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OPTIMAL INEQUALITIES FOR GRADIENTS OF SOLUTIONS OF ELLIPTIC EQUATIONS OCCURRING IN TWO-PHASE HEAT CONDUCTORS*

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Abstract. We consider solutions to divergence form partial differential equations that model steady state heat conduction in random two-phase composites. The coefficient representing the conductivity takes two scalar values. Optimal bounds on the L^2 norm of the gradient of the solution are found. The optimal upper bound is given in terms of the volume fraction occupied by each conducting phase. The optimal lower bound is independent of the volume fractions of the component conductors. The bounds follow from a Stieltjes integral representation for the L^2 norm of the gradient. Maximizing sequences of configurations are found using the corrector theory of homogenization.

Key words. homogenization, Stieltjes functions, spectral theorem, isoperimetric inequalities

AMS subject classifications. 35J, 35P, 74Q05

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1. Introduction. Consider a bounded region Ω of R^N with a sufficiently regular boundary containing two isotropic conductors subjected to a constant applied temperature gradient $\mathbf{E}$ in R^N. Here we consider any dimension N greater than or equal to 2. The conductivities of the two materials are written as α and β, and the indicator function of the β phase is denoted by χ, where $\chi = 1$ inside the β phase and 0 otherwise. We suppose that the set occupied by the β phase is Lebesgue measurable and that $\beta > \alpha$. The local conductivity of the two-phase conductor is described by $a(\chi) = \alpha(1 - \chi) + \beta\chi$. The temperature T inside the two-phase conductor is the solution of

$$-\text{div}\ (a(\chi)\nabla T) = 0 \tag{1.1}$$

subject to the boundary condition $T = \mathbf{E} \cdot \mathbf{x}$. Since the coefficient $a(\chi)$ is bounded and measurable, the equilibrium equation (1.1) is interpreted in the weak sense. Here we recall that the weak solution of (1.1) is defined to be the function T in $W^{1,2}(\Omega)$ that satisfies

$$\int_\Omega a(\chi)\nabla T \cdot \nabla\varphi \, dx = 0 \tag{1.2}$$

for all functions φ in $W_0^{1,2}(\Omega)$.

We suppose that the composite is random in that we specify only the volume fraction θ of the β phase and consider the ensemble of configurations that satisfy

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this isoperimetric constraint. The set of conductivities associated with this class is denoted by ad_θ and is written

$$ad_\theta = \left\{ a(\chi),\ \text{where } \chi \text{ satisfies} \int_\Omega \chi\, dx = \theta \times meas(\Omega),\ 0 \leq \theta \leq 1 \right\}. \tag{1.3}$$

In this paper we address the problem of extremizing

$$\|\nabla T\|_2^2 \triangleq \int_\Omega |\nabla T|^2\, dx \tag{1.4}$$

over the class ad_θ. We provide optimal upper and lower bounds on the quantity $\|\nabla T\|_2^2$. The upper bound depends explicitly upon the volume fraction occupied by the β phase. In order to state the bounds we set $\lambda = \frac{\beta}{\alpha}$ and introduce the function $f(z)$ defined by

$$f(z) = \frac{z}{\left(\frac{1}{1-\lambda} - z\right)^2} \tag{1.5}$$

and we give the following optimal inequality.

THEOREM 1.1 (optimal inequality for the L^2 norm of the gradient). *For any admissible conductivity $a(\chi)$ in ad_θ the associated temperature gradient ∇T satisfies*

$$meas(\Omega) \times |\mathbf{E}|^2 \leq \|\nabla T\|_2^2 \leq U(\theta, \mathbf{E}), \tag{1.6}$$

where $U(\theta,\mathbf{E})$ depends upon the contrast λ and is given by

$$U(\theta, \mathbf{E}) = meas(\Omega) \times (1 + \theta f(1-\theta))\, |\mathbf{E}|^2 \quad \text{for } \lambda \leq 2, \tag{1.7}$$

and for $\lambda \geq 2$

$$U(\theta, \mathbf{E}) = meas(\Omega) \times \begin{cases} (1 + \theta f(1/(\lambda-1)))\, |\mathbf{E}|^2 & \text{if } \theta \leq 1 - 1/(\lambda-1), \\ (1 + \theta f(1-\theta))\, |\mathbf{E}|^2 & \text{if } \theta \geq 1 - 1/(\lambda-1). \end{cases} \tag{1.8}$$

The upper bound is attained by a suitable extremal sequence of configurations in ad_θ. The lower bound is attained by a configuration made up of parallel layers of the β conductor with layer normals orthogonal to $\mathbf{E}$. These results hold for all bounded domains Ω of R^N, $N \geq 2$ with Lipschitz boundary.

Extremal sequences of configurations that attain the upper bound are found to be given by the well-known finite rank laminar microstructures. This class of configurations is known to give extremal effective conductivity properties; see [7] and [9]. They also arise in the study of minimization problems for integral functionals of the form $\int_\Omega W(\nabla\phi)\, dx$ with nonconvex energy densities W; see [1], [3], [4], and [6].

It is shown here that only laminates of the first and second rank appear in extremal sequences of configurations. In order to describe a second rank laminate we introduce two characteristic functions, one for each scale of oscillation. We consider the periodic function $\chi^1(t)$ defined on the real line with period $0 \leq t \leq 1$ such that $\chi^1 = 1$ for $0 \leq t \leq \theta_1$ and $\chi^1 = 0$ elsewhere. Similarly we introduce the unit periodic function χ^2 such that $\chi^2 = 1$ for $0 \leq t \leq \theta_2$ and $\chi^2 = 0$ elsewhere. We introduce unit vectors $\mathbf{n}^1$ and $\mathbf{n}^2$ representing layer directions and put

$$\chi_L^\varepsilon(x) = \left(1 - \chi^1\left(\frac{\mathbf{n}^1 \cdot x}{\varepsilon}\right)\right)\left(1 - \chi^2\left(\frac{\mathbf{n}^2 \cdot x}{\varepsilon^2}\right)\right). \tag{1.9}$$

