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A Saddle-Point Theorem with Application to Structural Optimization[1]

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Abstract. The relaxation for optimal compliance design is independent of whether the underlying elastic problem is formulated in terms of displacements or strains. For the purposes of numerical experimentation and computation, it may be advantageous to formulate optimal design problems in terms of displacements as is done in Ref. 1. The relaxed problem delivered by the displacement-based formulation is of min-min-max type. Because of this, efficient numerical implementation is hampered by the order of the last two min-max operations. We show here that the last two min-max operations may be exchanged, facilitating an efficient numerical algorithm. We remark that the rigorous results given here corroborate the numerical methods and experiments given in Ref. 1.

Key Words. Young measures, saddle points, finite-rank laminates, structures, optimization.

1. Introduction

It is now well known that optimal design of structures made from two dissimilar elastic materials may involve zones of composite formed from the two constituent materials; see Refs. 2–6. This observation motivates extension of the design space to incorporate the effective elastic properties of composites; see Refs. 4–6. This extension of the design space is commonly known as the relaxation of the original problem; cf. Refs. 7 and 8.

In this paper, we shall consider relaxation for problems of compliance optimization for three-dimensional structures made from two isotropic

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materials with elasticities specified by shear and bulk moduli $\mu_i, \kappa_i, i = 1, 2$, given by

$$C_i = 2\mu_i P_s + 3\kappa_i P_h, \tag{1}$$

such that $\mu_2 \geq \mu_1$, $\kappa_2 \geq \kappa_1$. Here, P_s and P_h are the projections onto deviatoric and hydrostatic strains, respectively. The structural domain is denoted by Ω and lies in $\mathbb{R}^3$. We remark that all results stated here immediately carry over to planar elastic problems.

We suppose that the relatively stiff material characterized by C_2 is more costly. Therefore, our goal is to minimize the compliance over all material layouts subject to a constraint on the amount of expensive material. The underlying elastic problem can be formulated variationally either in terms of stresses or elastic displacements.

It is easily seen (see Section 2) that the relaxation for this problem is achieved through the use of the well-known extremal class of effective elastic tensors corresponding to finite-rank laminar microstructures; see Refs. 7 and 9–14. The relaxation is independent of the formulation of the underlying elastic problem.

For purposes of numerical experimentation and computation, it may be advantageous to formulate the optimization problem in terms of displacements. The relaxed variational problem delivered by the displacement-based approach is of min-min-max type; see Section 2, Eq. (22). Because of this, an efficient numerical scheme is hampered by the order of the last two min-max operations. In this paper, we provide a saddle-point theorem justifying the exchange; see Theorem 4.1. The saddle-point theorem is established with the aid of a convexity property enjoyed by the effective tensors of finite-rank laminates (see Theorem 3.1) and the use of a tensor-valued family of Young measures; see Section 4.

Once the exchange is made, the compliance problem is of min-max-min type and the rightmost minimization reduces to the minimization of a local energy density at each point in the structural domain. This feature is computationally attractive, since the minimization of the local energy density can be done analytically; see Theorem 2.1. The saddle-point theorem and max-min exchange presented here have been incorporated in the recently developed numerical methods given in Ref. 1.

We illustrate the relationship between the relaxed Lagrangian for the displacement problem and its counterpart for the stress-based problem. The relaxed integrands appearing in both Lagrangians are nonquadratic functions of their arguments; nevertheless, there exists a duality relation between the two Lagrangians; see Section 5. We remark that the min-max interchange theorem easily generalizes to multiload problems. The associated saddle theorem and relations between relaxed Lagrangians are given in Section 6.

We remark that a simple change of variables shows that the saddle-point theorem can be written in terms of H-measures associated with minimizing sequences of designs; see Theorem 7.1. It follows from this observation that the relaxed controls are given in terms of the local volume fraction of the two materials and the H-measure; see Section 7.

To simplify the reading of the text, we list some definitions, vector spaces, and function spaces used in the presentation. All fourth-order tensors will be denoted by upper case letters $C, T, I, \ldots$; second-order tensors will be denoted using underlined lower case letters $\underline{\sigma}, \underline{e}, \ldots$; vectors will be denoted simply by lower case letters $f, g, \ldots$; scalar products and contractions are defined as follows:

$f \cdot g$ is the scalar with value $\sum_{i=1}^{3} f_i g_i$;

$\underline{\sigma} : \underline{e}$ is the scalar with value $\sum_{i,j=1}^{3} \sigma_{ij} e_{ij}$;

(T, V) is the scalar with value $\sum_{ijkl=1}^{3} T_{ijkl} V_{ijkl}$;

$|T|$ is the scalar with value $\sqrt{(T, T)}$;

$T\underline{e}$ is the second-order tensor with coefficients $(T\underline{e})_{ij} = \sum_{kl} T_{ijkl} e_{kl}$.

The following vector spaces are used

S_3 is the space of 3×3 symmetric matrices;
F is the space of fourth-order tensors such that $T_{ijkl} = T_{klij} = T_{jikl}$;
I is the identity tensor in F, given by

$$I_{ijkl} = (1/2)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk});$$

$E(\alpha, \beta)$ is the set of all T in F such that $\alpha I \leq \tau \leq \beta I$ in the sense of quadratic forms; here, $\alpha = \min(\kappa_1 \mu_1)$ and $\beta = \max(\kappa_2, \mu_2)$;
J is the set of all T in F such that T is positive definite, i.e., $T\underline{e} : \underline{e} \geq 0$ for all $\underline{e}$ in S_3.

The following function spaces are used throughout this presentation:

$L^\infty(\Omega, F)$ is the space of tensor-valued functions $T(x)$ defined on Ω such that each element $T_{ijkl}(x)$ is in $L^\infty(\Omega, \mathbb{R})$ and $T(x)$ takes values in F almost everywhere in Ω;
$L^\infty(\Omega, [0, 1])$ is the space of scalar-valued L^∞-functions mapping Ω into $[0, 1]$;

$L^p(\Omega, S_3)$ is the space of symmetric matrix-valued functions $\underline{m}$ defined on Ω such that each element $\underline{m}_{ij}$ is in $L^p(\Omega, \mathbb{R})$ and $\underline{m}$ takes values in S_3 almost everywhere;

$L^1(\Omega, J)$ is the space of tensor-valued functions $T(x)$ defined on Ω such that each element T_{ijkl} is in $L^1(\Omega, \mathbb{R})$ and $T(x)$ takes values in J almost everywhere in Ω.

