{C}_2 - \mathbb{C}_1))|\hat{\chi}_2(k)|^2\,\mathrm{d}k \tag{2.43}$$

and calculation shows that

$$I - \hat{T}_\infty^2(\hat{k})(\mathbb{C}_2 - \mathbb{C}_1) = [I + \hat{T}_\infty^1(\hat{k})(\mathbb{C}_2 - \mathbb{C}_1)]^{-1}. \tag{2.44}$$

Thus $\mathbb{C}^u$ may be written as

$$\mathbb{C}^u = \mathbb{C}_1 + (\mathbb{C}_2 - \mathbb{C}_1)\frac{1}{w_2}\int_{\mathbb{R}^3} [I + \hat{T}_\infty^1(\hat{k})(\mathbb{C}_2 - \mathbb{C}_1)]^{-1}|\hat{\chi}_2(k)|^2\,\mathrm{d}k. \tag{2.45}$$

Introducing the measure

$$M^2(\mathrm{d}n) = \left[\frac{1}{w_2}\int_{s>0} |\hat{\chi}_2^\infty(sn)|^2\,\mathrm{d}s\right]\mathrm{d}n \tag{2.46}$$

defined on the unit sphere S^2 it follows from (2.41), (2.42), (2.45) and (2.46) that the Wu tensor T_2 for grain E_2 is bounded by

$$(\mathbb{C}_2-\mathbb{C}_1)L_2 \leqslant (\mathbb{C}_2-\mathbb{C}_1)T_2 \leqslant (\mathbb{C}_2-\mathbb{C}_1)U_2 \tag{2.47}$$

where

$$L_2 = \left[I+\int_{S_2} \hat{T}_\infty^1(n)M^2(\mathrm{d}n)(\mathbb{C}_2-\mathbb{C}_1)\right]^{-1} \tag{2.48}$$

and

$$U_2 = \int_{S_2} [I+\hat{T}_\infty^1(n)(\mathbb{C}_2-\mathbb{C}_1)]^{-1}M^2(\mathrm{d}n). \tag{2.49}$$

Similar arguments show that if the inclusion grain E_2 is softer than the matrix (i.e. $\kappa_2 < \kappa_1$, $\mu_2 < \mu_1$) then the bounds switch and

$$(\mathbb{C}_2-\mathbb{C}_1)U_2 \leqslant (\mathbb{C}_2-\mathbb{C}_1)T_2 \leqslant (\mathbb{C}_2-\mathbb{C}_1)L_2. \tag{2.50}$$

Bounds on the polarization tensor $(\sigma_2-\sigma_1)P_2$ are obtained in the same way and are given by

$$(\sigma_2-\sigma_1)l_2 \leqslant (\sigma_2-\sigma_1)P_2 \leqslant (\sigma_2-\sigma_1)\mathscr{U}_2, \tag{2.51}$$

where

$$l_2 = \left[I+\int_{S^2} \hat{\Lambda}_\infty^1(n)M^2(\mathrm{d}n)(\sigma_2-\sigma_1)\right]^{-1}, \tag{2.52}$$

$$\mathscr{U}_2 = \int_{S^2} [I+\hat{\Lambda}_\infty^1(n)(\sigma_2-\sigma_1)]^{-1}M^2(\mathrm{d}n) \tag{2.53}$$

and

$$\hat{\Lambda}_\infty^1(n) = \frac{1}{\sigma^1}\, n\otimes n. \tag{2.54}$$

The bounds switch when the inclusion grain E_2 is less conducting than the matrix.

3. Inequalities for the Polarization Tensor

In this section we examine the bounds for the polarization $(\sigma_2-\sigma_1)P_2$ of a generalized inclusion E_2 of arbitrary shape χ_2^∞ embedded in an infinite matrix of conductivity σ_1. We show that the lower bound is precisely the polarization tensor of an ellipsoid with orientation and axes determined by the function $|\hat{\chi}_2^\infty(k)|^2$. The upper bound is shown to be a convex combination of polarization tensors associated with plate like inclusions.

We first examine the lower bound on the polarization tensor. To this end we introduce the polarization tensor for an ellipsoidal grain of conductivity σ_2 embedded in an infinite matrix of σ_1 material. Let $1+m_1$, $1+m_2$, $1+m_3$, $(m_i \geqslant 0)$ be the half lengths of the principal axes and let e_1, e_2, e_3 be the coordinate axes. The ellipsoidal axes are related to the coordinate axes through the rotation matrix Q. We define the diagonal matrix $\mathbf{L}$

$$\mathbf{L} = \begin{pmatrix} L_1 & & \\ & L_2 & \\ & & L_3 \end{pmatrix} \tag{3.1}$$

such that L_i, $i = 1, 2, 3$ are the depolarizing factors given by (cf. STRATTON, 1941),

$$L_i(m_1, m_2, m_3) = \frac{\prod_{i=1}^{3} \sqrt{1+m_i}}{2} \int_1^{\infty} \frac{ds}{(s+m_i)\sqrt{\prod_{i=1}^{3} (s+m_i)}}, \tag{3.2}$$

where

$$L_1 + L_2 + L_3 = 1 \quad \text{and} \quad L_i \geqslant 0, \quad i = 1, 2, 3. \tag{3.3}$$

The polarization $(\sigma_2 - \sigma_1) P_2^e$ of the ellipsoid is given by

$$(\sigma_2 - \sigma_1) P_2^e = (\sigma_2 - \sigma_1) \left[\mathbf{I} + QLQ^{-1} \frac{\sigma_2 - \sigma_1}{\sigma_1} \right]^{-1}. \tag{3.4}$$

We note that QLQ^{-1} is positive definite and $\operatorname{tr}(QLQ^{-1}) = 1$. We now consider the lower bound l_2. It follows from (2.52) and (2.54) that l_2 is of the form

$$l_2 = \left[I + A \frac{\sigma_2 - \sigma_1}{\sigma_1} \right]^{-1}, \tag{3.5}$$

where the geometric tensor A is given by

$$A = \int_{S^2} n \otimes n \, M^2(dn). \tag{3.6}$$

It is evident that A is positive definite and trace $A = 1$. The assertion that the lower bound $(\sigma_2 - \sigma_1) l_2$ is the polarization tensor $(\sigma_2 - \sigma_1) P_2^e$ for an ellipsoid follows immediately from Theorem (3.A).

Theorem (3.*A*)

Given any grain shape function $\chi_2^{\varepsilon}(x)$ there exists an ellipsoid with axes $1+m_1$, $1+m_2$, $1+m_3$ and orientation Q such that

$$QLQ^{-1} = A. \tag{3.7}$$

The proof is somewhat technical and is provided in the Appendix.