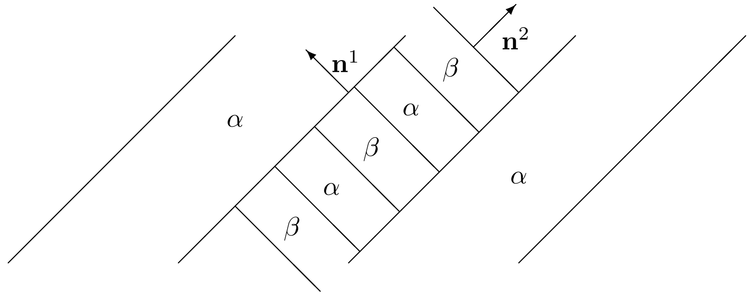

FIG. 1. *A laminate of second rank.*

The configurations associated with the sequence of characteristic functions $\{\chi_L^\varepsilon\}_{\varepsilon>0}$ are referred to as a laminate of the second rank. The conductivities for this sequence of configurations are given by $a(\chi_L^\varepsilon)$; see Figure 1. The laminate of first rank has one less scale of oscillation and is given by

$$\chi_L^\varepsilon(x) = \left(1 - \chi^1\left(\frac{\mathbf{n}^1 \cdot x}{\varepsilon}\right)\right). \tag{1.10}$$

The sequence of temperature gradients associated with laminates of rank one or two is written as $\{\nabla T_L^\varepsilon\}_{\varepsilon>0}$, where

$$-\mathrm{div}\ (a(\chi_L^\varepsilon)\nabla T_L^\varepsilon) = 0 \tag{1.11}$$

and $T_L^\varepsilon = \mathbf{E} \cdot \mathbf{x}$ on the boundary of Ω.

In general, we may consider any sequence of configurations $\{\chi^\varepsilon\}_{\varepsilon>0}$ indexed by ε and the associated sequence of temperature gradients $\{\nabla T^\varepsilon\}_{\varepsilon>0}$ satisfying $-\mathrm{div}\ (a(\chi^\varepsilon)\nabla T^\varepsilon) = 0$ and $T^\varepsilon = \mathbf{E} \cdot \mathbf{x}$ on the boundary of Ω. A sequence of configurations $\{\chi^\varepsilon\}_{\varepsilon>0}$ is said to be a maximizing sequence if

$$\lim_{\varepsilon\to 0} \|\nabla T^\varepsilon\|^2 = U(\theta, \mathbf{E}). \tag{1.12}$$

A configuration is said to be minimizing if the associated temperature T satisfies

$$\|\nabla T\|^2 = meas(\Omega) \times |\mathbf{E}|^2. \tag{1.13}$$

The next result identifies configurations that minimize the L^2 norm of the gradient.

THEOREM 1.2 (minimizing configurations for the L^2 norm). *Given* $\mathbf{E}$ *in* R^N*, a minimizing configuration is obtained by placing the* β *conductor in layers oriented so that the layer normals are orthogonal to* $\mathbf{E}$*. The number and thickness of the layers is constrained only by the requirement that the configuration be in* ad_θ*.*

It is easily shown that the temperature for this configuration is given by $T = \mathbf{E}\cdot\mathbf{x}$ everywhere in Ω; see section 4.

We now identify maximizing sequences of configurations.

THEOREM 1.3 (maximizing sequences of configurations for the L^2 norm). *Given* $\mathbf{E}$ *in* R^N

1. *If $\lambda \leq 2$, then a maximizing sequence of configurations is given by a laminate of the first rank in ad_θ with layer normal $\mathbf{n}^1$ parallel to $\mathbf{E}$ and $\theta^1 = 1 - \theta$.*
2. *If $\lambda > 2$, then*
   (a) *if $\theta \leq 1 - 1/(\lambda - 1)$, then a maximizing sequence of configurations is given by a laminate of the second rank in ad_θ with layer normal $\mathbf{n}^1$ parallel to $\mathbf{E}$, layer normal $\mathbf{n}^2$ orthogonal to $\mathbf{E}$, $\theta_1 = \frac{1}{1+\theta(\lambda-1)}$, and $\theta_2 = 1 - \theta - (\frac{1}{\lambda-1})$;*
   (b) *if $\theta \geq 1 - 1/(\lambda - 1)$, then a maximizing sequence of configurations is given by a laminate of the first rank in ad_θ with layer normal $\mathbf{n}^1$ parallel to $\mathbf{E}$ and $\theta_1 = 1 - \theta$.*

The geometry for minimizing configurations is independent of the volume fraction of the β phase and the contrast λ. On the other hand, for $\lambda > 2$, we see that the maximizing sequences of configurations change from laminates of rank one to laminates of rank two when the volume fraction of the β phase drops below $1 - \frac{1}{\lambda-1}$. When this happens the extremal configuration of α and β phases changes topology and the α phase occupies a connected set, while the β phase is in the form of thin rectangular inclusions.

In view of the applications, it is important to control the temperature gradient, as regions containing large temperature gradients are most often the first to suffer damage during service. Theorem 1.2 provides rigorous rules of thumb for the design of configurations for minimizing the L^2 norm of the temperature gradient, i.e., minimizing configurations are given by layering the two conductors in strips parallel to the applied field $\mathbf{E}$. On the other hand, the upper bound given in Theorem 1.1 provides the best possible upper bound on the L^2 norm of the temperature gradient when only the volume fraction of the β phase is known. We point out that the upper bound goes to infinity with the contrast λ.