2. Mathematical Formulation of the Problem

The compliance or work done in the structural domain against body forces and boundary tractions by the resulting elastic displacement u is given by

$$l(u) = \int_\Omega f \cdot u \, dx + \int_{\partial\Omega} g \cdot u \, dS, \tag{2}$$

where f is the distributed force density in $H^{-1}(\Omega)^3$ and g is the prescribed boundary traction in $H^{-1/2}(\partial\Omega)$. The displacement u is an element of $H^1(\Omega)^3$ and satisfies the equilibrium equations

$$-\operatorname{div} \underline{\sigma} = f, \qquad \text{in } \Omega, \tag{3}$$

$$\underline{\sigma} \cdot n = g, \qquad \text{on } \partial\Omega, \tag{4}$$

$$\underline{\sigma} = C(x)\underline{e}(u), \tag{5}$$

where $\underline{e}(u)$ is the strain tensor given by

$$\underline{e}(u) = (1/2)(u_{i,j} + u_{j,i}). \tag{6}$$

Here, the structural layout is prescribed by the piecewise constant elasticity tensor $C(x)$ given by

$$C(x) = \chi_1 C_1 + \chi_2 C_2, \tag{7}$$

where χ_1 is the indicator function of material 1 and $\chi_2 = 1 - \chi_1$.

The choices of body force and boundary tractions are consistent with the solvability requirement

$$\int_\Omega f \cdot v \, dx - \int_{\partial\Omega} g \cdot v \, ds = 0, \tag{8}$$

for all $v \in \mathscr{C}$, where

$$\mathscr{C} = \{v : \underline{e}(v) = 0\}. \tag{9}$$

We consider the problem of minimizing the compliance over all layouts of materials 1 and 2 subject to a volume constraint on the stiffer and more expensive material 2. We note that a particular layout is specified by $C(x)$ and the volume constraint on material 2 is given by

$$V_2 \geq \int_\Omega \chi_2 \, dx. \tag{10}$$

Here, V_2 is the maximum amount of material 2 allowed in the design and $V_2 < \mathrm{vol}(\Omega)$. The minimum compliance problem takes the form

$$\min_{C(x)} \quad l(u), \tag{11a}$$

$$\text{s.t.} \quad (3)-(6) \text{ and } (10). \tag{11b}$$

Here, we may view the above problem as one of distributed-parameter optimal control, where the control is $C(x)$. Problem (11) can be written variationally over the space $H^1(\Omega)^3$ of admissible displacement fields as

$$\min_{C(x)} \max_{u \in H^1(\Omega)^3} \left\{ 2l(u) - \int_\Omega C(x)\underline{\varepsilon}(u) : \underline{\varepsilon}(u) \, dx + \lambda \int_\Omega \chi_2 \, dx \right\}. \tag{12}$$

Here, λ denotes the positive multiplier associated with the volume fraction constraint. For a thorough treatment of Lagrange multipliers and their use in optimal design the reader is referred to the work of Kohn and Strang (Ref. 8). Alternatively, problem (11) may be written variationally over the space K of admissible stress fields $\underline{\tau}$ as

$$\min_{C(x)} \min_{\underline{\tau} \in K} \left\{ \int_\Omega C^{-1}(x)\underline{\tau} : \underline{\tau} \, dx + \lambda \int_\Omega \chi_2 \, dx \right\}. \tag{13}$$

Here, K is given by

$$K = \begin{cases} \underline{\tau} \text{ in } L^2(\Omega, S_3), & \text{s.t. } -\mathrm{div}\, \underline{\tau} = f, \text{ in } \Omega, \\ \underline{\tau} \cdot n = g, & \text{on } \partial\Omega. \end{cases}$$

It is well known, from the work of Refs. 2, 4–6, and 8, that problems of the type given by (11)–(13) are ill posed and require relaxation. This relaxation is accomplished through the extension of the design space. Indeed, the set of layouts is extended to include composites formed of the original constituents. We denote the set of effective elastic tensors associated with all composites formed using materials C_1 and C_2 by G. This set is most conveniently parameterized by the volume fraction of material 2 in the composite given by θ_2, $0 \leq \theta_2 \leq 1$. For fixed volume fraction θ_2, the set of associated effective elastic tensors is denoted by G_{θ_2}.

Although a complete characterization of G_{θ_2} is as yet unknown, certain boundaries of this set have been worked out (Refs. 10–14), and it follows from the theory of H-convergence (Ref. 5) that it is a closed and bounded set. Indeed, elementary estimates show that G_{θ_2} lies in $E(\alpha, \beta)$ for all values of θ_2. We now extend the design space to include composites.

Definition 2.1. A generalized layout is given by a local volume fraction $\theta_2(x)$ in $L^\infty(\Omega, [0, 1])$ and an associated tensor field $C(x)$ in $L^\infty(\Omega, F)$ taking values in the set $G_{\theta_2(x)}$. This set of tensor fields associated with generalized layouts is denoted by $\tilde{G}_{\theta_2(x)}$.

It follows from the remarks above that $\tilde{G}_{\theta_2(x)}$ lies on bounded subset of $L^\infty(\Omega, F)$ for all choices of the local volume fraction $\theta_2(x)$.

The relaxed versions of problems (12) and (13) are given by

$$\min_{\theta_2 \in L^\infty(\Omega,[0,1])} \min_{C \in \tilde{G}_{\theta_2(x)}} \max_{u \in H^1(\Omega)^3} \left\{ 2l(u) + \int_\Omega [\lambda\theta_2(x) - C(x)\underline{\varepsilon}(u) : \underline{\varepsilon}(u)]\, dx \right\} \tag{14}$$

and

$$\min_{\theta_2 \in L^\infty(\Omega,[0,1])} \min_{C \in \tilde{G}_{\theta_2(x)}} \min_{\underline{\tau} \in K} \left\{ \int_\Omega [C^{-1}(x)\underline{\tau} : \underline{\tau} + \lambda\theta_2(x)]\, dx \right\}. \tag{15}$$

Problems (14) and (15) are seen to be well posed by applying the arguments given in Murat and Tartar (Ref. 5) to the case of elasticity. We observe however that, to construct the optimal structural layout either analytically or numerically, we must possess some additional knowledge about the set of elastic tensors G_{θ_2}. This extra information is provided in the well-known extremal property of effective tensors of finite-rank laminar microstructures given in Ref. 13; see also Refs. 11 and 12. Simply put, the extremal property asserts that, for a prescribed local volume fraction $\theta_2(x)$ and strain $\underline{\varepsilon}(u(x))$, the local elastic energy is maximized over all elements of $G_{\theta_2(x)}$ by an effective tensor $\bar{C}$ in $G_{\theta_2(x)}$ associated with a finite-rank stiff laminate, i.e.,

$$\max_{C \in G_{\theta_2(x)}} \{C\underline{\varepsilon}(u) : \underline{\varepsilon}(u)\} = \bar{C}\underline{\varepsilon}(u) : \underline{\varepsilon}(u). \tag{16}$$

Similarly for a local strain $\underline{\tau}$, the local compliance is minimized over $G_{\theta_2(x)}$ by an effective tensor $\bar{C}$ associated with a stiff laminate, i.e.,

$$\min_{C \in G_{\theta_2(x)}} \{C^{-1}\underline{\tau} : \underline{\tau}\} = \bar{C}^{-1}\underline{\tau} : \underline{\tau}. \tag{17}$$

Definition 2.2. We denote the set of all elements in G_{θ_2} associated with stiff finite-rank laminar microstructures by $\overline{GL}_{\theta_2}$.