We now consider the upper bound $(\sigma_2-\sigma_1)\mathcal{U}_2$ on the polarization tensor to show that it is given by a convex combination of polarizations for plate-like inclusions. We observe that a plate-like inclusion is obtained by letting one of the ellipsoidal axes specified by the unit vector n go to zero. It follows from the work of WALPOLE (1967) and KINOSHITA and MURA (1971) that the polarization of a plate conductivity σ_2 embedded in an infinite matrix of σ_1 conductor is given by

$$(\sigma_2-\sigma_1)P_2 = (\sigma_2-\sigma_1)\left[\mathrm{I}-n\otimes n\,\frac{(\sigma_2-\sigma_1)}{\sigma^2}\right]. \tag{3.8}$$

Noting that

$$[\mathrm{I}+\hat{\Lambda}^{1}_{\infty}(n)(\sigma_2-\sigma_1)]^{-1} = \left[\mathrm{I}-n\otimes n\,\frac{(\sigma_2-\sigma_1)}{\sigma^2}\right],$$

the upper bound $(\sigma_2-\sigma_1)\mathcal{U}_2$ given in (2.51) can be written using (2.53) and (3.6) as

$$(\sigma_2-\sigma_1)\mathcal{U}_2 = (\sigma_2-\sigma_1)\left[\mathrm{I}-A\,\frac{(\sigma_2-\sigma_1)}{\sigma_2}\right]. \tag{3.9}$$

We denote the set of 3×3 positive definite matrices with unit trace by $S^{(1)}$. It is easily seen that $S^{(1)}$ forms a convex polyhedron with rank one matrices as vertices. Since A defined by (3.6) has trace 1 and is positive definite it is an element of $S^{(1)}$. Therefore there exists at most six rank-one matrices $n^i\otimes n^i$, $i = 1,\ldots,6$ such that

$$\mathrm{A} = \sum_{i=1}^{6} p_i n^i\otimes n^i,\quad p^i \geqslant 0,\quad \sum_{i=1}^{6} p^i = 1. \tag{3.10}$$

Therefore it is immediate from (3.8), (3.9) and (3.10) that the upper bound $(\sigma_2-\sigma_1)\mathcal{U}_2$ can be written as a convex combination of polarizations for plate-like inclusions.

4. ESTIMATES OF THE WU STRAIN TENSOR USING PLATE-LIKE INCLUSIONS AND CONVEXITY PROPERTIES OF BOUNDS

Because of the higher-order tensorial nature of the Wu strain tensor for elasticity the lower bounds (2.48) on T_2 have not yet been identified with Wu tensors associated with ellipsoidal inclusions. However, work is in progress along this direction. In contrast it is relatively easy to argue as before that the upper bound (2.49) on T_2 is associated with convex combinations of Wu tensors for plate-like geometries. It follows from the work of WALPOLE (1967) and KINOSHITA and MURA (1971) that the Wu strain tensor for a plate-like inclusion of elasticity $\mathbb{C}_2$ with "n" being the direction vector of the vanishingly small axes is given by

$$\mathrm{T}_2 = [I+\hat{T}^1(n)(\mathbb{C}_2-\mathbb{C}_1)]^{-1}, \tag{4.1}$$

where $\hat{T}^1(n)$ is given by (2.35) for $i = 1$. Calculation shows that $\mathbf{T}_2$ is also given by the expression

$$\mathbf{T}_2 = \mathbf{I} - \hat{T}^2(n)(\mathbb{C}_2 - \mathbb{C}_1), \tag{4.2}$$

where $\hat{T}^2(n)$ is given by (2.35), for $i = 2$.

The upper bound U_2 given by (2.47) can be written as

$$\mathrm{U}_2 = \mathbf{I} - \mathbf{A}^2(\mathbb{C}_2 - \mathbb{C}_1), \tag{4.3}$$

where

$$\mathbf{A}^2 = \int_{S^2} \hat{T}^2(n) M^2(\mathrm{d}n). \tag{4.4}$$

We observe that (4.4) amounts to a convex combination of tensors $\hat{T}^2(\hat{n})$. The set of tensors $\hat{T}^2(\hat{n})$ for $\hat{n} \in S^2$ constitute a surface in the 14-dimensional vector space of totally symmetric tensors. It is evident from (4.4) that $\mathbf{A}^2$ lies within the convex hull of the surface. It follows immediately that extreme points on the hull are of the form $\hat{T}^2(n)$ for $n \in S^2$. Thus from Carathedory's theorem we see that $\mathbf{A}^2$ can be expressed as a convex combination of at most 15 extreme points.

Indeed, there exist at most 15 directions n^i, $i = 1, \ldots, j \leqslant 15$ and weights $\rho_i \geqslant 0$ such that $\sum_{i=1}^{j} \rho_i = 1$ and

$$\mathbf{A}^2 = \sum_{i=1}^{j} \rho_i \hat{T}^2(n^i). \tag{4.5}$$

It follows immediately from these arguments and (4.3) that the upper bound U_2 can be written as a convex combination of Wu strain tensors for plate-like inclusions.

We state here for future use a convexity property of the lower bound in terms of the geometric measure M^2. Indeed, given the parameter $0 \leqslant \rho \leqslant 1$ and two measures $M_a^2(\mathrm{d}n)$, $M_b^2(\mathrm{d}n)$ associated with two shape functions $\chi_{2,a}^{\infty}$, $\chi_{2,b}^{\infty}$ we have

$$\begin{aligned}(\mathbb{C}_2 - \mathbb{C}_1)\left[I + \left\{\rho \int_{S^2} \hat{T}_\infty^1(\hat{n}) M_a^2(\mathrm{d}n) + (1-\rho)\int_{S^2} \hat{T}_\infty^1(n) M_b^2(\mathrm{d}n)\right\}(\mathbb{C}_2 - \mathbb{C}_1)\right]^{-1} \\ \leqslant \rho(\mathbb{C}_2 - \mathbb{C}_1)\left[I + \int_{S^2} \hat{T}_\infty^1(n) M_a^2(\mathrm{d}n)(\mathbb{C}_2 - \mathbb{C}_1)\right]^{-1} \\ + (1-\rho)(\mathbb{C}_2 - \mathbb{C}_1)\left[I + \int_{S^2} \hat{T}_\infty^1(n) M_b^2(\mathrm{d}n)(\mathbb{C}_2 - \mathbb{C}_1)\right]^{-1}.\end{aligned} \tag{4.6}$$

We remark that the lower bound for the polarization tensor exhibits an identical convexity property in terms of the geometric measure $M^2(\mathrm{d}n)$.