The basic idea behind our approach is to encode the constraint given by the equilibrium condition (1.1) directly into the cost functional $\|\nabla T\|_2^2$. To do this we follow Golden and Papanicolaou [5] and introduce a scattering theory formalism to express ∇T in terms of the solution operator for (1.1). We then substitute the representation for ∇T into the L^2 norm to obtain the desired Stieltjes representation for $\|\nabla T\|_2^2$ in terms of a matrix valued measure. Using perturbation theory we see as in [5] that there are an infinite number of constraints on the matrix valued measure. We judiciously choose a subset of these constraints associated with the first and second moments of the measure. Our choice is motivated by the corrector theory of homogenization for laminates of finite rank given by Briane [2]. Subject to these constraints we extremize the Stietljes representation formula over all associated matrix valued measures to obtain the bounds given in Theorem 1.1. The attainability of the upper bound is established by comparing it to the limits of the L^2 norms associated with laminates of rank one or two. The comparison is facilitated using an explicit Stieltjes integral representation formula for these limits. We are confident that the approach developed here will be successful for investigating analogous problems in the elasticity setting.

The paper is organized as follows. In section 2 we review the recent results of Briane [2] that give the explicit form of corrector matrices for laminates of finite rank. We apply this theory to write the limit of the L^2 norms for finite rank laminates as Stieltjes functions. In section 3 we develop a Stieltjes representation formula for the L^2 norm of the gradient for any admissible configuration. In section 4 we use the representation formula to obtain the bounds stated in Theorem 1.1 and to establish their optimality.

2. Correctors and the L^2 norm for layered materials. In this section we obtain an explicit formula for

$$\lim_{\varepsilon \to 0} \|\nabla T_L^\varepsilon\|_2^2.$$

We start by reviewing the notion of H convergence due to Spagnolo [10] and Murat and Tartar [8]. We consider the sequence of conductivities $\{a(\chi^\varepsilon)\}_{\varepsilon>0}$ associated with the sequence of configurations $\{\chi^\varepsilon\}_{\varepsilon>0}$. The sequence $\{a(\chi^\varepsilon)\}_{\varepsilon>0}$ is said to H converge to A if for any function f of $H^{-1}(\Omega)$ the solutions $u^\varepsilon \in W_0^{1,2}(\Omega)$ of

$$-\text{div}\ (a(\chi^\varepsilon)\nabla u^\varepsilon) = f$$

satisfy $u^\varepsilon \to u^0$ weakly in $W^{1,2}(\Omega)$ and $a(\chi^\varepsilon)\nabla u^\varepsilon \to A\nabla u^0$ weakly in $L^2(\Omega;\, R^N)$, where u^0 is the solution of $-\text{div}\left(A\nabla u^0\right) = f$ and $u^0 \in W_0^{1,2}(\Omega)$. In fact more can be said about the convergence of the sequence $\{\nabla u^\varepsilon\}_{\varepsilon>0}$. There exists a matrix field $\mathbf{P}^\varepsilon$, called a corrector, for which

$$\nabla u^\varepsilon = \mathbf{P}^\varepsilon \nabla u^0 + z^\varepsilon,$$

where $z^\varepsilon \to 0$ strongly in $L^1(\Omega;\, R^N)$. Tartar [11] and Murat and Tartar [8] prove there always exists such a sequence of correctors $\mathbf{P}^\varepsilon$.

We choose layering directions $\mathbf{n}^1$ and $\mathbf{n}^2$ so that they are orthogonal to each other and put $\chi_1^\varepsilon = \chi^1(\frac{\mathbf{n}^1\cdot\mathbf{x}}{\varepsilon})$ and $\chi_2^\varepsilon = \chi^2(\frac{\mathbf{n}^2\cdot\mathbf{x}}{\varepsilon^2})$. We invoke Theorem 2.1 of Briane [2], and a straightforward calculation shows that the correctors are of the form

$$\mathbf{P}^\varepsilon = \chi_1^\varepsilon \mathbf{P}^1 + (1-\chi_1^\varepsilon)\left[\chi_2^\varepsilon \mathbf{P}^2 + (1-\chi_2^\varepsilon)\mathbf{P}^3\right], \tag{2.1}$$

where the constant matrices $\mathbf{P}^1$, $\mathbf{P}^2$, and $\mathbf{P}^3$ are given by

$$\mathbf{P}^1 = \mathbf{I} + (1-\theta_1)\left(\frac{(1-\theta_2)(\lambda-1)}{1-\theta_1(1-\theta_2)+\theta_1(1-\theta_2)\lambda}\right)\mathbf{n}^1\otimes\mathbf{n}^1, \tag{2.2}$$

$$\mathbf{P}^2 = \mathbf{I} - \theta_1\left(\frac{(1-\theta_2)(\lambda-1)}{1-\theta_1(1-\theta_2)+\theta_1(1-\theta_2)\lambda}\right)\mathbf{n}^1\otimes\mathbf{n}^1 + (1-\theta_2)\left(\frac{\lambda-1}{(1-\theta_2)+\theta_2\lambda}\right)\mathbf{n}^2\otimes\mathbf{n}^2, \tag{2.3}$$

and

$$\mathbf{P}^3 = \mathbf{I} - \theta_1\left(\frac{(1-\theta_2)(\lambda-1)}{1-\theta_1(1-\theta_2)+\theta_1(1-\theta_2)\lambda}\right)\mathbf{n}^1\otimes\mathbf{n}^1 - \theta_2\left(\frac{\lambda-1}{(1-\theta_2)+\theta_2\lambda}\right)\mathbf{n}^2\otimes\mathbf{n}^2, \tag{2.4}$$

where $\mathbf{I}$ is the $N\times N$ identity and $\mathbf{n}^1\otimes\mathbf{n}^1$ and $\mathbf{n}^2\otimes\mathbf{n}^2$ are the rank one matrices $\mathbf{n}_i^1\mathbf{n}_j^1$ and $\mathbf{n}_i^2\mathbf{n}_j^2$, respectively. The H limit for the sequence $\{a(\chi_L^\varepsilon)\}_{\varepsilon>0}$ is a constant $N\times N$ matrix denoted by A^L [7], [8]. For the boundary value problem treated here we have that

$$\begin{aligned} T^\varepsilon &\to \mathbf{E}\cdot\mathbf{x} \ \text{ weakly in } \ W^{1,2}(\Omega) \text{ and} \\ a(\chi^\varepsilon)\nabla T^\varepsilon &\to A^L\mathbf{E} \ \text{ weakly in } \ L^2(\Omega;\, R^N). \end{aligned} \tag{2.5}$$