It easily follows from the stress-based formulation and the extremal property (17) that the relaxation is accomplished by extending the design space to include only effective tensors associated with finite-rank stiff laminar microstructures. Indeed, denoting by R the common value of (11) through (15), it is evident from the stress-based variational formulation given in (15) that

$$R = \min_{\theta_2 \in L^\infty(\Omega,[0,1])} \min_{\underset{\sim}{\tau} \in K} \min_{C \in \bar{G}_{\theta_2(x)}} \int_\Omega [C^{-1}(x)\underset{\sim}{\tau} : \underset{\sim}{\tau} + \lambda\theta_2(x)]\, dx$$

$$= \min_{\theta_2 \in L^\infty(\Omega,[0,1])} \min_{\underset{\sim}{\tau} \in K} \int_\Omega \left[\min_{C \in G_{\theta_2(x)}} \{C^{-1}(x)\underset{\sim}{\tau} : \underset{\sim}{\tau}\} + \lambda\theta_2(x) \right] dx. \quad (18)$$

We observe that minimization over $\bar{G}_{\theta_2(x)}$ may be exchanged with integration. Once exchanged, the minimization over the control variable C becomes pointwise, and from the extremal property (17) we observe that

$$R = \min_{\theta_2 \in L^\infty(\Omega,[0,1])} \min_{\underset{\sim}{\tau} \in K} \int_\Omega [H(\underset{\sim}{\tau}, x) + \lambda\theta_2(x)]\, dx, \quad (19)$$

where

$$H(\underset{\sim}{\tau}, x) = \min_{C \in \overline{GL}_{\theta_2(x)}} \{C^{-1}\underset{\sim}{\tau} : \underset{\sim}{\tau}\}, \quad (20)$$

and the relaxation using finite-rank laminates follows.

This relaxation is now well known and was first established in two dimensions in Ref. 10. To see that the same relaxation applies to the displacement-based formulation (14), we introduce the set of generalized layouts taking values in $\overline{GL}_{\theta_2}$.

Definition 2.3. For prescribed local volume fraction $\theta_2(x)$, the set of all tensor fields in $L^\infty(\Omega, F)$ taking values almost everywhere in $\overline{GL}_{\theta_2(x)}$ is denoted by $\overline{GL}_{\bar{\theta}_2(x)}$.

We now observe from (19) and Definition 2.3 that

$$R = \min_{\theta_2 \in L^\infty(\Omega,[0,1])} \min_{\underset{\sim}{\tau} \in K} \min_{C \in \overline{GL}_{\bar{\theta}_2(x)}} \int_\Omega [C^{-1}(x)\underset{\sim}{\tau} : \underset{\sim}{\tau} + \lambda_2\theta_2(x)]\, dx$$

$$= \min_{\theta_2 \in L^\infty(\Omega,[0,1])} \min_{C \in \overline{GL}_{\bar{\theta}_2(x)}} \min_{\underset{\sim}{\tau} \in K} \int_\Omega [C^{-1}(x)\underset{\sim}{\tau} : \underset{\sim}{\tau} + \lambda_2\theta_2(x)]\, dx$$

$$= \min_{\theta_2 \in L^\infty(\Omega,[0,1])} \min_{C \in \overline{GL}_{\bar{\theta}_2(x)}} \max_{u \in H^1(\Omega)^3} \left\{ 2l(u) + \int_\Omega [\lambda\theta_2(x) - C(x)\underset{\sim}{e}(u) : \underset{\sim}{e}(u)]\, dx \right\}. \quad (21)$$

Here, the last equality follows from the duality between the stress and displacement formulations of the elastic compliance. It is evident from (21) that the relaxation for the displacement formulation is also accomplished using finite-rank laminates. We collect our observations and write the relaxed compliance optimization in displacement formulation as

$$R = \min_{\theta_2 \in L^\infty(\Omega,[0,1])} \min_{C \in \overline{GL}_{\theta_2(x)}} \max_{u \in H^1(\Omega)^3} \left\{ 2l(u) + \int_\Omega [\lambda\theta_2(x) - C(x)\underline{e}(u) : \underline{e}(u)]\, dx \right\}. \tag{22}$$

For the purposes of numerical implementation, it is advantageous to switch the orders of minimization and maximization. Indeed, if the last two operations are interchanged, the subsequent minimization may be done analytically; see Ref. 1. In what follows, we show that it is possible to exchange the two rightmost operations of min-max; see Theorem 4.1. In this way, we arrive at the fundamental result of this exposition.

Theorem 2.1. The relaxation for the displacement-based optimal compliance design problem (12) is given by

$$\min_{\theta_2 \in L^\infty(\Omega,[0,1])} \max_{u \in H^1(\Omega)^3} \left\{ 2l(u) + \int_\Omega [\lambda\theta_2(x) + F(\underline{e}(u), x)]\, dx \right\}, \tag{23}$$

where for any constant strain $\underline{\zeta}$, we have

$$F(\underline{\zeta}, x) = -\max_{C \in \overline{GL}_{\theta_2(x)}} C\underline{\zeta} : \underline{\zeta}. \tag{24}$$

We observe that $F(\zeta, x)$ is a nonlinear function of the strain variable $\underline{\zeta}$. Here, $F(\zeta, x)$ can be computed analytically or numerically using explicit formulas for tensors in $\overline{GL}_{\theta_2}$. We remark that $F(\zeta, x)$ has been calculated explicitly for the two-dimensional case in Ref. 1. Similar strain energy functions have been computed earlier in the context of three-dimensional incompressible elasticity and two-dimensional elasticity; see Refs. 10, 11, and 14. The proof of Theorem 2.1 is given in Section 4.