5. INEQUALITIES FOR EMA

5.1. *Generalized DEM theories and the EMA attractor*

The generalized DEM theory provides an incremental process under which a hierarchical aggregate of different elastic or conducting phases is constructed over an infinite number of scales. For grains of ellipsoidal shape the resulting overall properties are given by explicit algebraic equations. These equations are precisely those given by the EMA (cf. NORRIS, 1985; BRUGGEMAN, 1935). For grains other than ellipsoidal, one still obtains formulas which agree with the EMA equations. However, the field equations associated with the EMA must now be solved numerically.

To fix ideas we consider a two phase hierarchical aggregate of grains with elasticity $\mathbb{C}_1$ and $\mathbb{C}_2$. We describe the construction process for the generalized DEM theory developed by NORRIS (1985). We consider a volume of space V_0 in which the composite is to be constructed. The volume V_0 is initially composed of an anisotropic elastic "backbone" material with elasticity specified by $\mathbb{C}_0$. Grains of materials 1 and 2 are "embedded" into the volume V_0 by removing an infinitesimal volume fraction of the backbone material. The resulting composite has a new overall or effective elasticity $\mathbb{C}^e$. The volume fractions of the backbone and materials 1 and 2 are given by φ_0, φ_1, φ_2 such that $\varphi_0+\varphi_1+\varphi_2 = 1$. The insertion removal process continues by removing the present material and replacing it with inclusions of materials 1 and 2. However, the replacement grains are always an order of magnitude greater in size than those at the previous removal replacement step. This insures that at each step the material in V_0 is homogeneous. (For isotropic composites we assume in addition that at each step the grains are dispersed with random orientation.) This process may be parameterized by the variable t so that the volume fractions and effective elasticity are of the form $\varphi_1 = \varphi_1(t)$, $\varphi_2 = \varphi_2(t)$, $\mathbb{C}^e = \mathbb{C}^e(t)$. The process continues until there is no backbone material left. For a composite containing volume fractions θ_1 and θ_2 of materials 1 and 2 respectively; we have

$$\lim_{t\to\infty}\varphi_1(t) = \theta_1 \quad \text{and} \quad \lim_{t\to\infty}\varphi_2(t) = \theta_2.$$

We introduce the parameters $v_1(t)$ and $v_2(t)$ that measure the total volumes of materials 1 and 2 added during the construction process. It follows that $\dot{v}_1$ and $\dot{v}_2$ are both positive as materials 1 and 2 are always added not taken out. The construction process can be described by a path in the (v_1, v_2) plane.

Given an admissible path in the (v_1, v_2) plane such that

$$\lim_{t\to\infty}\varphi_1(t) = \theta_1 \quad \text{and} \quad \lim_{t\to\infty}\varphi_2(t) = \theta_2,$$

the generalized DEM of NORRIS (1985) states that the effective elasticity $\mathbb{C}^e(t)$ satisfies the initial value problem.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{C}^e(t) = (\mathbb{C}_1-\mathbb{C}^e)\langle \mathbb{T}_1^{\mathrm{eff}}\rangle \frac{\dot{v}_1}{V_0} + (\mathbb{C}_2-\mathbb{C}^e)\langle \mathbb{T}_2^{\mathrm{eff}}\rangle \frac{\dot{v}_2}{V_0}, \tag{5.1}$$

$$\mathbb{C}^e(0) = \mathbb{C}_0. \tag{5.2}$$

Here $\mathbb{T}_1^{\mathrm{eff}}$ and $\mathbb{T}_2^{\mathrm{eff}}$ are the Wu strain tensors for a single grain of material 1 and for a grain of material 2 embedded in an infinite matrix of elasticity $\mathbb{C}^e$, respectively. The symbol $\langle\cdot\rangle$ denotes averaging over all prescribed grain orientations. NORRIS (1985) showed that in the limit as t goes to ∞ the resulting $\mathbb{C}^e(t)$ converges to $\overset{*}{\mathbb{C}}{}^e$ where $\overset{*}{\mathbb{C}}{}^e$ solves the EMA equations

$$\theta_1(\mathbb{C}_1-\overset{*}{\mathbb{C}}{}^e)\langle\overset{*}{\mathbf{T}}{}_1^{\text{eff}}\rangle+\theta_2(\mathbb{C}_2-\overset{*}{\mathbb{C}}{}^e)\langle\overset{*}{\mathbf{T}}{}_2^{\text{eff}}\rangle = 0. \tag{5.3}$$

Here $\overset{*}{\mathbf{T}}{}_1^{\text{eff}}$ and $\overset{*}{\mathbf{T}}{}_2^{\text{eff}}$ are the Wu strain tensors for grains of material 1 and of material 2 specified by shape functions χ_1^∞, χ_2^∞ embedded in an infinite matrix of elasticity $\overset{*}{\mathbb{C}}{}^e$. AVELLANEDA (1987) showed that $\mathbb{C}^e(t)$ converges to the attractor $\overset{*}{\mathbb{C}}{}^e$ as a power of the residual volume fraction φ_0 independently of grain shape.

5.2. *Inequalities for EMA theories for hierarchical elastic aggregates and suspensions using spheres and plate-like grains*

In this section we obtain upper and lower bounds on EMAs for isotropic hierarchical elastic composites made from two "well ordered" materials $\mathbb{C}_2 > \mathbb{C}_1$. These bounds depend only on volume fraction and are insensitive to further details of the geometry of the grains. We show that these bounds are the best possible bounds given this limited information. Indeed, we show that the bounds correspond to EMAs made from spheres and randomly oriented plates.

We first recall the lower bound on the Wu strain tensor appearing in (2.47) for a grain E_2 of elasticity $\mathbb{C}_2$ given by

$$(\mathbb{C}_2-\mathbb{C}_1)\mathbf{L}_2 \leqslant (\mathbb{C}_2-\mathbb{C}_1)\mathbf{T}_2. \tag{5.4}$$

Averaging (5.4) over all orientations gives

$$(\mathbb{C}_2-\mathbb{C}_1)\langle\mathbf{L}_2\rangle \leqslant (\mathbb{C}_1-\mathbb{C}_1)\langle\mathbf{T}_2\rangle, \tag{5.5}$$

where $\langle\mathbf{L}_2\rangle$ is written as

$$\langle\mathbf{L}_2\rangle = \int_{SO^3} Q\otimes Q[\mathbf{I}+\mathbf{A}^2(\mathbb{C}_2-\mathbb{C}_1)]^{-1}Q\otimes Q\,\mathrm{d}Q \tag{5.6}$$

and $\mathbf{A}^2$ is given by

$$\mathbf{A}^2 = \int_{S^2}\hat{T}^1_\infty(n)M^2(\mathrm{d}n). \tag{5.7}$$

Here $Q\otimes Q\mathbf{F}Q\otimes Q = \mathbf{F}_{ijkl}Q_{im}Q_{jn}Q_{ko}Q_{lp}$ denotes the rotation of any symmetric fourth-order tensor $\mathbf{F}$ and $\mathrm{d}Q$ is the Haar measure for SO^3. Applying the convexity property (4.6) to (5.6) yields the inequality

$$(\mathbb{C}_2-\mathbb{C}_1)\bar{\mathbf{L}}_2 \leqslant (\mathbb{C}_2-\mathbb{C}_1)\langle\mathbf{L}_2\rangle, \tag{5.8}$$

where

$$\bar{\mathbf{L}}_2 = \left[\mathrm{I}+\left(\left(\int_{SO^3}(Q\otimes Q\mathbf{A}^2 Q\otimes Q\,\mathrm{d}Q)(\mathbb{C}_2-\mathbb{C}_1)\right)\right]^{-1}. \tag{5.9}$$