From the corrector theory we have

$$\nabla T_L^\varepsilon = \mathbf{P}^\varepsilon\mathbf{E} + z^\varepsilon. \tag{2.6}$$

It is evident from the formulas describing $\mathbf{P}^\varepsilon$ that the sequence $\{\mathbf{P}^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^\infty(\Omega;\, R^{N\times N})$. Thus we appeal to Theorem 3 of Murat and Tartar [8] to find that $z^\varepsilon \to 0$ strongly in $L^2(\Omega;\, R^N)$. We note that because of the separation of scales, the sequence of products $\{\chi_1^\varepsilon\chi_2^\varepsilon\}_{\varepsilon>0}$ converges in a weak L^∞ *star* to the product $\theta_1\theta_2$. Collecting our results and taking limits we find that

$$\lim_{\varepsilon\to 0}\|\nabla T_L^\varepsilon\|_2^2 = \mathbf{C}_{ij}^L\mathbf{E}_i\mathbf{E}_j, \tag{2.7}$$

where the matrix $\mathbf{C}^L$ is given by

$$\begin{aligned}\mathbf{C}^L &= meas(\Omega)\mathbf{I}\\ &+meas(\Omega)(1-\theta_1)(1-\theta_2)\left(\frac{\theta_1(1-\theta_2)}{(\frac{1}{1-\lambda}-\theta_1(1-\theta_2))^2}\right)\mathbf{n}^1\otimes\mathbf{n}^1\\ &+meas(\Omega)(1-\theta_1)(1-\theta_2)\left(\frac{\theta_2}{(\frac{1}{1-\lambda}-\theta_2)^2}\right)\mathbf{n}^2\otimes\mathbf{n}^2.\end{aligned} \tag{2.8}$$

Here we note that the total volume fraction of the β phase is given by $\theta=(1-\theta_1)(1-\theta_2)$, and we can rewrite (2.8) as

$$\mathbf{C}^L=\mathbf{C}(\boldsymbol{\mu}^L)=meas(\Omega)\times\left(\mathbf{I}+\int_0^{1-\theta}f(z)\boldsymbol{\mu}^L(dz)\right), \tag{2.9}$$

where the matrix valued measure $\boldsymbol{\mu}$ is given by

$$\boldsymbol{\mu}^L(dz)=\left(\theta\delta(z-\theta_1(1-\theta_2))\mathbf{n}^1\otimes\mathbf{n}^1+\theta\delta(z-\theta_2)\mathbf{n}^2\otimes\mathbf{n}^2\right)dz. \tag{2.10}$$

Equations (2.9) and (2.10) provide the desired Stieltjes integral formula for the limit given in (2.7).

3. Stieltjes integral representation formula for the L^2 norm. In this section we develop a representation formula for $\|\nabla T\|_2^2$, where T is the solution of (1.1) and $T=\mathbf{E}\cdot\mathbf{x}$ on the boundary of Ω. Motivated by perturbation theory we shall first rewrite the constraint given by (1.1). To this end we introduce the solution operator $(-\Delta)^{-1}$ mapping $H^{-1}(\Omega)$ onto $W_0^{1,2}(\Omega)$ for the problem given by $w\in W_0^{1,2}(\Omega)$ and

$$-\Delta w = f \quad \text{on } \Omega. \tag{3.1}$$

Next we introduce the subspace $\mathcal{E}$ of $L^2(\Omega;\, R^N)$ defined by

$$\mathcal{E}=\{\eta\in L^2(\Omega;\, R^N)\mid \eta=\nabla\varphi,\ \varphi\in W_0^{1,2}(\Omega)\},$$

and we introduce the operator P defined by $P=\partial_{x_i}(\Delta)^{-1}\partial_{x_j}$. It is easily checked that the operator P is a projection from $L^2(\Omega;\, R^N)$ into the subspace $\mathcal{E}$. We introduce the field perturbation $\phi=T-\mathbf{E}\cdot\mathbf{x}$ and rewrite the conductivity $a(\chi)$ as a positive perturbation from the uniform state α, i.e., $a(\chi)=\alpha+(\beta-\alpha)\chi$. Expanding T and $a(\chi)$ in (1.1) gives

$$-\alpha\Delta\phi=\text{div}\,\left((\beta-\alpha)\chi(\nabla\phi+\mathbf{E})\right). \tag{3.2}$$

Dividing both sides by α, applying $(-\Delta)^{-1}$ to both sides, and manipulating gives

$$\nabla\phi+\mathbf{E}+P\left[\,(\lambda-1)\chi\,(\nabla\phi+\mathbf{E})\,\right]=\mathbf{E} \tag{3.3}$$

or

$$[\mathbf{I} + (\lambda - 1)\,\Lambda\,]\,\nabla T = \mathbf{E}, \tag{3.4}$$

where $\Lambda = P\chi$. From this we obtain the desired expression

$$\nabla T = [\,\mathbf{I} + (\lambda - 1)\,\Lambda\,]^{-1}\,\mathbf{E}. \tag{3.5}$$

It is clear that the equilibrium constraint (1.1) is now explicitly encoded in the formula for ∇T as given by (3.5). The next step is to rewrite $\|\nabla T\|_2^2$ in a way that exploits the spectral properties of the operator Λ. To do this we expand the energy dissipation denoted by Q in two different ways. Here

$$Q = \frac{1}{meas(\Omega)} \int_\Omega a(\chi)\,\nabla T \cdot \nabla T\,dx. \tag{3.6}$$