3. Convexity Properties of Finite-Rank Laminates

The necessary new tool for deducing the saddle-point theorem is a convexity property enjoyed by the effective elastic tensors for finite-rank laminar microstructures. Before stating the convexity property, we introduce the formulas for the effective tensors of finite-rank laminar composites. They are given by

$$\bar{C} = C_2 - (1-\theta_2)[(C_2 - C_1)^{-1} - \theta^2 \overset{*}{T}_2]^{-1} \tag{25}$$

and

$$\underset{\sim}{C} = C_1 + \theta_2[(\mathbb{C}_2 - \mathbb{C}_1)^{-1} + (1-\theta_2)\overset{*}{T}_1]^{-1} \tag{26}$$

a/jota/vol.81-3/P279.p09
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for stiff and compliant composites, respectively. Here, the tensors $\overset{*}{T}_i$, $i = 1, 2$, are of the form

$$\overset{*}{T}_i = \sum_{r=1}^{j} \rho_r \hat{\Gamma}_i(n_r), \qquad 1 \leq j < \infty, \tag{27}$$

where

$$\rho_r \geq 0 \quad \text{and} \quad \sum_{r=1}^{j} \rho_r = 1, \tag{28}$$

and $\hat{\Gamma}_i(v)$ are tensor-valued functions of vectors v defined on the unit sphere S^2, given by

$$\hat{\Gamma}_i(v)_{mnop} = (1/4\mu_i)(v_m v_o \delta_{qs} + v_m v_p \delta_{mn} + v_n v_o \delta_{mp} + v_n v_p \delta_{mo}) + [3/(3\kappa_i + 2\mu_i) - 1/\mu_i] v_m v_n v_o v_p, \qquad i = 1, 2. \tag{29}$$

These formulas were derived in Ref. 9 and have been rewritten in notation convenient for this exposition.

We introduce the convex sets of tensors Δ_1, Δ_2 formed by considering all convex combinations $\overset{*}{T}_1$ and $\overset{*}{T}_2$ delivered by formula (27). To understand the geometry of the sets Δ_1 and Δ_2, we regard $\hat{\Gamma}_1(v)$ and $\hat{\Gamma}_2(v)$ given by (29) as tensor-valued maps mapping the surface of the unit sphere to surfaces in the space F of fourth-order tensors. It is now evident from (27) that Δ_1 and Δ_2 correspond to the closed convex hulls of these surfaces.

We provide here a useful but equivalent definition of the set $\overline{GL}_{\theta_2}$ given in Section 2.

Definition 3.1. Fixing θ_2 in [0, 1], the set of tensors $\overline{GL}_{\theta_2}$ corresponds to the set of all tensors $\bar{C}$ delivered by (25) as the tensor $\overset{*}{T}_2$ ranges through Δ_2.

Remark 3.1. It follows from (25) and elementary estimates that $\bar{C}$ lies in $E(\alpha, \beta)$. In this way, we see that $\overline{GL}_{\theta_2}$ is bounded independently of θ_2.

We indicate the dependence of the effective tensors $\bar{C}$ and $\underset{\sim}{C}$ on $\overset{*}{T}_1$, $\overset{*}{T}_2$, and θ_2 by writing

$$\bar{C} = \bar{C}(\overset{*}{T}_2, \theta_2), \tag{30}$$

$$\underset{\sim}{C} = \underset{\sim}{C}(\overset{*}{T}_1, \theta_2). \tag{31}$$

For any finite set of 3×3 symmetric matrices $\zeta^1, \zeta^2, \ldots, \zeta^l$, $l < \infty$, we form

$$\bar{f}(\overset{*}{T}_2) = \sum_{j=1}^{l} (\bar{C}(\overset{*}{T}_2, \theta_2)\zeta^j : \zeta^j) \tag{32}$$

and

$$\underset{\sim}{f}(\overset{*}{T}_1) = \sum_{j=1}^{l} \underset{\sim}{C}(\overset{*}{T}_1, \theta_1)\zeta^j : \zeta^j. \tag{33}$$

We now state the following convexity property for laminates.

Theorem 3.1. Convexity Property. For fixed θ_2 and for $\overset{*}{T}_1 \in \Delta_1$ and $\overset{*}{T}_2 \in \Delta_2$, the sum of the energies $\bar{f}(\overset{*}{T}_2)$ is concave in $\overset{*}{T}_2$ and the sum $\underset{\sim}{f}(\overset{*}{T}_1)$ is convex in $\overset{*}{T}_1$; i.e., for $\overset{*}{T}_2$ and $\overset{*}{T}'_2$ in Δ_2 and t in [0, 1], we have

$$\bar{f}(t\overset{*}{T}_2 + ((1-t)\overset{*}{T}'_2) \geq t\bar{f}(\overset{*}{T}_2) + (1-t)\bar{f}(\overset{*}{T}'_2), \tag{34}$$

while for $\overset{*}{T}_1$ and $\overset{*}{T}'_1$ in Δ_1 and t in [0, 1], we have

$$\underset{\sim}{f}(t\overset{*}{T}_1 + (1-t)\overset{*}{T}'_1) \leq t\underset{\sim}{f}(\overset{*}{T}_1) + (1-t)\underset{\sim}{f}(\overset{*}{T}'_1). \tag{35}$$

We provide the proof of concavity of $\bar{f}(\overset{*}{T}_2)$ noting that the convexity of $\underset{\sim}{f}(\overset{*}{T}_1)$ follows from similar arguments.

Proof. We introduce the function $\bar{g}(t)$ for t in [0, 1] defined by

$$\bar{g}(t) = \bar{f}(t\overset{*}{T}_2 + (1-t)\overset{*}{T}'_2), \tag{36}$$

for $\overset{*}{T}_2$ and $\overset{*}{T}'_2$ in Δ_2. We prove the theorem by showing that

$$\partial_t^2 \bar{g} \leq 0, \qquad \text{for } t \text{ in } [0, 1].$$

From (25) and (32), we may write

$$\bar{g}(t) = \sum_{j=1}^{l} \{C_2\zeta^j : \zeta^j - (1-\theta_2)[A + tB]^{-1}\zeta^j : \zeta^j\}, \tag{37}$$

where

$$A = (\mathbb{C}_2 - \mathbb{C}_1)^{-1} - \overset{*}{T}'_2 \quad \text{and} \quad B = \overset{*}{T}'_2 - \overset{*}{T}_2. \tag{38}$$

Differentiation of (37) yields

$$\partial_t^2 \bar{g}(t) = -(1-\theta_2)2 \sum_{j=1}^{l} [A + tB]^{-1/2}B[A + tB]^{-1}\zeta^j : [A + tB]^{-1/2}B[A + tB]^{-1}\zeta^j \leq 0, \tag{39}$$

and concavity follows. □

Remark 3.2. In the sequel, we shall make use of contractions of the form $(\bar{C}, B)$, where B is any positive-definite tensor in the set J. Expanding B in its eigentensors $\eta^1, \eta^2, \ldots, \eta^6$ and eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_6$, we find that

$$(\bar{C}, B) = \sum_{l=1}^{6} \bar{C}\zeta^l : \zeta^l, \tag{40}$$

where

$$\zeta^l = \sqrt{\lambda_l}\eta^l, \qquad l = 1, \ldots, 6.$$

Thus, we have the following corollary.