The group average of $\mathbf{A}^2$ over SO^3 is given by

$$\int_{SO^3} Q\otimes Q\mathbf{A}^2 Q\otimes Q\,\mathrm{d}Q = \left(\frac{6\mu_1+3k_1}{5\mu_1(3k_1+4\mu_1)}\mathbf{I}-\frac{\mu_1+3k_1}{15\mu_1(3k_1+4\mu_1)}I\otimes I\right)\equiv \mathrm{S}_1, \tag{5.10}$$

and $\bar{L}_2$ becomes $\bar{L}_2 = [I+\mathrm{S}_1(\mathbb{C}_2-\mathbb{C}_1)]^{-1}$. We note from (5.9) and (5.10) that $\bar{\mathbf{L}}_2$ is independent of inclusion shape. Thus the shape independent lower bound $(\mathbb{C}_2-\mathbb{C}_1)\bar{\mathbf{L}}_2$ on the averaged Wu tensor is given by

$$(\mathbb{C}_2-\mathbb{C}_1)\bar{\mathbf{L}}_2 \leqslant (\mathbb{C}_2-\mathbb{C}_1)\langle \mathbf{L}_2\rangle \leqslant (\mathbb{C}_2-\mathbb{C}_1)\langle \mathbf{T}_2\rangle. \tag{5.11}$$

As $\mathrm{S}_1 = \mathrm{S}\mathbb{C}_1^{-1}$ where S is Eshelby's fourth-order tensor (cf. WU, 1966), one readily verifies that $\bar{\mathbf{L}}_2$ is also the Wu strain tensor for spherical inclusions of elasticity $\mathbb{C}_2$ embedded in an infinite matrix of elasticity $\mathbb{C}_1$ (cf. WILLIS, 1982). In this way we see that the lower bound given by $(\mathbb{C}_2-\mathbb{C}_1)\bar{\mathbf{L}}_2$ is the best possible lower bound on the orientation averaged Wu strain tensor.

A similar shape independent upper bound on the average Wu tensor $\langle \mathbf{T}_1\rangle$ for a grain E_1 embedded in a matrix of $\mathbb{C}_2$ is given by

$$(\mathbb{C}_1-\mathbb{C}_2)\langle \mathbf{T}_1\rangle \leqslant (\mathbb{C}_1-\mathbb{C}_2)\bar{\mathbf{U}}_1. \tag{5.12}$$

Here

$$\bar{\mathbf{U}}_1 = [\mathrm{I}+\mathrm{S}_2(\mathbb{C}_1-\mathbb{C}_2)]^{-1}, \tag{5.13}$$

where

$$\mathrm{S}_2 = \left(\frac{6\mu_2+3k_2}{5\mu_2(3k_2+4\mu_2)}\mathbf{I}-\frac{\mu_2+3k_2}{15\mu_2(3k_2+4\mu_2)}I\otimes I\right). \tag{5.14}$$

Here $\mathrm{S}_2 = \mathrm{S}\mathbb{C}_2^{-1}$ where S is Eshelby's tensor. One readily verifies that $\bar{\mathbf{U}}_1$ is the Wu strain tensor for spherical inclusions of elasticity $\mathbb{C}_1$ embedded in an infinite matrix of elasticity $\mathbb{C}_2$ (cf WILLIS, 1982).

We introduce the EMA for two special isotropic aggregates: we first consider an isotropic aggregate of spherical grains $\bar{E}_1$ and the plate like grains $\bar{E}_2$ in the concentrations θ_1 and θ_2 respectively, and denote its effective tensor by $\bar{\mathbb{C}}$. We also consider the EMA for the dual isotropic aggregate made of plate like grains $\underline{E}_1$ and spherical grains $\underline{E}_2$ in the concentrations θ_1 and θ_2 respectively, and denote its effective tensor by $\underline{\mathbb{C}}$. We now show that the EMAs for these two special aggregates place upper and lower bounds on all EMAs independent of grain shape.

Theorem (5.A). Inequalities for an isotropic two phase elastic EMA

All isotropic EMAs $\mathbb{C}^e$ defined by (5.54) for arbitrary grains E_1 and E_2 in the proportions θ_1 and θ_2 are bounded below by the EMA, $\underline{\mathbb{C}}$, with grains $\underline{E}_2$ spherical and $\underline{E}_1$ plate-like and bounded above by the EMA, $\bar{\mathbb{C}}$, with grains $\bar{E}_2$ plate-like and $\bar{E}_1$ spherical, i.e.

$$\underline{\mathbb{C}} \leqslant \overset{*}{\mathbb{C}}^{e} \leqslant \bar{\mathbb{C}}. \tag{5.15}$$

The proof of the theorem follows immediately from Section 4 and the inequalities (2.47), (2.50), (5.11) and (5.12). Indeed for the homogenization path (v_1, v_2) associated with the initial value problem (5.1) we have that $\bar{\mathbb{C}}$ and $\underline{\mathbb{C}}$ given in the hypothesis are the EMA attractors for the initial value problems

$$\frac{d\mathbb{C}^e}{dt} = (\mathbb{C}_1 - \mathbb{C}^e)\langle \bar{T}_1^{\text{eff}} \rangle \frac{\dot{v}_1}{v_0} + (\mathbb{C}_2 - \mathbb{C}^e)\langle \bar{T}_2^{\text{eff}} \rangle \frac{\dot{v}_2}{v_0}, \tag{5.16}$$

$$\mathbb{C}^e = \mathbb{C}_0 \tag{5.17}$$

and

$$\frac{d\mathbb{C}^e}{dt} = (\mathbb{C}_1 - \mathbb{C}^e)\langle \underline{T}_1^{\text{eff}} \rangle \frac{\dot{v}_1}{v_0} + (\mathbb{C}_2 - \mathbb{C}^e)\langle \underline{T}_2^{\text{eff}} \rangle \frac{\dot{v}_2}{v_0}, \tag{5.18}$$

