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THE SECOND STEKLOFF EIGENVALUE AND ENERGY DISSIPATION INEQUALITIES FOR FUNCTIONALS WITH SURFACE ENERGY*

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Abstract. A functional with both bulk and interfacial surface energy is considered. It corresponds to the energy dissipated inside a two-phase electrical conductor in the presence of an electrical contact resistance at the two-phase interface. The effect of embedding a highly conducting particle into a matrix of lesser conductivity is investigated. We find the criterion that determines when the increase in surface energy matches or exceeds the reduction in bulk energy associated with the particle. This criterion is general and applies to any particle with Lipschitz continuous boundary. It is given in terms of the of the second Stekloff eigenvalue of the particle. This result provides the means for selecting energy-minimizing configurations.

Key words. Stekloff eigenvalue, heat conduction, size effects, isoperimetric inequalities

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1. Introduction.

We consider a suspension of electrically conducting particles embedded in a matrix with a lower electrical conductivity. The two-phase conductor fills out a domain $\Omega \subset R^3$ with Lipschitz continuous boundary $\partial\Omega$. The electric conductivity tensor associated with the particle is denoted by σ_r and that of the matrix by σ_m. Here, both conductors are assumed anisotropic, and σ_r, σ_m are given by 3×3 symmetric, positive definite matrices. The tensors satisfy the inequality $\sigma_r > \sigma_m$ in the sense of quadratic forms. We suppose that there is an interfacial contact resistance between the two phases. The contact resistance is characterized by a scalar β with dimensions of conductivity per unit length.

The region occupied by the better conductor is denoted by A_r, and the region occupied by the matrix is denoted by A_m. The interface separating them is assumed Lipschitz continuous and is denoted by Γ and $\Omega = A_r \cup A_m \cup \Gamma$. The resistivity tensor inside the composite is described by $\sigma^{-1}(x) = \sigma_r^{-1}\chi_{A_r} + \sigma_m^{-1}(1-\chi_{A_r})$, where χ_{A_r} equals one in A_r and zero otherwise. For a prescribed current $g \in H^{-1/2}(\partial\Omega)$, such that $\int_{\partial\Omega} g ds = 0$, the thermal energy dissipated inside the composite is given by $E(A_r, g)$, where

$$E(A_r, g) = \min\{C(A_r, j) : j \in L^2(\Omega)^3, \mathrm{div} j = 0,\ j \cdot n = g \text{ on } \partial\Omega\} \tag{1.1}$$

and

$$C(A_r, j) = \int_\Omega \sigma^{-1}(x)\ j \cdot j dx + \beta^{-1}\int_\Gamma (j \cdot n)^2 ds. \tag{1.2}$$

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Here div $j = 0$ holds in the sense of distributions, ds is the element of surface area, and the vector n is the unit normal pointing into the matrix phase. The first term of the functional $C(A_r, j)$ is associated with bulk energy dissipation, while the second term gives the energy dissipation at the two-phase interface. The minimizer j_{A_r} is precisely the current in the composite and is related to the potential u_{A_r} by the constituitive law: $j_{A_r} = \sigma(x)\nabla u_{A_r}$ and

$$\operatorname{div}(\sigma(x)\nabla u_{A_r}) = 0 \text{ in } A_r \cup A_m. \tag{1.3}$$

Across the interface one has

$$[j_{A_r} \cdot n] = 0 \text{ on } \Gamma, \tag{1.4}$$

and

$$j_{A_r} \cdot n_{|_2} = -\beta[u_{A_r}] \text{ on } \Gamma,\ \sigma_m \nabla u_{A_r} \cdot n = g \text{ on } \partial\Omega. \tag{1.5}$$

Here $u_{A_r} \in H^1(\Omega\backslash\Gamma)$ and $[u_{A_r}] = u_{A_r|_2} - u_{A_r|_1}$, where the subscripts indicate the side of the interface where the trace is taken. The requirement $\int_{\partial\Omega} g ds = 0$ is the solvability condition for the equation of state, and the potential u_{A_r} is determined uniquely up to a constant. To expedite the presentation we denote the subspace of all elements $g \in H^{-1/2}(\partial\Omega)$ such that $\int_{\partial\Omega} g ds = 0$ by $H^{-1/2}(\partial\Omega) \setminus R$.