Expanding $a(\chi)$ as $a(\chi) = \alpha + \chi(\beta - \alpha)$ and substitution into (3.6) gives

$$Q = \frac{1}{meas(\Omega)} \int_\Omega \alpha|\nabla T|^2\,dx + \frac{(\beta - \alpha)}{meas(\Omega)} \int_\Omega \chi\,|\nabla T|^2\,dx. \tag{3.7}$$

We expand ∇T as $\nabla T = \nabla\phi + \mathbf{E}$ in (3.6) to obtain

$$\begin{aligned} Q &= \frac{1}{meas(\Omega)} \int_\Omega a(\chi)\,\nabla T \cdot \mathbf{E}\,dx \\ &= \alpha|\mathbf{E}|^2 + \frac{(\beta - \alpha)}{meas(\Omega)} \int_\Omega \chi\,\nabla T \cdot \mathbf{E}\,dx. \end{aligned} \tag{3.8}$$

Here the first equality in (3.8) follows from (1.2), and the second follows from expansion of $a(\chi)$ and $\int_\Omega \nabla\phi \cdot \mathbf{E}\,dx = 0$. Eliminating Q using (3.7) and (3.8) gives

$$\|\nabla T\|_2^2 = meas(\Omega)\left(|\mathbf{E}|^2 + \frac{(\lambda - 1)}{meas(\Omega)} \int_\Omega \chi\,(\nabla T \cdot \mathbf{E})\,dx - \frac{(\lambda - 1)}{meas(\Omega)} \int_\Omega \chi\,|\nabla T|^2\,dx\right). \tag{3.9}$$

For vector fields η and ψ in $L^2(\Omega;\,R^N)$ we introduce the bilinear form $\langle \eta, \psi \rangle$ defined by

$$\langle \eta, \psi \rangle = \frac{1}{meas(\Omega)} \int_\Omega \chi\,(\eta \cdot \psi)\,dx,$$

and $\langle \eta, \psi \rangle$ is an inner product for the Hilbert space $\mathcal{H}$ defined by

$$\begin{aligned} \mathcal{H} = \Big\{ &\psi \in L^2(\Omega;\,R^N) \text{ modulo the equivalence class of elements}\,\psi \\ &\text{such that } \int_\Omega \chi\psi\,dx = 0 \Big\}. \end{aligned}$$

Substitution of (3.5) into (3.9) gives

$$\begin{aligned} \|\nabla T\|_2^2 = {} & meas(\Omega)|\mathbf{E}|^2 \\ & + meas(\Omega)(\lambda - 1)\left\langle [\,\mathbf{I} + (\lambda - 1)\,\Lambda\,]^{-1}\,\mathbf{E}, \mathbf{E} \right\rangle \\ & - meas(\Omega)(\lambda - 1)\left\langle [\,\mathbf{I} + (\lambda - 1)\,\Lambda\,]^{-1}\,\mathbf{E}, [\,\mathbf{I} + (\lambda - 1)\,\Lambda\,]^{-1}\,\mathbf{E} \right\rangle. \end{aligned} \tag{3.10}$$

It is easily seen that Λ is a positive symmetric operator on $\mathcal{H}$ with norm less than or equal to 1. From spectral theory we immediately obtain the existence of a projection valued measure $R(dz)$ with support on $[0,1]$ for which

$$\left\langle [\,\mathbf{I} + (\lambda - 1)\,\Lambda\,]^{-1}\,\mathbf{E}, \mathbf{E}\right\rangle = \left\langle \int_0^1 \frac{1}{1+z(\lambda-1)}\,R(dz)\mathbf{E}, \mathbf{E}\right\rangle \tag{3.11}$$

and

$$\left\langle [\,\mathbf{I} + (\lambda - 1)\,\Lambda\,]^{-1}\,\mathbf{E}, [\,\mathbf{I} + (\lambda - 1)\,\Lambda\,]^{-1}\,\mathbf{E}\right\rangle = \left\langle \int_0^1 \frac{1}{(1+z(\lambda-1))^2}\,R(dz)\mathbf{E}, \mathbf{E}\right\rangle. \tag{3.12}$$

Collecting our results we arrive at the Stieltjes integral representation formula given by the following theorem.

THEOREM 3.1 (Stieltjes integral representation formula).

$$\|\nabla T\|_2^2 = \mathbf{C}_{ij}(\boldsymbol{\mu})\mathbf{E}_i\mathbf{E}_j, \tag{3.13}$$

where

$$\mathbf{C}(\boldsymbol{\mu}) = meas(\Omega)\left(\mathbf{I} + \int_0^1 f(z)\,\boldsymbol{\mu}(dz)\right) \tag{3.14}$$

and

$$\boldsymbol{\mu}_{ij}(dz) = \langle R(dz)\,\mathbf{e}^i, \mathbf{e}^j\rangle. \tag{3.15}$$

Here $\mathbf{e}^i$, $i = 1, 2\ldots, N$ is an orthonormal basis for R^N. Moreover, $\boldsymbol{\mu}_{ij} = \boldsymbol{\mu}_{ji}$, since $R(dz)$ is symmetric and for all $\mathbf{E}$ in R^N we have that the measures $\boldsymbol{\mu}(dz)\mathbf{E}\cdot\mathbf{E}$ are positive.

It is evident from Theorem 3.1 that the geometric information is stored in the measure $\boldsymbol{\mu}$ while the ratio of conductivities is contained in $f(z)$. The extremal behavior of $\|\nabla T\|_2^2$ is governed by the global maxima and minima of f on $[0,1]$.