$$\mathbb{C}^e = \mathbb{C}_0, \tag{5.19}$$

respectively. Here $\langle \bar{T}_1^{\text{eff}} \rangle$ is the average Wu tensor for a spherical grain E_1 embedded in an effective elastic medium $\mathbb{C}^e$ and $\langle \bar{T}_2^{\text{eff}} \rangle$ is the average Wu tensor for a plate-like grain E_2 embedded in the effective medium $\mathbb{C}^e$. Similarly $\langle \underline{T}_1^{\text{eff}} \rangle$ is the average Wu tensor of a plate-like grain E_1 embedded in an effective medium and $\langle \underline{T}_2^{\text{eff}} \rangle$ is the average Wu tensor corresponding to a spherical grain embedded in an effective elastic medium $\mathbb{C}^e$. Denoting the right-hand sides of (5.1), (5.16), (5.18) by $\bar{f}(t, \mathbb{C}^e)$, $f(t, \mathbb{C}^e)$, and $\underline{f}(t, \mathbb{C}^e)$ respectively, it now follows from the inequalities (2.47), (2.50), (5.11) and (5.12) that

$$\underline{f}(t, \mathbb{C}^e) \leqslant f(t, \mathbb{C}^e) \leqslant \bar{f}(t, \mathbb{C}^e), \tag{5.20}$$

and the theorem is proved by noting that the above inequality holds for all time.

5.3. *Inequalities for EMA theory for hierarchical random conductivity models*

We consider the EMA for an isotropic hierarchical composite made up of two grains E_1 and E_2 of isotropically conducting materials σ_1 and σ_2 such that $0 < \sigma_1 < \sigma_2$. We suppose that the grains have shapes given by χ_1^x, χ_2^x and occur in the proportions θ_1, θ_2, and denote the isotropic effective conductivity by $\overset{*}{\sigma}^e$.

We now introduce the EMA for two special aggregates: the first is an isotropic hierarchical aggregate composed of ellipsoidal grains $\underline{E}_2$ of material 2 and plate-like grains $\underline{E}_1$ of material 1. The depolarizing factors for the ellipsoidal grain are denoted by $\underline{L}_i$, $i = 1, 2, 3$ and the effective conductivity is denoted by $\underline{\sigma}$. The second isotropic hierarchical aggregate is composed of ellipsoidal grains $\bar{E}_1$ of material 1 and plate-like grains $\bar{E}_2$ of material 2 and we denote the associated effective conductivity by $\bar{\sigma}$. The depolarizing factors for the ellipsoidal grain are denoted by $\bar{L}_i$, $i = 1, 2, 3$.

Lastly, we introduce the geometric tensors $\bar{A}$ and $\underline{A}$ associated with grains E_1 and E_2 defined by

$$\bar{A} = \int_{S^2} n \otimes n M_1(\mathrm{d}n) \quad \text{and} \quad \underline{A} = \int_{S^2} n \otimes n M_2(\mathrm{d}n). \tag{5.21}$$

Here M^1 and M^2 are the measures associated with shape functions $\chi_1^\infty, \chi_2^\infty$ respectively. Letting $\underline{\lambda}_i$ and $\bar{\lambda}_i$ $(i = 1, 2, 3)$ represent the eigenvalues of the tensors $\underline{A}$ and $\bar{A}$ respectively, we have the following theorem.

Theorem (5.*B*) *Inequalities for an isotropic two phase electrically conducting EMA*

Given the isotropic EMA $\mathring{\sigma}^e$ for arbitrary grains E_1 and E_2 of shape χ_1^∞ and χ_2^∞ respectively in the proportions θ_1 and θ_2, there exists an EMA of conductivity $\underline{\sigma}$ with grains $\underline{E}_2$ ellipsoidal and $\underline{E}_1$ plate-like with volume fractions θ_1, θ_2 and an EMA of conductivity $\bar{\sigma}$ with grains $\bar{E}_1$ ellipsoidal and $\bar{E}_2$ plate-like with volume fractions θ_1, θ_2 such that

$$\underline{\sigma} \leqslant \mathring{\sigma}^e \leqslant \bar{\sigma}, \tag{5.22}$$

where the depolarizing factors of the ellipsoidal grains $\bar{E}_1$ and $\underline{E}_2$ are determined by the geometric tensors $\bar{A}$ and $\underline{A}$ through the equations

$$\bar{L}_i = \bar{\lambda}_i, \quad i = 1, 2, 3, \tag{5.23}$$

and

$$\underline{L}_i = \underline{\lambda}_i, \quad i = 1, 2, 3, \tag{5.24}$$

respectively.

The proof of theorem is analogous to the proof of Theorem 5.A and follows from the inequalities (2.51).

We now fix the grain shapes E_1, E_2 and allow the volume fraction θ_2 to vary from zero to one. We consider the associated family $\mathscr{F}(E_1, E_2)$ of EMAs with homogenization paths terminating at θ_2 and $\theta_1 = 1 - \theta_2$. For θ_2 in [0, 1] we denote any EMA in this family by $\mathring{\sigma}^e(\theta_2)$. Analogously we define the families $\mathscr{F}(\bar{E}_1, \bar{E}_2)$ and $\mathscr{F}(\underline{E}_1, \underline{E}_2)$. Here $\mathscr{F}(\bar{E}_1, \bar{E}_2)$ is the family of EMAs associated with ellipsoidal grains $\bar{E}_1$ of material 1 and plate like grains $\bar{E}_2$ of material 2, and $\mathscr{F}(\underline{E}_1, \underline{E}_2)$ is the family associated with plate-like grains $\underline{E}_1$ of material 1 and ellipsoidal grains $\underline{E}_2$ of material 2. The EMAs associated with the families $\mathscr{F}(\bar{E}_1, \bar{E}_2)$ and $\mathscr{F}(\underline{E}_1, \underline{E}_2)$ are denoted by $\bar{\sigma}(\theta_2)$ and $\underline{\sigma}(\theta_2)$ respectively.

We observe that the estimates given by (2.51) are independent of volume fraction and depend only upon the grain shapes E_1 and E_2, therefore it follows that we have bounds on EMAs with prescribed grain shape. Indeed, given the family $\mathscr{F}(E_1, E_2)$ of EMAs there exist extremal families $\mathscr{F}(\bar{E}_1, \bar{E}_2)$ and $\mathscr{F}(\underline{E}_1, \underline{E}_2)$ such that

$$\underline{\sigma}(\theta_2) \leqslant \overset{*}{\sigma}{}^{e}(\theta_2) \leqslant \bar{\sigma}(\theta_2) \tag{5.25}$$

for $0 \leqslant \theta_2 \leqslant 1$.