The replacement of a region of matrix denoted by "Σ" with material of better conductivity amounts to a nonlocal perturbation of the functional $C(A_r, j)$. The region Σ is assumed to be compactly contained within the matrix (i.e., $\Sigma \subset A_m$ and $\partial\Sigma \cap \partial A_m = \emptyset$). The perturbed functional is written as

$$C(A_r \cup \Sigma, j) = \int_\Omega \tilde{\sigma}^{-1}(x)\ j \cdot j dx + \beta^{-1} \int_{\Gamma\cup\partial\Sigma} (j \cdot n)^2 ds, \tag{1.6}$$

where $\partial\Sigma$ is the reinforcement (or particle) boundary and

$$\tilde{\sigma}^{-1}(x) = \sigma_r^{-1}\chi_{A_r\cup\Sigma} + \sigma_m^{-1}(1 - \chi_{A_r\cup\Sigma}). \tag{1.7}$$

In this article we present the geometric criterion that determines when effects due to surface energy overcome the benefits of a highly conducting particle. This criterion is general and applies to any particle with Lipschitz continuous boundary. In order to give the criterion, we introduce the 3×3 symmetric matrix $\mathbf{R_{cr}}$ given by

$$\mathbf{R_{cr}} = \beta^{\mathbf{-1}}(\sigma_{\mathbf{m}}^{\mathbf{-1}} - \sigma_{\mathbf{r}}^{\mathbf{-1}})^{\mathbf{-1}}. \tag{1.8}$$

Here each element of $\mathbf{R_{cr}}$ has dimensions of length. This tensor provides a measure of the relative magnitude of the interfacial barrier resistance with respect to the mismatch between the resistivity tensors of the matrix and particle. For a given particle occupying the set "Σ," the geometric parameter of interest is its second Stekloff eigenvalue ρ_2. The second Stekloff eigenvalue has dimensions of conductivity per unit length and we write $\rho_2(\Sigma, \sigma_r)$ to indicate its dependence on the conductivity and geometry of the particle. When Σ has Lipschitz continuous boundary the variational formulation for the second Stekloff eigenvalue is given by

$$\rho_2(\Sigma, \sigma_r) = \min_{\operatorname{div}(\sigma_r\nabla\varphi)=0} \frac{\int_{\partial\Sigma}(\sigma_r\nabla\varphi \cdot n)^2 ds}{\int_\Sigma \sigma_r \nabla\varphi \cdot \nabla\varphi dx}; \tag{1.9}$$

cf. Kuttler and Sigillito [9] and Alessandrini and Magnanini [1]. Equality in (1.9) holds for the second Stekloff eigenfunction φ_2, where div $(\sigma_r \nabla \varphi_2) = 0$ in Σ, $\int_{\partial\Sigma} \varphi_2 ds = 0$, and

$$\sigma_r \nabla \varphi_2 \cdot n = \rho_2(\Sigma, \sigma_r)\varphi_2 \text{ on } \partial\Sigma. \tag{1.10}$$

The study of this eigenvalue problem was initiated in the work of Stekloff [17]. It is evident that the second Stekloff eigenvalue and boundary traces of the Stekloff eigenfunction correspond to the first nonzero eigenvalue and eigenfunction of the Dirichlet to Neumann map on $\partial\Sigma$.