Corollary 3.1. For fixed θ_2 and for $\overset{*}{T}_2$, $\overset{*}{T}_2'$ in Δ_2 and for any B in J and t in [0, 1], we have

$$(\bar{C}(t\overset{*}{T}_2 + (1-t)\overset{*}{T}_2'), B) \geq t(\bar{C}(\overset{*}{T}_2), B) + (1-t)(\bar{C}(\overset{*}{T}_2'), B). \tag{41}$$

4. Saddle-Point Theorem

In this section, we establish the following theorem.

Theorem 4.1. For fixed local volume fraction θ_2 in $L^\infty(\Omega, [0, 1])$ and the Lagrangian $L(C, u)$ defined by

$$L(C, u) \equiv 2l(u) + \int_\Omega [\lambda\theta_2(x) - C(x)\underline{e}(u) : \underline{e}(u)]\, dx, \tag{42}$$

we have

$$\min_{C \in \overline{GL}_{\theta_2(x)}} \max_{u \in H^1(\Omega)^3} L(C, u) = \max_{u \in H^1(\Omega)^3} \min_{C \in \overline{GL}_{\theta_2(x)}} L(C, u). \tag{43}$$

We remark that Theorem 2.1 follows immediately from Theorem 4.1. Indeed, we may write

$$\min_{C \in \overline{GL}_{\theta_2(x)}} L(C, u) = 2l(u) + \int_\Omega \left[\lambda\theta_2(x) - \max_{C \in \overline{GL}_{\theta_2(x)}} \{C(x)e(u) : e(u)\}\right] dx,$$

and Theorem 2.1 follows.

The Lagrangian appearing in Theorem 4.1 has arguments in $L^\infty(\Omega, F)$ and $H^1(\Omega)^3$. Thus, the saddle-point theorem is proved using the $L^\infty(\Omega, F)$ weak* and weak $H^1(\Omega)^3$ topologies. This theorem is established with the aid of the Young measures and follows from the convexity property given in Theorem 3.1.

To proceed further, we introduce the following fourth-order tensor fields.

Definition 4.1. We define the set D_2 to be the set of all fourth-order tensor fields in $L^\infty(\Omega, F)$ taking values in Δ_2 almost everywhere in Ω.

We now give a useful alternative definition of the set of controls given by $\overline{GL}_{\bar\theta_2(x)}$.

Definition 4.2. For prescribed local volume fraction θ_2 in $L^\infty(\Omega, [0, 1])$, the set $\overline{GL}_{\bar\theta_2(x)}$ is defined to be the set of all fourth-order tensor fields $\bar{C}(\overset{*}{T}_2(x), \theta_2(x))$ associated with finite-rank laminar microstructures given by the formula

$$C(\overset{*}{T}_2(x), \theta_2(x)) \equiv C_2 - (1-\theta_2(x))[(C_2 - C_1)^{-1} - \theta_2(x)\overset{*}{T}_2(x)]^{-1}, \tag{44}$$

where $\overset{*}{T}_2(x)$ is an element of D_2.

Remark 4.1. It follows from Remark 3.1 that $\overline{GL}_{\bar\theta_2(x)}$ is a bounded subset of $L^\infty(\Omega, F)$.

We now use Definition 4.3 to transform the Lagrangian given by (42). Indeed, writing

$$D(\overset{*}{T}_2, u) \equiv 2l(u) + \int_\Omega [\lambda\theta_2(x) - \bar{C}(\overset{*}{T}_2(x), \theta_2(x))\underset{\sim}{e}(u) : \underset{\sim}{e}(u)]\, dx, \tag{45}$$

we see that

$$\min_{C\in\overline{GL}_{\bar\theta_2(x)}} \max_{u\in H^1(\Omega)^3} L(C, u) = \min_{\overset{*}{T}_2\in D_2} \max_{u\in H^1(\Omega)^3} D(\overset{*}{T}_2, u) \tag{46}$$

and

$$\max_{u\in H^1(\Omega)^3} \min_{C\in\overline{GL}_{\bar\theta_2(x)}} L(C, u) = \max_{u\in H^1(\Omega)^3} \min_{\overset{*}{T}_2\in D_2} D(\overset{*}{T}_2, u). \tag{47}$$

Thus, Theorem 4.1 is established if we can demonstrate that the transformed Lagrangian $D(\overset{*}{T}_2, u)$ has a saddle point in $H^1(\Omega)^3 \times D_2$.

To obtain that $D(\overset{*}{T}_2, u)$ has a saddle point, we may use standard arguments of the kind given in Ref. 15 [see Propositions 2.1, 2.2, and (2.24), page 174], provided that the following properties hold:

(P1) D_2 is convex and sequentially compact in the $L^\infty(\Omega, F)$ weak* topology;
(P2) for every u in $H^1(\Omega)^3$ and $\overset{*}{T}_2$ in D_2, the Lagrangian $D(\overset{*}{T}_2, u)$ is convex and lower semicontinuous in $\overset{*}{T}_2$ in the $L^\infty(\Omega, F)$ weak* topology;
(P3) for each $\overset{*}{T}_2$ in D_2, $D(\overset{*}{T}_2, u)$ is concave and upper semicontinuous in u for u in $H^1(\Omega)^3$;
(P4) there exists $\overset{*}{T}_2$ in D_2 such that $\lim_{|u|_{H^1}\to\infty} D(\overset{*}{T}_2, u) = -\infty$.

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We remark that the properties (P3) and (P4) are standard and are easily established; their proofs are postponed until the end of the section. Properties (P1) and (P2) are special to this problem. Indeed, standard saddle-point theorems of the type given in Ref. 15 apply to Lagrangians defined over reflexive Banach spaces. However, the Lagrangian $D(\overset{*}{T}_2, u)$ has one of its arguments in a bounded subset of $L^\infty(\Omega, F)$. Because of this, we must use the $L^\infty(\Omega, F)$ weak* topology. Properties (P1) and (P2) will be established with the aid of the Young measures.