Here the depolarizing factors of the ellipsoidal grains $\bar{E}_1$ and $\underline{E}_2$ are determined by the geometric tensors $\bar{A}$ and $\underline{A}$ through the equations

$$\bar{L}_i = \bar{\lambda}_i, \quad i = 1, 2, 3, \tag{5.26}$$

and

$$\underline{L}_i = \underline{\lambda}_i, \quad i = 1, 2, 3, \tag{5.27}$$

respectively.

It is interesting to plot the upper and lower bounds given by (5.25). For very rotund grains the associated upper and lower bounds are far apart as in Fig. 2. As one or

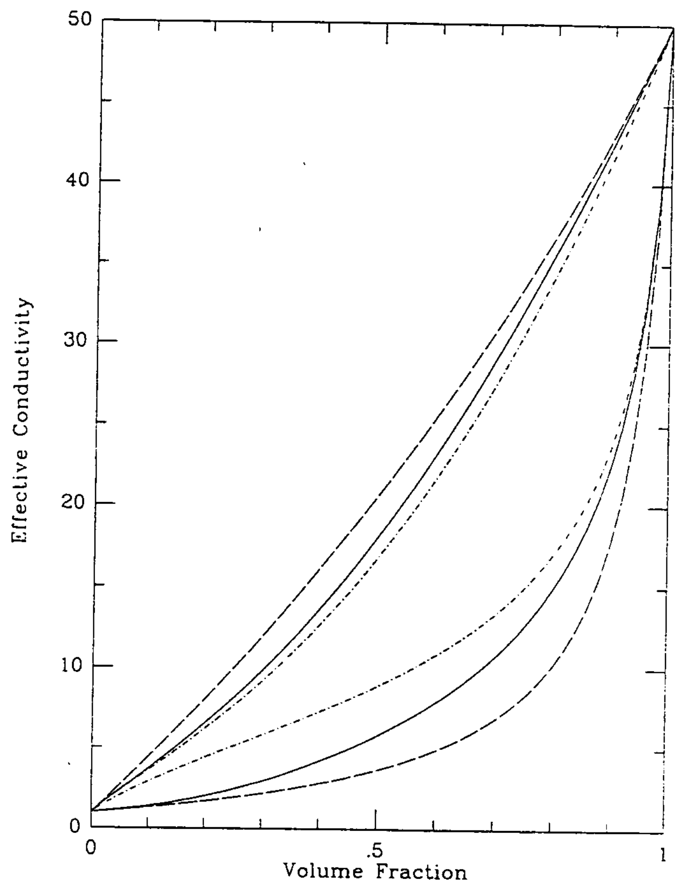

FIG. 2. Isoperimetric bounds are plotted for $\sigma_1 = 1$ and $\sigma_2 = 50$. The dashed lines represent the Hashin–Shtrikman upper and lower bounds. The solid curves represent the geometry independent bounds on the EMA. The upper solid curve corresponds to spheres of σ_1 and plate-like inclusions of σ_2. The lower solid curve corresponds to spheres of σ_2 and plate-like inclusions of σ_2. The curves $\cdot - \cdot - \cdot$ indicate geometry dependent bounds on the effective conductivity. These upper bounds correspond to ellipsoidal inclusions of σ_1 with depolarizations $L_1 = 0.125$, $L_2 = 0.25$, $L_3 = 0.625$, and plates of σ_2; the lower bounds correspond to ellipsoidal inclusions of σ_2 with depolarizations $L_1 = 0.125$, $L_2 = 0.375$, $L_3 = 0.5$, and plates of σ_1.

both of the grains become flat the upper and lower bounds approach each other as in Fig. 3. The upper and lower bounds converge and agree in the limiting case of plate-like grains. It is evident that these bounds gauge the rotundness of the grain shape. We remark that the EMAs appearing in Figs 2 and 3 were found by computing the attractors of the generalized DEM of NORRIS (1985) for homogenization paths $(\varphi_1(t))$, $(\varphi_2(t))$ given by $\varphi_1 = t$, $\varphi_2 = t\theta_2/(1-\theta_2)$, for $0 \leqslant t \leqslant 1-\theta_2$.

6. A DESCRIPTION OF THE SET OF EFFECTIVE CONDUCTIVITIES PREDICTED BY THE EMA

Here we describe the set of effective conductivities that the EMA is capable of modeling as we consider all possible grain shapes with Lipschitz boundaries. In this

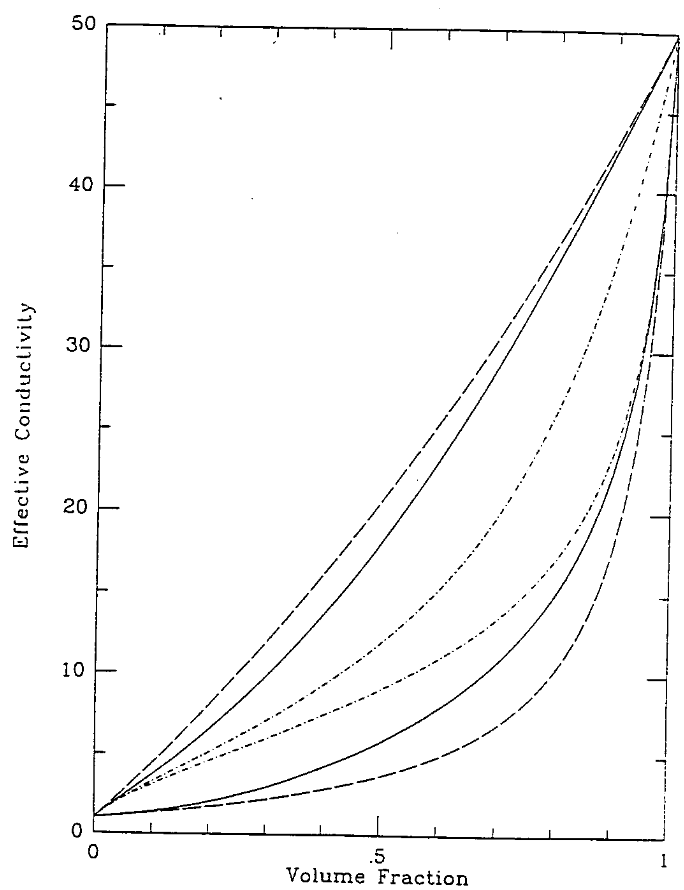

FIG. 3. In Fig. 3 the upper bounds are those for ellipsoids of σ_1 with polarizations $L_1 = 0.1$, $L_2 = 0.1$, $L_3 = 0.8$, and $L_1 = 0.01$, $L_2 = 0.01$, $L_3 = 0.98$, respectively. The lower bounds are those for ellipsoids of σ_2 with polarizations $L_1 = 0.1$, $L_2 = 0.1$, $L_3 = 0.8$, and $L_1 = 0.01$, $L_2 = 0.01$, $L_3 = 0.98$, respectively.

way we find the set of all effective conductivities of the hierarchical models realizing the EMA.