Let $\mathrm{E}(\mathrm{A_r} \cup \Sigma, \mathrm{g})$ denote the associated energy dissipation obtained by replacing a region Σ compactly contained inside A_m with the better conductor. It is given by

$$\mathrm{E}(\mathrm{A_r} \cup \Sigma, \mathrm{g}) = \min\{C(A_r \cup \Sigma, j) : j \in L^2(\Omega)^3, \mathrm{div}\ j = 0,\ j \cdot n = g \text{ on } \partial\Omega\} \tag{1.11}$$

We state the following theorem.

THEOREM 1.1 (energy dissipation inequality). *Let Σ be a set with Lipschitz continuous boundary that is compactly contained in A_m. If $\rho_2(\Sigma, \sigma_r)$ satisfies*

$$\mathbf{R_{cr}^{-1}} \leq \sigma_{\mathbf{r}}^{\mathbf{-1}} \rho_{\mathbf{2}}(\mathbf{\Sigma}, \sigma_{\mathbf{r}}), \tag{1.12}$$

then

$$E(A_r \cup \Sigma, g) \geq E(A_r, g) \tag{1.13}$$

for all $g \in H^{-1/2}(\partial\Omega) \setminus R$.

Here (1.12) holds in the sense of quadratic forms. No assumptions on the topological nature of the particle domain Σ is made. Indeed it can be a disjoint union of multiply-connected components. The proof of this theorem is provided in section 2. We emphasize that (1.13) holds for every current $g \in H^{-1/2}(\partial\Omega) \setminus R$.

When the particle is made from an isotropic conductor, one can readily compute ρ_2 for spheres and rectangular fibers; cf. Kuttler and Sigillito [9]. For starlike domains and domains with smooth boundary, isoperimetric inequalities bounding ρ_2 from below have been obtained in the work of Payne [15], Bramble and Payne [2]: see also the review article of Payne [16]. These observations are applied in section 3, where heat dissipation inequalities are given in terms of the physical dimensions of the reinforcement. Such *size effect* inequalities predict the existence of a critical particle dimension below which the particle will no longer reduce the total heat dissipated inside the composite. These results show that the size of the domain Ω must be taken into consideration. Indeed, if the domain is "too thin," then the particle will have to have dimensions below the critical value in order to fit inside it. For such domains, the addition of highly conducting particles will not reduce the energy.

Theorem 1.1 can be applied to problems of energy minimization over various classes of configurations. We consider mixtures of two isotropically conducting materials. For this case, the particle and matrix phases have scalar conductivities and we continue to denote them as σ_r and σ_m, respectively, where $\sigma_r > \sigma_m$. The admissible class is chosen to be all suspensions of spheres of conductivity σ_r suspended in a matrix of σ_m. Here we allow the suspension to contain spheres of different radii. This class of suspensions is referred to as the class of *polydisperse* suspensions of spheres. We assume that each suspension consists of a finite number of spheres and that the spheres do not intersect. It is emphasized that no lower bound is placed on the size of the spheres appearing in the suspension. We suppose that the total amount of good conductor occupies no more than a prescribed volume fraction θ_r of the domain denoted by Ω. Theorem 4.1 shows that one needs only to consider suspensions of spheres with radii greater than or equal to $R_{cr} = \beta^{-1}(\sigma_m^{-1} - \sigma_r^{-1})^{-1}$ when looking for energy-minimizing configurations. This result rules out the appearance of fine scale mixtures of spheres (i.e., minimizing sequences of suspensions made with progressively smaller spheres). An existence proof of optimal designs within this class follows from a suitable Poincaré inequality together with the theory of Chenais [3], [4] for shape optimization problems over a restricted class of Lipschitz domains. This topic is pursued elsewhere and will appear in [10]. These results are in striking contrast to what is seen when there is perfect bonding between the two conductors. For this situation it is often the case that no optimal design exists. Instead, minimizing sequences of designs exhibit regions consisting of progressively finer mixtures of the two conductors; see Lurie and Cherkaev [13] and Murat and Tartar [14].