We now establish property (P1).

Lemma 4.1. D_2 is convex and sequentially compact in the $L^\infty(\Omega, F)$ weak* topology.

Proof. We see that the convexity of D_2 follows from Definition 4.1 and the fact that Δ_2 is convex. To prove compactness, we consider the sequence $\overset{*}{T}{}_2^k$ in D_2. Since the set Δ_2 is bounded, it follows that $\overset{*}{T}{}_2^k$ is bounded in $L^\infty(\Omega, F)$ and there exists a subsequence $\overset{*}{T}{}_2^{k'}$ converging in $L^\infty(\Omega, F)$ weak* to $\overset{*}{T}{}_2^\infty$. We prove compactness by showing that $\overset{*}{T}{}_2^\infty$ is an element of D_2. We introduce the family of Young measures $\{\nu_x(Y)\}_{x\in\Omega}$ associated with the sequence. As $\overset{*}{T}{}_2^k$ only takes values in Δ_2, we observe that $\{\nu_x(Y)\}_{x\in\Omega}$ is a family of probability measures with support inside Δ_2. Therefore,

$$\overset{*}{T}{}_2^\infty = \int_{\Delta_2} Y \, d\nu_x(Y). \tag{48}$$

It is evident from the convexity of Δ_2 and (48) that $\overset{*}{T}{}_2^\infty$ is an element of D_2, and Lemma 4.1 follows. □

We prove the weak* lower semicontinuity stated in property (P2) with the aid of the following lemma.

Lemma 4.2. Given a sequence $\overset{*}{T}{}_2^k$ defined on the set D_2 converging in $L^\infty(\Omega, F)$ weak* to an element $\overset{*}{T}{}_2^\infty$ in D_2, there exists a subsequence $\overset{*}{T}{}_2^{k'}$ such that, for any $B(x)$ in $L^1(\Omega, J)$,

$$\lim_{k'\to\infty} \int_\Omega (\bar{C}(\overset{*}{T}{}_2^{k'}, \theta_2), B) \, dx \le \int_\Omega (\bar{C}(\overset{*}{T}{}_2^\infty, \theta_2) B) \, dx. \tag{49}$$

Proof. For the sequence given in the hypothesis of Lemma 4.2, the associated family of Young measures $\{\nu_x(Y)\}_{x\in\Omega}$ are probability measures with support contained in Δ_2, and we may represent the weak* limit $\overset{*}{T}{}_2^\infty$ as

$$\overset{*}{T}{}_2^\infty(x) = \int_{\Delta_2} Y \, d\nu_x(Y). \tag{50}$$

We observe from (44) that $\bar{C}(Y, \theta_2)$ is a Carathéodory function, in the sense that $\bar{C}(Y, \theta_2)$ is continuous in Y for almost all x in Ω, and is bounded and measurable in x for every Y in Δ_2. Therefore, applying a version of the fundamental theorem of Young measures [see Eq. 15 of Ref. 16], there exists a subsequence $\overset{*}{T}{}_2^k$ converging weak* $L^\infty(\Omega, F)$ to $\overset{*}{T}{}_2^\infty$ such that

$$\lim_{k \to \infty} \int_\Omega (\bar{C}(\overset{*}{T}{}_2^k(x), \theta_2(x)), B)\, dx = \int_\Omega \left(\left(\int_{\Delta_2} \bar{C}(Y, \theta_2(x))\, d\nu_x(Y), B \right) dx. \quad (51)$$

It follows immediately from Corollary 3.1, Eq. (41), and (50) that

$$\int_\Omega \left(\left(\int_{\Delta_2} \bar{C}(Y, \theta_2(x))\, d\nu_x(Y), B \right) dx \le \int_\Omega (\bar{C}(\overset{*}{T}{}_2^\infty(x), \theta_2(x)), B)\, dx, \quad (52)$$

and the lemma is proved. □

It follows from Lemma 4.2 that, for fixed u in $H^1(\Omega)^3$ and $\overset{*}{T}{}_2^k \to \overset{*}{T}{}_2^\infty$ weak* in $L^\infty(\Omega, F)$, there exists a subsequence $\overset{*}{T}{}_2^k$ converging in $L^\infty(\Omega, F)$ weak* to $\overset{*}{T}{}_2^\infty$ such that

$$\lim_{k \to \infty} D(\overset{*}{T}{}_2^k, u) \ge D(\overset{*}{T}{}_2^\infty, u). \quad (53)$$

We also observe from Remark 3.1 that, for u fixed in $H^1(\Omega)^3$, the Lagrangian $D(\overset{*}{T}_2, u)$ is bounded below for $\overset{*}{T}_2$ in D_2. Therefore, it follows that, for any sequence $\overset{*}{T}{}_2^k$ converging to a limit $\overset{*}{T}{}_2^\infty$ in $L^\infty(\Omega, F)$ weak*,

$$\lim_{k \to \infty} \inf D(\overset{*}{T}{}_2^k, u) > -\infty. \quad (54)$$

In this way, we see that the $L^\infty(\Omega, F)$ weak* lower semicontinuity for $D(\overset{*}{T}_2, u)$ follows from (53) and (54). Lastly, the convexity of $D(\overset{*}{T}{}_2^k, u)$ in $\overset{*}{T}_2$ follows immediately from Theorem 3.1 and property (P2) is established.

We conclude the section by noting that property (P3) follows from inspection and (P4) is a consequence of Korn's inequality.

5. Relaxed Lagrangians and Duality

The saddle-point Theorem 4.1 is used to provide a duality relation between partially relaxed Lagrangians appearing in the stress and displacement based optimal compliance design problems (19) and (23).

Indeed, we have the following proposition.

Proposition 5.1. For prescribed volume fraction $\theta_2(x)$ in $L^\infty(\Omega, [0, 1])$,

$$\min_{\underline{\tau}\in K} \int_\Omega H(\underline{\tau}, x)\, dx = \max_{u\in H^1(\Omega)^3} \left\{2l(u) + \int_\Omega + \int_\Omega F(e(\underline{u}), x)\, dx\right\}, \quad (55)$$

where $H(\underline{\tau}, x)$ and $F(\underline{\zeta}, x)$ are nonlinear functions of $\underline{\tau}$ and $\underline{\zeta}$ and are given by formulas (20) and (24), respectively.