We start by describing the boundary of the set. We introduce two extremal families of EMA. The first family comprises those constructed from spherical grains $\bar{E}_1$ of material 1 and plate-like grains $\bar{E}_2$ of material 2. The associated EMAs are parameterized by volume fraction θ_2 only, and are given by $\bar{\sigma}^{sp}(\theta_2)$ for $0 \leqslant \theta_2 \leqslant 1$. The second class of EMAs are those constructed from plate-like grains $\underline{E}_1$ of material 1 and spherical grains $\underline{E}_2$ of material 2. The associated EMAs are denoted by $\underline{\sigma}^{sp}(\theta_2)$ for $0 \leqslant \theta_2 \leqslant 1$.

It follows immediately from the convexity properties of the bounds on the electric polarization tensors noted in Section 4, that

$$\underline{\sigma}^{sp}(\theta_2) \leqslant \sigma^{e}(\theta_2) \leqslant \bar{\sigma}^{sp}(\theta_2) \tag{6.1}$$

for any effective tensor $\sigma^e(\theta_2)$ associated with an EMA made from grains E_1 and E_2 in the proportions $1-\theta_2$ and θ_2 respectively.

We see that this places upper and lower bounds on all EMAs independent of grain shape. We plot the bounds $\underline{\sigma}^{sp}(\theta_2)$ and $\bar{\sigma}^{sp}(\theta_2)$ for conductivities $\sigma_1 = 1$ and $\sigma_2 = 50$ in Fig. 2. We see that these bounds are narrower than the well-known Hashin–Shtrikman bounds also plotted in Fig. 2.

We consider the region lying between two functions $\underline{\sigma}^{sp}(\theta_2)$ and $\bar{\sigma}^{sp}(\theta_2)$ for $0 \leqslant \theta_2 \leqslant 1$. In the following we show that every point in this region is the effective conductivity for a hierarchical EMA made from spheroidal grains. In this way we show that the set of effective conductivities modeled by the EMA may be realized by only considering those for spheroidal grains! This gives us some feel for the robustness of the EMA with regard to grain shape.

To show that every point is realized, we fix θ_2 and show that the interval $[\underline{\sigma}^{sp}(\theta_2), \bar{\sigma}^{sp}(\theta_2)]$ is swept out by EMAs associated with spheroidal grains. Indeed, for an admissible homogenization path such that

$$\lim_{t\to\infty} \varphi_1(t) = \theta_1, \quad \lim_{t\to\infty} \varphi_2(t) = \theta_2,$$

the EMA "$\mathring{\sigma}^e(\theta_2, A)$" is defined as the attractor for the initial value problem given by

$$\frac{d\sigma^e(t,\theta_2,A)}{dt} = f(t,\sigma^e,A), \tag{6.2}$$

$$\sigma^e(0) = \sigma_0. \tag{6.3}$$

Here, $0 \leqslant A \leqslant \frac{1}{3}$ and $f(t,\sigma^e,A)$ is given by

$$f(t,\sigma^e,A) = \frac{(\sigma_1-\sigma^e)}{3}\langle P_1^{e,A}\rangle \frac{\dot{v}_1(t)}{V_0} + \frac{(\sigma_2-\sigma^e)}{3}\langle P_2^{e,A}\rangle \frac{\dot{v}_2(t)}{V_0}, \tag{6.4}$$

where $(\sigma_1-\sigma^e)\langle P_1^{e,A}\rangle$ and $(\sigma_2-\sigma^e)\langle P_2^{e,A}\rangle$ are the orientation averaged polarization tensors for spheroidal grains with depolarizing factors given by

$L_1 = L_2 = A, \quad L_3 = 1-2A, \quad$ and $\quad L_1 = (\frac{1}{3}-A), \quad L_2 = (\frac{1}{3}-A),$

and $\quad L_3 = (\frac{1}{3}+2A)$

respectively. One readily checks that $\overset{*}{\sigma}^e(\theta_2, 0) = \underline{\sigma}^{sp}(\theta_2)$ and $\overset{*}{\sigma}^e(\theta_2, \frac{1}{3}) = \bar{\sigma}^{sp}(\theta_2)$. Therefore, to show that every point between $\underline{\sigma}^{sp}(\theta_2)$ and $\bar{\sigma}^{sp}(\theta_2)$ is attained we prove that $\overset{*}{\sigma}^e(\theta_2, A)$ is continuous in the parameter A. We observe from the result of AVELLANEDA (1987) that $\sigma^e(t, \theta_2, A)$ converges to the attractor $\overset{*}{\sigma}^e(\theta_2, A)$ as a power of the residual volume independently of the parameter A. Next, we observe for $0 \leqslant t \leqslant \infty$, $0 \leqslant A \leqslant \frac{1}{3}$, and $\sigma_1 \leqslant \sigma^e \leqslant \sigma_2$ that the right-hand side $f(t, \sigma^e, A)$ is continuous in t, σ^e and A, and locally Lipshitzian in σ^e independently of A. Thus for $0 \leqslant t < \infty$ the solution $\sigma^e(t, \theta_2, A)$ is continuous in A. Continuity of the EMA

$$\overset{*}{\sigma}^e(\theta_2, A) = \lim_{t \to \infty} \sigma^e(t, \theta_2, A)$$

now follows immediately from the remarks given above and from the estimate

$$|\overset{*}{\sigma}^e(\theta_2, A) - \overset{*}{\sigma}^e(\theta_2, A')| \leqslant |\overset{*}{\sigma}^e(\theta_2, A) - \overset{*}{\sigma}(t, \theta_2, A)| + |\overset{*}{\sigma}^e(t, \theta_2, A') - \overset{*}{\sigma}^e(t, \theta_2, A)|$$
$$+ |\overset{*}{\sigma}^e(t, \theta_2, A') - \overset{*}{\sigma}^e(\theta_2, A')|. \quad (6.5)$$

Here A' lies in $[0, \frac{1}{3}]$ and is sufficiently close to A.