More generally, we consider Lipschitz domains A_r of good conductor compactly contained within the design domain Ω. As before, we place no constraints on the topological nature of the reinforcing set A_r. We show, subject to the resource constraint $\text{meas}(A_r) \leq \theta_r \text{ meas}(\Omega)$, that all energy minimizing configurations lie within a subclass of domains determined by bounds on $\rho_2(A_r, \sigma_r)$: see Theorem 4.2.

2. Energy dissipation inequalities. In this section we establish Theorem 1.1.

For any $g \in H^{-1/2}(\partial\Omega) \setminus R$ we write the difference $\Delta E = E(A_r \cup \Sigma, g) - E(A_r, g)$ as

$$\Delta E = C(A_r, \tilde{j}) - C(A_r, \hat{j}) + D(\Sigma, \tilde{j}), \tag{2.14}$$

where $\tilde{j} = \text{argmin}\{C(A_r \cup \Sigma, j)\}$, $\hat{j} = \text{argmin}\{C(A_r, j)\}$, and $D(\Sigma, \tilde{j})$ is given by

$$D(\Sigma, \tilde{j}) = \beta^{-1}\left\{\int_{\partial\Sigma} (\tilde{j} \cdot n)^2 ds - \int_{\Sigma} \beta(\sigma_m^{-1} - \sigma_r^{-1})\tilde{j} \cdot \tilde{j} dx\right\}. \tag{2.15}$$

Noting that the field $\tilde{j}$ is an admissible trial for the variational principle (1.1), we have

$$C(A_r, \tilde{j}) - C(A_r, \hat{j}) \geq 0. \tag{2.16}$$

Thus

$$\Delta E \geq D(\Sigma, \tilde{j}). \tag{2.17}$$

Now, the equations of state for the potential $\tilde{u} \in H^1(\Omega \backslash (\Gamma \cup \partial\Sigma))$ imply that $\tilde{j} = \sigma_r \nabla \tilde{u}$ in Σ, $[\sigma \nabla \tilde{u} \cdot n] = 0$ on $\partial\Sigma$, and $\tilde{j} \cdot n_{|_2} = \sigma_r \nabla \tilde{u} \cdot n_{|_2}$ on $\partial\Sigma$. Thus from (2.15) and (2.17) we obtain

$$\Delta \text{E} \geq \beta^{-1}\left\{\int_{\partial\Sigma} (\sigma_r \nabla \tilde{u} \cdot n)^2 ds - \int_{\Sigma} \beta(\sigma_m^{-1} - \sigma_r^{-1})\sigma_r \nabla \tilde{u} \cdot \sigma_r \nabla \tilde{u} dx\right\}. \tag{2.18}$$

From (1.9), it follows that

$$\int_{\partial\Sigma} (\sigma_r \nabla\varphi \cdot n)^2 ds - \rho_2(\Sigma, \sigma_r) \int_{\Sigma} \sigma_r \nabla\varphi \cdot \nabla\varphi dx \geq 0 \tag{2.19}$$

for all $\varphi \in H^{3/2}(\Sigma)$ such that $\text{div}(\sigma_r \nabla\varphi) = 0$ in Σ.

Comparing the right-hand side of (2.18) with (2.19), we discover that

$$\Delta E \geq 0 \tag{2.20}$$

for

$$\sigma_r \beta(\sigma_m^{-1} - \sigma_r^{-1})\sigma_r \leq \sigma_r \rho_2(\Sigma, \sigma_r), \tag{2.21}$$

and the theorem follows.

We observe that strict inequality in (2.20) follows from strict inequality in (2.21), provided that $\nabla \tilde{u}$ is not identically equal to zero on Σ.

3. The second Stekloff eigenvalue for simple shapes and size effects. The second Stekloff eigenvalue for a sphere of radius a filled with isotropic conductor σ_r is given by $\rho_2 = \sigma_r/a$. It follows immediately from Theorem 1.1 that if both conducting phases are isotropic and if Σ is a sphere of radius a, then we have the following theorem.