4. Derivation of the isoperimetric inequalities. In view of the Stieltjes formula for the gradient we can replace the extremal problems

$$\mathrm{A} = \inf_{a(\chi)\in ad_\theta}\left\{\|\nabla T\|_2^2\,;\ -\mathrm{div}\ (a(\chi)\nabla T) = 0,\ T = \mathbf{E}\cdot\mathbf{x} \text{ on } \partial\Omega\right\} \tag{4.1}$$

and

$$\mathrm{B} = \sup_{a(\chi)\in ad_\theta}\left\{\|\nabla T\|_2^2\,;\ -\mathrm{div}\ (a(\chi)\nabla T) = 0,\ T = \mathbf{E}\cdot\mathbf{x} \text{ on } \partial\Omega\right\} \tag{4.2}$$

with the equivalent problems given by

$$\mathrm{A} = \inf_{\boldsymbol{\mu}\in\mathcal{A}_\theta}\{\mathbf{C}(\boldsymbol{\mu})\mathbf{E}\cdot\mathbf{E}\} \tag{4.3}$$

and

$$\mathrm{B} = \sup_{\boldsymbol{\mu}\in\mathcal{A}_\theta}\{\mathbf{C}(\boldsymbol{\mu})\mathbf{E}\cdot\mathbf{E}\}. \tag{4.4}$$

Here the set $\mathcal{A}_\theta$ is the set of measures $\boldsymbol{\mu}$ given by (3.15) associated with any configuration of the β phase described by a characteristic function χ subject to the isoperimetric constraint $\int_\Omega \chi\, dx = \theta\, meas(\Omega)$. Instead of attempting an explicit characterization of $\mathcal{A}_\theta$ we introduce a larger set of measures $\overline{\mathcal{A}_\theta}$ and compute the lower and upper bounds

$$\underline{\mathrm{A}} = \inf_{\boldsymbol{\mu}\in\overline{\mathcal{A}_\theta}} \{\mathbf{C}(\boldsymbol{\mu})\mathbf{E}\cdot\mathbf{E}\} \tag{4.5}$$

and

$$\overline{\mathrm{B}} = \sup_{\boldsymbol{\mu}\in\overline{\mathcal{A}_\theta}} \{\mathbf{C}(\boldsymbol{\mu})\mathbf{E}\cdot\mathbf{E}\}\,. \tag{4.6}$$

The goal here is to find a suitable choice for $\overline{\mathcal{A}_\theta}$ that delivers optimal bounds. We start by examining constraints on the measure $\boldsymbol{\mu}^L(dz)$ associated with laminates of the second rank defined in (2.10). One readily sees that

$$\int_0^1 \boldsymbol{\mu}^L(dz) = \theta\mathbf{I}, \tag{4.7}$$

and since $1-\theta = \theta_2 + \theta_1(1-\theta_2)$ we have

$$\mathbf{T}^L \triangleq \int_0^1 z\,\boldsymbol{\mu}^L(dz) \le (\max\{\theta\theta_1(1-\theta_2), \theta\theta_2\}) \times \mathbf{I} \le \theta(1-\theta)\mathbf{I}. \tag{4.8}$$

Next, for comparison, we apply perturbation expansions to look for constraints on $\boldsymbol{\mu}(dz)$. Expansion about $\lambda = 1$ gives

$$[\,\mathbf{I} + (\lambda-1)\,\Lambda\,]^{-1} = \mathbf{I} + (1-\lambda)\,\Lambda + (1-\lambda)^2\,\Lambda^2 + (1-\lambda)^3\,\Lambda^3 + \cdots \tag{4.9}$$

and

$$\frac{1}{1+z(\lambda-1)} = 1 + (1-\lambda)\,z + (1-\lambda)^2\,z^2 + (1-\lambda)^3\,z^3 + \cdots. \tag{4.10}$$

Substituting these expansions into (3.11) and equating like powers of $\lambda - 1$ gives

$$\int_0^1 z^n\,\boldsymbol{\mu}_{ij}(dz) = \langle \Lambda^n \mathbf{e}^i, \mathbf{e}^j\rangle, \quad n = 0, 1 \ldots. \tag{4.11}$$

Focusing on the cases $n = 0$ and $n = 1$ we have

$$\int_0^1 \boldsymbol{\mu}_{ij}(dz) = \theta\,\mathbf{I}_{ij} \tag{4.12}$$

and

$$\int_0^1 z\,\boldsymbol{\mu}_{ij}(dz) = \langle \Lambda\mathbf{e}^i, \mathbf{e}^j\rangle. \tag{4.13}$$

We estimate the largest and smallest eigenvalues for the tensor $\mathbf{T}_{ij} \triangleq \langle\Lambda\mathbf{e}^i, \mathbf{e}^j\rangle$. We note that constant vectors lie in the null space of the operator P, and we introduce $\overline{\chi} = \chi - \theta$ to find that

$$\begin{aligned} 0 \le \mathbf{T}_{ij}\mathbf{E}_i\mathbf{E}_j &= \frac{1}{meas(\Omega)}\int_\Omega (P\chi)\mathbf{E}\cdot\chi\mathbf{E}\,dx \\ &= \frac{1}{meas(\Omega)}\int_\Omega (P\overline{\chi})\mathbf{E}\cdot\overline{\chi}\mathbf{E}\,dx \\ &\le \frac{1}{meas(\Omega)}\int_\Omega (\overline{\chi})^2\,dx|\mathbf{E}|^2 = \theta(1-\theta)|\mathbf{E}|^2. \end{aligned} \tag{4.14}$$

Thus the spectrum of the tensor $\mathbf{T}$ lies in the interval $[0, \theta(1-\theta)]$. Motivated by (4.7), (4.8), (4.12), and (4.14) we define $\overline{\mathcal{A}_\theta}$ to be given by all $N \times N$ symmetric matrices with elements given by finite Borel measures such that for any vector $\mathbf{v}$ the measure given by $\boldsymbol{\mu}(dz)\mathbf{v} \cdot \mathbf{v}$ is positive and the matrix of measures satisfies the moment constraints

$$\int_0^1 \boldsymbol{\mu}(dz) = \theta\,\mathbf{I} \tag{4.15}$$

and

$$\int_0^1 z\,\boldsymbol{\mu}(dz) = \mathbf{T}, \tag{4.16}$$

where $\mathbf{T}$ is a symmetric $N \times N$ matrix with eigenvalues contained in the interval $[0, \theta(1-\theta)]$. Its clear from the definition of $\overline{\mathcal{A}_\theta}$ that this set of measures contains $\mathcal{A}_\theta$.