7. BOUNDS ON LOW VOLUME EXPANSIONS FOR ANISOTROPIC CONDUCTING AND ELASTIC COMPOSITES

The low volume expansion for the effective conductivity of a dilute random suspension of particles of conductivity σ_2 in a matrix of σ_1 is given by

$$\boldsymbol{\sigma}^e = \sigma_1 \mathbf{I} + \theta_2(\sigma_2 - \sigma_1)\langle P_2(\sigma_1, \sigma_2)\rangle + O(\theta_2^2). \quad (7.1)$$

Here the effective conductivity may be anisotropic. The single grain polarization tensor $(\sigma_2 - \sigma_1)P_2(\sigma_1, \sigma_2)$ is defined by (2.19). The average polarization tensor of the suspension is $\langle P_2(\sigma_1, \sigma_2)\rangle$ where the average is taken over prescribed grain shape and orientation distributions. We now exhibit upper and lower bounds on σ^e valid up to second order in the volume fraction. Indeed, it follows immediately from the inequalities on single grain polarization tensors given in (2.51) that there exists a low volume fraction expansion of conductivity $\bar{\sigma}$ of plate-like inclusions of material 2 and a low volume expansion of conductivity $\bar{\sigma}$ of ellipsoidal inclusions of material 2 for which the inequality

$$\underline{\sigma} \leqslant \sigma^e \leqslant \bar{\sigma}^e \quad (7.2)$$

holds to second order in the volume fraction θ_2. A similar inequality holds to second order for the effective conductivity of a suspension of a σ_2 inclusion in a σ_1 matrix with the convention $\sigma_1 > \sigma_2$. Here the upper bound is given by a suspension of ellipsoids and the lower bound corresponds to a suspension of plate-like inclusions. It now follows that for isotropic suspensions of material 2 grains in a matrix of material 1 with $\sigma_2 > \sigma_1$ that the low volume fraction expansion of MAXWELL (1873) for isotropic suspensions of spheres.

$$\overset{*}{\sigma} = \sigma_1 + \frac{\theta_2 3(\sigma_2 - \sigma_1)\sigma_1}{\sigma_2 + 2\sigma_1}, \tag{7.3}$$

is to second order (in θ_2), a lower bound on the effective conductivity of isotropic suspensions of arbitrary particle shape. If $\sigma_2 < \sigma_1$, it is an upper bound. In this way we see that random isotropic dilute suspensions of spheres have extremal effective conducting properties. Arguing in the same fashion we see for $\mathbb{C}_2 > \mathbb{C}_1$, that the effective elastic tensor $\mathbb{C}^e$ of a random low volume fraction suspension of particles of elasticity $\mathbb{C}_2$ embedded in a matrix $\mathbb{C}_1$ is bounded to second order in the volume fraction θ_2 by

$$\mathbb{C}_1 + \theta_2(\mathbb{C}_2 - \mathbb{C}_1)\langle \mathbf{L}_2 \rangle \leqslant \mathbb{C}^e \leqslant \mathbb{C}_1 + \theta_2(\mathbb{C}_2 - \mathbb{C}_1)\langle \mathbf{U}_2 \rangle. \tag{7.4}$$

Here $\mathbf{L}_2$ and $\mathbf{U}_2$ are given by (2.48) and (2.49) respectively and the average $\langle \cdot \rangle$ is taken over the prescribed grain shape and orientation distributions appearing in the suspension. We remark that for the case $\mathbb{C}_1 > \mathbb{C}_2$ the bounds are reversed.

For isotropic suspensions of material 2 grains in a matrix of material 1 with $\mathbb{C}_2 > \mathbb{C}_1$, it follows that the low volume fraction expansion for an isotropic suspension of spheres given by

$$\overset{*}{\mathbb{C}} = \mathbb{C}_1 + \theta_2(\mathbb{C}_2 - \mathbb{C}_1)[\mathbf{I} + \mathbf{S}_1(\mathbb{C}_2 - \mathbb{C}_1)]^{-1} \tag{7.5}$$

[where $\mathbf{S}_1$ is given by (5.10)], is to second order, a lower bound on the effective elasticity of isotropic suspensions of arbitrary particle shape. If $\mathbb{C}_2 < \mathbb{C}_1$, it is upper bound.

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APPENDIX

Here we establish Theorem (3.A). It follows from the positive definiteness of A and tr $A = 1$ that it is sufficient to show that for any point (a_1, a_2, a_3) on the polyhedron $\Delta = \{a_1, a_2, a_3;\ a_1+a_2+a_3 = 1,\ a_i \geqslant 0\}$ there exist points $m = (m_1, m_2, m_3)$ in $\mathbb{R}^3_+$ such that

$$L_i = a_i. \tag{A.1}$$

Proof. We define the vector m^{-1} in $\mathbb{R}^3_+$ by $m^{-1} = (m_1^{-1}, m_2^{-1}, m_3^{-1})$ and let $\underline{B}(m) = L_1(m^{-1}),$ $L_2(m^{-1}),\ L_3(m^{-1})$. Clearly $\underline{B}$ lies on the polyhedron Δ. We show that $\underline{B}$ attains every point on Δ as m ranges over $\mathbb{R}^3_+$. This will establish (A.1).

To prove this we introduce the auxiliary function $T(m)$ given by

$$T(m) = \left(\sum_{k=1}^{3} m_k\right)\underline{B}(m). \tag{A.2}$$

We state the following

Lemma A.1. Given any point P in $\mathbb{R}^3_+$ there exists a point (m_1, m_2, m_3) in $\mathbb{R}^3_+$ such that

$$T(m) = P. \tag{A.3}$$

It follows from Lemma A.1 that $\underline{B}(m)$ attains all points on Δ. To see this consider any point $\underline{a} = (a_1, a_2, a_3)$ on Δ. We form $\lambda\underline{a}$ where λ is in $\mathbb{R}_+$. From Lemma A.1 there exists an m in $\mathbb{R}^3_+$ such that

$$T(m) = \lambda\underline{a}. \tag{A.4}$$

We see immediately from (3.9) and (3.11) that

$$\sum_{k=1}^{3} T_k(m) = \sum_{k=1}^{3} (m_k) = \lambda \tag{A.5}$$

and therefore

$$\underline{B}(m) = \underline{a}. \tag{A.6}$$

The proof of Lemma A.1 follows from an application of topological degree theory. Given any point P in $\mathbb{R}^3_+$ we show that $T^{-1}(P)$ is a compact subset of $\mathbb{R}^3_+$. To see this we note that $m_j = 0$ implies $T_j = 0$ and $m_j \to \infty$ implies $T(m) \to \infty$. Therefore there exists a positive number R and a set

$$B_R = \left\{ (m_1, m_2, m_3) \,\middle|\, \sum_{i=1}^{3} m_i \leqslant R, m \in \mathbb{R}^3_+ \right\}$$

such that $T^{-1}(P) \subseteq B_R$ and $T^{-1}(P)$ does not intersect the boundary of B_R. We now apply the topological degree to the bounded set B_R. We consider the homotopy $h(t, m) = tI(m) + (1-t)T(m)$. Here $I(m)$ is the identity transform. One readily verifies that $P \notin h(t, \partial B_R)$, and thus homotopy invariance of the degree implies $\mathrm{d}(T(m), B_R, P) = 1$, and Lemma A.1 is proved.