THEOREM 3.1 (size effect for spheres). *For any current flux $g \in H^{-1/2}(\partial\Omega) \setminus R$,*

$$E(A_r \cup \Sigma, g) \geq E(A_r, g) \tag{3.22}$$

if

$$a \leq R_{cr} = \beta^{-1}(\sigma_m^{-1} - \sigma_r^{-1})^{-1}. \tag{3.23}$$

Other size-effect theorems have been obtained in the context of effective properties for isotropic suspensions of isotropically conducting spheres in an isotropic matrix. In that context the results have focused on critical radii for monodisperse suspensions of spheres; see Lipton and Vernescu [11]. Here the critical radius is precisely R_{cr} and is that for which the conductivity of the composite equals that of the matrix.

Results involving various averages of sphere radii have been found in the context of isotropic polydisperse suspensions of spheres; see Lipton and Vernescu [12]. There it is shown that if the harmonic mean of the sphere radii lies above R_{cr}, then the effective conductivity is greater than the matrix conductivity. Moreover, the effective conductivity lies below that of the matrix when the arithmetic mean of the radii lies below R_{cr}.

For size effects in the context of isotropic dilute suspensions of spheres, see Chiew and Glandt [5]. Prediction of size effects for isotropic monodisperse suspensions of spheres, by way of micromodels such as the effective medium theory and differential effective medium theory, can be found in the work of Every, Tzou, Hasselman, and Raj [7], Hasselman and Johnson [8], and Davis and Artz [6].

More generally, we consider starlike inclusions Σ filled with isotropic conductor σ_r embedded in an isotropic matrix with conductivity σ_m. Fixing the origin inside Σ, we denote by h_m the minimum distance from the origin to a tangent plane on $\partial\Sigma$. The maximum and minimum distance from the origin to $\partial\Sigma$ are denoted by r_M and r_m, respectively. For such shapes, Bramble and Payne [2] show

$$\sigma_r^{-1}\rho_2(\Sigma, \sigma_r) \geq \frac{1}{r_M}\left[\left(\frac{r_m}{r_M}\right)^2 \frac{h_m}{r_M}\right]. \tag{3.24}$$

It is evident from (3.24) and Theorem 1.1 that we have the following size effect theorem for starlike reinforcements.

THEOREM 3.2 (size effect theorem for starlike particles). *If the reinforcement Σ is starlike with geometric parameters r_m, r_M, and h_m, then for any $g \in H^{-1/2}(\partial\Omega) \setminus R$, we have*

$$E(A_r \cup \Sigma, g) \geq E(A_r, g) \tag{3.25}$$

if

$$\left(\frac{1}{r_M}\left[\left(\frac{r_m}{r_M}\right)^2 \frac{h_m}{r_M}\right]\right)^{-1} \leq R_{cr}. \tag{3.26}$$

To fix ideas we apply this theorem to an ellipsoidal particle. Here we suppose that the half-lengths of the major and minor axes are specified by a and c, respectively. For this case Theorem 3.2 implies the following corollary.

COROLLARY 3.3 (size effect theorem for ellipsoidal particles). *Given an ellipsoidal particle Σ with major and minor axes specified by a and c, respectively, then for any current flux $g \in H^{-1/2}(\partial\Omega) \setminus R$*

$$E(A_r \cup \Sigma, g) \geq E(A_r, g) \tag{3.27}$$

if

$$a\left(\frac{a}{c}\right)^3 \leq R_{cr}. \tag{3.28}$$

We consider an ellipsoidal inclusion such that $c = a(1-\lambda)$ for $0 < \lambda < 1$. It follows from the corollary that the introduction of an ellipsoidal inclusion will not lower the energy dissipated inside the composite when a lies below $R_{cr}(1-\lambda)^3$.