For the purpose of computing the bounds $\underline{\mathrm{A}}$ and $\overline{\mathrm{B}}$ we characterize the range of the map $\mathbf{H}(\boldsymbol{\mu})$ given by

$$\mathbf{H}(\boldsymbol{\mu}) = \int_0^1 f(z)\,\boldsymbol{\mu}(dz) \tag{4.17}$$

for $\boldsymbol{\mu}$ in $\overline{\mathcal{A}_\theta}$. We introduce the set $\overline{\mathcal{V}}_\theta$ of vectors $(\nu_1, \nu_2, \dots, \nu_N)$ whose elements are positive finite Borel measures supported on $[0,1]$ that satisfy the constraints

$$\int_0^1 \nu_i(dz) = \theta, \quad i = 1, \dots, N, \tag{4.18}$$

and

$$\int_0^1 z\,\nu_i(dz) = m_i, \quad \text{where } 0 \le m_i \le \theta(1-\theta), \quad i = 1, \dots, N. \tag{4.19}$$

We now state the following theorem.

Theorem 4.1 (the matrix range of $\mathbf{H}(\boldsymbol{\mu})$). *Let* R *be the image of* $\overline{\mathcal{A}_\theta}$ *under the map* $\mathbf{H} : \overline{\mathcal{A}_\theta} \to R^{N\times N}$. *Then* R *is given by*

$$\mathrm{R} = \left\{ \begin{array}{c} M \in R^{N\times N}\,;\ M = \Sigma_{i=1}^N \lambda_i \mathbf{e}^i \otimes \mathbf{e}^i, \\ \textit{where}\ \lambda_i = \int_0^1 f(z)\,\nu_i(dz), \textit{and}\ (\nu_1, \nu_2 \dots \nu_N)\ \textit{in}\ \overline{\mathcal{V}}_\theta, \\ \textit{and}\ \mathbf{e}^i,\ i = 1, \dots, N, \quad \textit{is any orthonormal basis for}\ R^N. \end{array} \right\}. \tag{4.20}$$

Proof. We denote the right-hand side of (4.20) by S and show $R = S$. One sees that $S \subset R$ by writing $M = \int_0^1 f(z)\,P(dz)$, where $P(dz) = \Sigma_i \nu_i(dz)\mathbf{e}^i \otimes \mathbf{e}^i$, and checking (4.15) and (4.16). To show $R \subset S$ we consider the matrix M given by $M = \int_0^1 f(z)\boldsymbol{\mu}(dz)$. Since M is symmetric it has an orthonormal system of eigenvectors $\mathbf{v}^i$, $i = 1\dots, N$, and $M = \Sigma_i \lambda_i \mathbf{v}^i \otimes \mathbf{v}^i$. From this one deduces that $\lambda_i = \int_0^1 f(z)\mu_i(dz)$, where μ_i are the positive measures given by $\mu_i(dz) = \boldsymbol{\mu}(dz)\mathbf{v}^i \cdot \mathbf{v}^i$. Next we observe that

$$\int_0^1 \mu_i(dz) = \int_0^1 \boldsymbol{\mu}(dz)\mathbf{v}^i \cdot \mathbf{v}^i = \theta \tag{4.21}$$

and

$$m_i = \int_0^1 z\,\mu_i(dz) = \int_0^1 z\,\boldsymbol{\mu}(dz)\mathbf{v}^i \cdot \mathbf{v}^i \le \theta(1-\theta) \tag{4.22}$$

to discover that $(\mu_1, \dots \mu_N)$ lies in $\overline{\mathcal{V}}_\theta$, and the theorem is proved. $\square$

We now establish the explicit formulas for the bounds given by the following theorem.

THEOREM 4.2 (bounds on the L^2 norm of the gradient).

$$\underline{\mathrm{A}} = meas(\Omega)|\mathbf{E}|^2 \leq \|\nabla T\|_2^2 \leq \overline{\mathrm{B}} = U(\theta, \mathbf{E}). \tag{4.23}$$

Before establishing the theorem we note that the lower bound $meas(\Omega)|\mathbf{E}|^2$ can be found directly. Indeed, we can write $T = \phi + \mathbf{E} \cdot \mathbf{x}$, where $\phi = 0$ on the boundary of Ω. Then expanding $\|\nabla T\|_2^2$ and noting that $\int_\Omega \nabla\phi \cdot \mathbf{E}\, dx = 0$, we have

$$\|\nabla T\|_2^2 = meas(\Omega)|\mathbf{E}|^2 + \int_\Omega |\nabla \phi|^2\, dx, \tag{4.24}$$

and the lower bound follows immediately.

Proof of Theorem 4.2. We start by proving $\underline{\mathrm{A}} = meas(\Omega)|\mathbf{E}|^2$. From Theorem 4.1 it follows that

$$\underline{\mathrm{A}} = \inf_{M \in R} \left\{ meas(\Omega)(\mathbf{I} + M)\, \mathbf{E} \cdot \mathbf{E} \right\}. \tag{4.25}$$

It is evident from (4.25) that for $(\nu_1, \dots, \nu_N)$ fixed the minimum occurs when $\mathbf{E}$ lies in the eigenspace of the smallest eigenvalue of M. Without loss of generality we assume that $\lambda_1 = \int_0^1 f(z)\, \nu_1(dz)$ is the smallest eigenvalue of M, and we choose $\mathbf{e}^1 = \mathbf{E}/|\mathbf{E}|$ to find that

$$\underline{\mathrm{A}} = \inf_{\substack{\nu_1 \geq 0, \\ \int_0^1 \nu_1(dz) = \theta, \int_0^1 z\nu_1(dz) \leq \theta(1-\theta)}} \left\{ meas(\Omega) \left(1 + \int_0^1 f(z)\, \nu_1(dz) \right) |\mathbf{E}|^2 \right\}. \tag{4.26}$$

To finish the minimization we note that for $\lambda > 1$ the function $f(z)$ is strictly positive on $0 < z < \infty$ with $f(0) = 0$ and $\lim_{z \to \infty} f(z) = 0$. Moreover, $f(z)$ has a global maximum over $[0, \infty)$ at $z = 1/(\lambda - 1)$ with $\acute{f}(z) \geq 0$ for $z \leq 1/(\lambda - 1)$ and $\acute{f}(z) \leq 0$ for $z \geq 1/(\lambda - 1)$. With this in mind we choose $\nu_1(dz) = \theta\delta(z)dz$. Since this choice is admissible we have established that $\underline{\mathrm{A}} = meas(\Omega)|\mathbf{E}|^2$.