Explicit formulas for $H(\underline{\tau}, x)$ and $F(\underline{\zeta}, x)$ have been worked out for the two-dimensional design problem and are given in Refs. 1 and 14.

Proof. The proof of Proposition 5.1 follows from the following string of equalities:

$$\begin{aligned}\min_{\underline{\tau}\in K} \int_\Omega H(\underline{\tau}, x)\, dx &= \min_{\underline{\tau}\in K} \min_{C\in \overline{GL}_{\bar\theta_2(x)}} \int_\Omega C^{-1}\underline{\tau} : \underline{\tau}\, dx \\ &= \min_{C\in \overline{GL}_{\bar\theta_2}} \min_{\underline{\tau}\in K} \int_\Omega C^{-1}\underline{\tau} : \underline{\tau}\, dx \\ &= \min_{C\in \overline{GL}_{\bar\theta_2(x)}} \max_{u\in H^1(\Omega)^3} \left\{2l(u) - \int_\Omega Ce(u) : e(u)\, dx\right\} \\ &= \max_{u\in H^1(\Omega)^3} \min_{C\in \overline{GL}_{\bar\theta_2(x)}} \left\{2l(u) - \int_\Omega Ce(u) : e(u)\, dx\right\} \\ &= \max_{u\in H^1(\Omega)^3} \left\{2l(u) + \int_\Omega F(\underline{e}(\underline{u}), x)\, dx\right\}. \quad (56)\end{aligned}$$

The second to the last equality in (56) is an application of Theorem 4.1. □

We note that the integrands $H(\tau, x)$ and $F(\zeta, x)$ have been portrayed in the literature as nonlinear constitutive laws for smart elastic materials; see Refs. 1 and 7. These materials are smart in the sense that they provide the optimal local elastic response for prescribed stress or displacement fields. We point out that Proposition 5.1 provides dual variational principles for such materials.

6. Multiload Problems

Theorem 2.1 and Proposition 5.1 can be easily extended to multiload optimal compliance design problems. Since the extension is straightforward and uses the techniques developed in earlier sections, we shall only state the results.

We consider N load cases prescribed by the body force densities f^i and boundary tractions g^i, $i = 1, 2, \ldots, N$. Associated with each load case (f^i, g^i) is a displacement field u^i satisfying equilibrium equations of the kind given by (3)–(6).

We consider minimizing a weighted sum of the compliances

$$l^i(u^i) = \int_\Omega f^i \cdot u^i\, dx + \int_{\partial\Omega} g^i \cdot u^i\, dS, \tag{57}$$

given by

$$L = \sum_{i=1}^{N} w_i l^i(u^i), \tag{58}$$

where

$$w_i \geq 0,\ i = 1, \ldots, N, \quad \text{and} \quad \sum_{i=1}^{N} w_i = 1.$$

The goal here is to minimize L subject to a volume constraint on the stiff elastic material characterized by elasticity tensor C_2. Defining

$$U^N = \otimes^N H^1(\Omega)^3,$$

the constrained optimization problem written in terms of displacements has the variational formulation

$$\min_{C(x)} \max_{(u^1, u^2, \ldots, u^N) \in U^N} \left[\sum_{i=1}^{N} w_i \left\{ 2l^i(u^i) - \int_\Omega C(x)\underline{e}(u^i) : \underline{e}(u^i)\, dx \right\} + \lambda V_2 \right]; \tag{59}$$

here, V_2 is the volume of stiff material in the design and λ is the Lagrange multiplier associated with the volume constraint. For N independently chosen constant strains $\underline{\zeta}^i, i = 1, \ldots, N$, we define the function $J(\underline{\zeta}^1, \underline{\zeta}^2, \ldots, \underline{\zeta}^N, x)$ by

$$J(\underline{\zeta}^1, \underline{\zeta}^2, \ldots, \underline{\zeta}^N, x) = -\max_{C \in \overline{GL}_{\theta_2(x)}} \sum_{i=1}^{N} w_i C\underline{\zeta} : \underline{\zeta}^i. \tag{60}$$

Then the relaxation is given by the following theorem.

Theorem 6.1. The relaxation for the multiload constrained compliance optimization problem (59) is

$$\min_{\theta_2(x) \in L^\infty(\Omega, [0,1])} \max_{(u^1, u^2, \ldots, u^N) \in U^N} \left[\sum_{i=1}^{N} 2w_i l^i(u^i) \right.$$
$$\left. + \int_\Omega [J(\underline{e}(u^1), \underline{e}(u^2), \ldots, e(u^N), x) + \lambda\theta_2(x)]\, dx \right]. \tag{61}$$

Theorem 6.1 is the extension of Theorem 2.1 to the multiload case. Defining

$$K^N = \otimes^N K,$$

the constrained optimization problem written in terms of stresses has the variational formulation

$$\min_{C(x)} \min_{(\underline{\tau}^1, \underline{\tau}^2, \ldots, \underline{\tau}^N) \in K^N} \left\{ \sum_{i=1}^{N} w_i \int_\Omega C^{-1}(x)\underline{\tau}^i : \underline{\tau}^i \, dx + \lambda V_2 \right\}. \tag{62}$$

Arguing as in Ref. 7 or as in Section 2, the relaxation of the compliance problem given in the stress-based variational formulation is

$$\min_{\theta_2(x) \in L^\infty(\Omega, [0,1])} \min_{(\underline{\tau}^1, \underline{\tau}^2, \ldots, \underline{\tau}^N) \in K^N} \int_\Omega R(\underline{\tau}^1, \underline{\tau}^2, \ldots, \underline{\tau}^N, x) + \lambda \theta_2(x) \, dx, \tag{63}$$

where for any set of constant strains $\underline{\tau}^1, \ldots, \underline{\tau}^N$,

$$R(\underline{\tau}^1, \underline{\tau}^2, \ldots, \underline{\tau}^N, x) = \min_{C \in \overline{GL}_{\theta_2(x)}} \sum_{i=1}^{N} w_i C^{-1} \underline{\tau}^i : \underline{\tau}^i. \tag{64}$$

One also has a duality relation between the relaxed Lagrangians for both formulations.