4. Energy minimizing configurations. We consider the problem of minimizing the thermal energy dissipation over the class of polydisperse suspensions of spheres of good conductor immersed in a matrix of lesser conductivity. The matrix and spheres are made from isotropically conducting material with conductivities specified by σ_m and σ_r, respectively. Here the suspensions consist of a finite number of nonintersecting spheres and we assume no lower bound on the sphere radii. Denoting the ith sphere by B_i, we write $A_r = \cup B_i$. We suppose that the suspension takes up no more than a prescribed volume fraction θ_r of the total composite; i.e., meas$(A_r) \leq \theta_r$ meas(Ω). We denote this class of suspensions by $\mathcal{C}_{\theta_r}$. We consider the subclass $\mathcal{SC}_{\theta_r}$ of $\mathcal{C}_{\theta_r}$, defined to be all suspensions with minimum sphere radii greater than or equal to R_{cr}. For a prescribed heat flux $g \in H^{-1/2}(\partial\Omega) \setminus R$ on the boundary, we consider the problem

$$\min\{E(A_r, g) : A_r \in \mathcal{C}_{\theta_r}\}. \tag{4.29}$$

Theorem 4.1 follows from Theorem 3.1.

THEOREM 4.1. *If a minimizer of problem* (4.1) *exists, then it can be found in the class $\mathcal{SC}_{\theta_r}$ or $A_r = \emptyset$. Moreover, if Ω has dimensions for which $\mathcal{SC}_{\theta_r}$ is empty, then the minimum energy dissipation is given by $E(\emptyset, g)$.*

Proof. We consider any suspension in the class $\mathcal{C}_{\theta_r}$. If there exist spheres of radius less than R_{cr}, then Theorem 3.1 shows that there is no advantage to keeping them in the suspension. When $\mathcal{SC}_{\theta_r}$ is empty, we see that no reinforcement is needed, and the minimum is attained for $A_r = \emptyset$. ☐

Next we consider energy minimization over a wide class of particle configurations. We suppose that σ_m and σ_r are anisotropic and let $\mathcal{CL}_{\theta_r}$ be the class of Lipschitz continuous sets A_r compactly contained inside Ω for which $\text{meas}(A_r) \leq \theta_r \ \text{meas}(\Omega)$. Here we assume that A_r is the union of one or more components and we make no assumption on the topological nature of each component. For a given reinforcement set A_r, we denote its ith component by A_r^i. The subclass $\mathcal{SCL}_{\theta_r}$ of $\mathcal{CL}_{\theta_r}$ is defined to be all $A_r \in \mathcal{CL}_{\theta_r}$ for which every component A_r^i satisfies

$$\sigma_r^{-1} \rho_2(A_r^i, \sigma_r) \leq \mathbf{R_{cr}^{-1}}. \tag{4.30}$$

For $g \in H^{-1/2}(\partial\Omega) \setminus R$ we consider the problem

$$\min\{E(A_r, g) : A_r \in \mathcal{CL}_{\theta_r}\}. \tag{4.31}$$

Theorem 4.2 follows immediately from Theorem 1.1.

THEOREM 4.2. *If a minimizer of problem* (4.3) *exists, then it can be found in* $\mathcal{SCL}_{\theta_r}$ *or* $A_r = \emptyset$*. Moreover, if* Ω *has dimensions for which* $\mathcal{SCL}_{\theta_r}$ *is empty, then the minimum energy dissipation is given by* $E(\emptyset, g)$*.*

5. Conclusions. The second Stekloff eigenvalue associated with the reinforcement phase is shown to be a basic tool for the study of nonlocal perturbations of functionals with bulk and surface energies associated with imperfectly bonded composite conductors. The associated energy dissipation inequalities establish a means for selecting energy minimizing configurations. For the problem treated in section 4, it is found that fine scale oscillations are rendered superfluous due to the electrical contact resistance associated with the interface.

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