Next we establish the upper bound. From Theorem 4.1 it follows that

$$\overline{\mathrm{B}} = \sup_{M \in R} \left\{ meas(\Omega)(\mathbf{I} + M)\, \mathbf{E} \cdot \mathbf{E} \right\}. \tag{4.27}$$

Here it is evident that for $(\nu_1, \dots, \nu_N)$ fixed the maximum occurs when $\mathbf{E}$ lies in the eigenspace of the largest eigenvalue of M. Without loss of generality we assume that $\lambda_1 = \int_0^1 f(z)\, \nu_1(dz)$ is the largest eigenvalue of M, and we choose $\mathbf{e}^1 = \mathbf{E}/|\mathbf{E}|$ to find that

$$\overline{\mathrm{B}} = \sup_{\substack{\nu_1 \geq 0, \\ \int_0^1 \nu_1(dz) = \theta, \int_0^1 z\nu_1(dz) \leq \theta(1-\theta)}} \left\{ meas(\Omega) \left(1 + \int_0^1 f(z)\, \nu_1(dz) \right) |\mathbf{E}|^2 \right\}. \tag{4.28}$$

To proceed we normalize and write $\nu_1(dz) = \theta p(dz)$. The extremal problem becomes

$$\overline{\mathrm{B}} = meas(\Omega) \left(|\mathbf{E}|^2 + \theta \sup_{p \in \mathcal{C}} \left(\int_0^1 f(z)\, p(dz) \right) |\mathbf{E}|^2 \right), \tag{4.29}$$

where $\mathcal{C}$ is the set of probability measures for which $0 \leq \bar{z} = \int_0^1 z\, p(dz) \leq (1-\theta)$. Noting that the function $f(z)$ is strictly concave over an interval that includes $[0, 1/(\lambda-1)]$, strictly increasing on $[0, 1/(\lambda-1)]$, and strictly decreasing on $(1/(\lambda-1), \infty)$, we have for $(1-\theta) \leq 1/(\lambda-1)$ that

$$\int_0^1 f(z)\, p(dz) < f(\bar{z}) \leq f(1-\theta). \tag{4.30}$$

It is evident that the best choice is $p(dz) = \delta(z-(1-\theta))$, and we find that

$$\overline{\mathrm{B}} = meas(\Omega) \times (1 + \theta f(1-\theta))\, |\mathbf{E}|^2. \tag{4.31}$$

We observe that for $1 < \lambda \leq 2$ we have $1/(\lambda-1) \geq 1$ and for $0 \leq \theta \leq 1$ we always have $(1-\theta) \leq 1/(\lambda-1)$. On the other hand, when $(1-\theta) \geq 1/(\lambda-1)$ it is evident that the best choice corresponds to the global maximum of f, i.e., $p(dz) = \delta(z - 1/(\lambda-1))$, and we find that

$$\overline{\mathrm{B}} = meas(\Omega) \times (1 + \theta f(1/(\lambda-1)))\, |\mathbf{E}|^2, \tag{4.32}$$

and the theorem follows. $\square$

We conclude by proving Theorems 1.2 and 1.3. To prove Theorem 1.2 we show that the lower bound $\underline{\mathrm{A}}$ is attained by configurations made up of layers of the β phase with layer normals orthogonal to $\mathbf{E}$. We recall (1.2) and write it in the equivalent form

$$\begin{aligned} \alpha\Delta T &= 0 \quad \text{in the } \alpha \text{ phase},\\ \beta\Delta T &= 0 \quad \text{in the } \beta \text{ phase, and}\\ \beta\nabla T\cdot\mathbf{n} &= \alpha\nabla T\cdot\mathbf{n} \quad \text{on the layer interface.} \end{aligned} \tag{4.33}$$

When $\mathbf{n}$ is perpendicular to $\mathbf{E}$ we easily see that the affine function $T = \mathbf{E}\cdot\mathbf{x}$ is a solution of (4.33) and optimality follows. To prove Theorem 1.3 we first suppose that that $1-\theta \leq 1/(\lambda-1)$ and show that the upper bound $\overline{\mathrm{B}}$ is saturated by a laminate of rank one. We refer to formulas (2.7), (2.9), and (2.10) and choose $\mathbf{n}^1$ parallel to $\mathbf{E}$ and set $\theta_1 = 1-\theta$ and $\theta_2 = 0$ to obtain

$$\lim_{\varepsilon\to 0} \|\nabla T_L^\varepsilon\|_2^2 = \overline{\mathrm{B}}. \tag{4.34}$$

Last we suppose that $1-\theta \geq 1/(\lambda-1)$. Here we choose $\mathbf{n}^1$ parallel to $\mathbf{E}$ and $\mathbf{n}^2$ orthogonal to $\mathbf{E}$ and choose $\theta_1 = \frac{1}{1+\theta(\lambda-1)}$ and $\theta_2 = 1-\theta-(\frac{1}{\lambda-1})$. This choice gives $\theta_1(1-\theta_2) = 1/(\lambda-1), (1-\theta_1)(1-\theta_2) = \theta$, and

$$\lim_{\varepsilon\to 0} \|\nabla T_L^\varepsilon\|_2^2 = \overline{\mathrm{B}} \tag{4.35}$$

follows from (2.7), (2.9), and (2.10).

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