Proposition 6.1. For prescribed volume fraction $\theta_2(x)$ in $L^\infty(\Omega, [0, 1])$,

$$\min_{(\underline{\tau}^1, \underline{\tau}^2, \ldots, \underline{\tau}^N) \in K^N} \int_\Omega R(\underline{\tau}^1, \underline{\tau}^2, \ldots, \underline{\tau}^N, x) \, dx$$

$$= \max_{(u^1, u^2, \ldots, u^N) \in U^N} \left\{ \sum_{i=1}^{N} 2 w_i l^i(u^i) + \int_\Omega J(\underline{e}(u^1), \underline{e}(u^2), \ldots, \underline{e}(u^N), x) \, dx \right\}. \tag{65}$$

7. Formulation in Terms of H-Measures

The H-measure introduced independently by Gerard (Ref. 17) and Tartar (Ref. 18) is a mathematical tool used to record the local anisotropy of weakly convergent sequences. To see how to interpret Theorems 4.1 in the context of H-measures, we note from Definition 4.2 that the control $\overset{*}{T}_2$ can be written as

$$\overset{*}{T}_2(x) = \int_{S^2} \hat{\Gamma}_2(v) \, d\mu_x(v), \tag{66}$$

where $\mu_x(v)$ is a probability measure on the unit sphere for almost all x in Ω. The measure $\mu_x(v)$ can be interpreted as the H-measure for the weakly converging sequence

$$\chi_2^n(x) - \theta_2(x) \rightharpoonup 0, \qquad \text{in } L^\infty \text{ weak}^*, \tag{67}$$

associated with a minimizing sequence $\chi_2^n(x)$ of layouts. As indicated earlier, these sequences can always be associated with finite-rank laminar microstructures. We change variables in the transformed Lagrangian $D(\hat{T}_2, u)$ given by (45) and define the new Lagrangian $\tilde{D}(\mu_x, u)$ by

$$\tilde{D}(\mu_x, u) = D\left(\int_{S^2} \hat{\Gamma}_2(v)\, d\mu_x(v), u\right). \tag{68}$$

Denoting by P the set of all probability measures on the unit sphere, we see from (45)–(47) and (68) that Theorem 4.1 is equivalent to the following theorem.

Theorem 7.1. We have

$$\min_{\mu_x \in P} \max_{u \in H^1(\Omega)^3} \tilde{D}(\mu_x, u) = \max_{u \in H^1(\Omega)^3} \min_{\mu_x \in P} \tilde{D}(\mu_x, u). \tag{69}$$

Similar statements can be made for the Lagrangians associated with multiload optimization problems.

In view of Theorem 7.1, we see that the relaxation of the minimum compliance problem as stated in (12) requires the introduction of two new relaxed controls associated with chattering sequences $\{\chi_2^n(x)\}_{n=1}^\infty$ of layouts, namely: the local volume fraction of material 2, given by $\theta_2(x)$, i.e. the weak limit of χ_2^n; and the H-measure associated with the sequence $\chi^n(x) - \theta_2(x)$, given by $\mu_x(v)$.

8. Conclusions

We remark that in general it is not possible to exchange minimization over volume fraction $\theta_2(x)$ and maximization over displacement fields in (23) or (61), as the resulting integrand may not be quasiconcave in the displacement. This observation is seen in the numerical results of Jog, Haber, and Bendsoe (see Ref. 1).

References

1. Jog, C. S., Haber, R. B., and Bendsoe, M. P., *A Displacement-Based Topology Design Method with Self-Adaptive Materials*, Topology Design of

plenum / jota / vol.81-3 / P279.p19
07:19:01 1994 Page 1

Structures, Edited by M. P. Bendsoe, Kluwer, Dordrecht, The Netherlands, NATO ASI Series, Vol. 227, pp. 219–247, 1993.

2. CHENG, K. T., and OLHOFF, N., *An Investigation Concerning Optimal Design of Solid Elastic Plates*, International Journal of Solids and Structures, Vol. 17, pp. 305–323, 1981.

3. KLOSOWICZ, B., and LURIE, K. A., *On the Optimal Nonhomogeneity of a Torsional Elastic Bar*, Archives of Mechanics, Vol. 24, pp. 239–249, 1971.

4. LURIE, K. A., and CHERKAEV, A. V., *Optimal Structural Design and Relaxed Controls*, Optimal Control Applications and Methods, Vol. 4, pp. 387– , 1983.

5. MURAT, F., and TARTAR, L., *Calcul des Variations et Homogénéisation*, Les Methodes de l'Homogénéisation: Théorie et Applications en Physique, Collection de la Direction des Etudes et Recherches, Electricité de France, Eyrolles, Paris, France, pp. 319–370, 1985.

6. LURIE, K. A., CHERKAEV, A. V., and FEDOROV, A. V., *Regularization of Optimal Design Problems for Bars and Plates*, Journal of Optimization Theory and Applications, Vol. 37, pp. 499–523, 1982.

7. KOHN, R., *Recent Progress in the Mathematical Modeling of Composite Materials*, Composite Material Response: Constitutive Relations and Damage Mechanisms, Edited by G. Sih *et al.*, Elsevier, Amsterdam, Holland, pp. 155–177, 1988.

8. KOHN, R. V., and STRANG, G., *Optimal Design and Relaxation of Variational Problems, Parts 1–3*, Communications in Pure and Applied Mathematics, Vol. 34, pp. 113–137, 1986; Vol. 34, pp. 139–182, 1986; Vol. 34, pp. 353–377, 1986.

9. FRANCFORT, G. A., and MURAT, F., *Homogenization and Optimal Bounds in Linear Elasticity*, Archives of Rational Mechanics and Analysis, Vol. 94, pp. 307–334, 1986.

10. GIBIANSKII, L., and CHERKAEV, A., *Design of Composite Plates of Extremal Rigidity*, Preprint, Ioffe Physicotechnical Institute, St. Petersburg, Russia, 1984 (in Russian).

11. KOHN, R. V., and LIPTON, R., *Optimal Bounds for the Effective Energy of a Mixture of Isotropic, Incompressible, Elastic Materials*, Archives of Rational Mechanics and Analysis, Vol. 102, pp. 331–350, 1988.

12. MILTON, G. W., and KOHN, R. V., *Variational Bounds on the Effective Moduli of Anisotropic Composites*, Journal of the Mechanics and Physics of Solids, Vol. 36, pp. 597–629, 1988.

13. AVELLANEDA, M., *Optimal Bounds and Microgeometries for Elastic Two-Phase Composites*, SIAM Journal on Applied Mathematics, Vol. 47, pp. 1216–1228, 1987.

14. ALLARE, G., and KOHN, R. V., *Optimal Bounds on the Effective Behavior of a Mixture of Two Well-Ordered Elastic Materials*, Quarterly of Applied Mathematics, Vol. 51, pp. 643–683, 1993.

15. EKELAND, I., and TEMAM, R., *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, Holland, 1976.

AUTHOR'S
B. S. PAGE 30