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Optimal Bounds for the Effective Energy of a Mixture of Isotropic, Incompressible, Elastic Materials

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1. Introduction

An isotropic, incompressible elastic material is characterized by just one parameter, its shear modulus. When two such materials are mixed on a fine scale, an *elastic composite* results, always incompressible but generally anisotropic. This *effective* or *homogenized* material is described by its stress-strain tensor C_{ijkl}, a symmetric linear map on the space of trace-free symmetric matrices, or equivalently by its *effective energy quadratic form* $(C\xi, \xi)$. Our goal is to bound this quadratic form above and below in terms of the shear moduli $\mu_1 < \mu_2$ of the two component materials, their relative proportions θ_1 and $\theta_2 = 1 - \theta_1$ in the mixture, and the strain tensor ξ. Our conclusion has the form

$$f_- \leqq (C\xi, \xi) \leqq f_+, \tag{1.1}$$

where $f_\pm = f_\pm(\mu_1, \mu_2; \theta_1, \theta_2; \xi)$ are explicitly computable functions. These bounds are in fact *optimal*, in the sense that no better estimation of $(C\xi, \xi)$ is possible. In other words for any choices of the parameters $0 < \mu_1 < \mu_2 < \infty$, $0 < \theta_1 < 1$ and any trace-free, symmetric matrix ξ, there exists a pair of stress-strain laws C_+, C_-, each corresponding to a material made by mixing the given components in the specified proportions, such that

$$(C_+\xi, \xi) = f_+ \quad \text{and} \quad (C_-\xi, \xi) = f_-. \tag{1.2}$$

There is a long history to the study of the effective moduli of composites, see *e.g.* [5, 15, 53]. Among the earliest results for elasticity were PAUL's bounds [42], which place C between the harmonic and arithmetic means of $2\mu_1$ and $2\mu_2$:

$$2(\mu_1^{-1}\theta_1 + \mu_2^{-1}\theta_2)^{-1} |\xi|^2 \leqq (C\xi, \xi) \leqq 2(\mu_1\theta_1 + \mu_2\theta_2) |\xi|^2. \tag{1.3}$$

In fact, our upper bound is precisely PAUL's:

$$f_+ = 2(\mu_1\theta_1 + \mu_2\theta_2) |\xi|^2. \tag{1.4}$$

Thus the bound itself is not new, though its optimality (1.2) seems not to have been noted before. Our lower bound, on the other hand, is new. Proposition 2.3 gives the comparision with (1.3):

$$f_- \geqq 2(\mu_1^{-1}\theta_1 + \mu_2^{-1}\theta_2)^{-1} |\xi|^2, \text{ with equality exactly when } \xi \text{ has rank two}. \tag{1.5}$$

Thus PAUL's lower bound is optimal in dimension two, but not in dimensions three or more.

If the composite in question is *isotropic*, then it is characterized by a single parameter, the effective shear modulus: $C\xi = \mu_{\text{eff}}\,\xi$. HASHIN & SHTRIKMAN gave bounds for the effective shear modulus of an isotropic composite in the early 1960's [13, 14, 16], and their bounds are now known to be optimal [9, 35, 40]. Unlike HASHIN & SHTRIKMAN, we place no restriction of isotropy on the tensor C in (1.1).

Our work should be viewed in the context of other recent progress in bounding the effective moduli of anisotropic, two-phase composites. The resurgence of interest in this topic has been motivated in large part by applications to structural optimization [23, 30, 31, 38, 48]. Having bounds alone is not enough for these purposes; it is crucially important that they be optimal. The ultimate goal is to characterize what is called the G-closure of a given set of materials—in other words, to specify precisely which effective tensors correspond to composites that can be made by mixing the available materials in prescribed proportions. This has been achieved for a number of scalar problems [8, 10, 11, 26–29, 49, 50], and, by use of essentially the methods of the present paper, for incompressible elasticity in two spatial dimensions [25]. Bounds on the effective energy alone, such as (1.1), do not determine the G-closure. Nevertheless they are useful for certain structural optimization problems involving the minimization or maximization of the compliance under a specified load [23]. Before this work, optimal upper and lower bounds analogous to (1.1) had been proved in the context of plate theory (which is essentially isomorphic to two-dimensional linear elasticity) [11]. Afterward, by building upon the ideas developed here and in [24], M. AVELLANEDA has given optimal upper and lower bounds not only for the effective energy $(C\xi, \xi)$ but also for any finite sum of energies $\Sigma\,(C\xi_i, \xi_i)$ [1].

Besides their interest for structural optimization, bounds such as (1.1) are also relevant to the averaging of equations for two-phase flow [21]. Indeed, if surface tension is ignored then the stationary Stokes equations describing a mixture of two viscous fluids and equations of incompressible elastostatics are identical. This suggests that the *effective viscosity* of such a mixture should satisfy (1.1). We must acknowledge, however, that it is unclear under what circumstances surface tension can really be ignored, since the surface area of the interface in a fine-scale mixture is generally very large [19].

Now a word about methods. Our proof of the optimal lower bound in (1.1) is based on the Hashin-Shtrikman variational principle. The key to its use lies in estimating a certain nonlocal term. From general considerations it is enough to consider the spatially periodic case, and the nonlocal term is then easily represented as a Fourier series. The necessary bound follows from a bit of linear algebra in Fourier space. This is essentially the method of [22]. Several other techniques have recently been developed for bounding effective moduli. One, based on compensated compactness, was applied to linear elasticity in [9], and to plate theory in [11]. We do not know whether or not our lower bound could be proved this way; an attempt to do so might reasonably use the lower semicontinuous quadratic functionals studied in [33]. Another approach, based on analytical continuation, has been applied to linear elasticity in [18]. That method is known to be linked to the Hashin-Shtrikman principle, at least for scalar equations [34], so we suppose that it could be used to give an alternate proof of our bound. However we have not attempted to carry this through.

Our proof that the bounds are optimal uses sequentially laminated composites. This fundamental construction, apparently discovered by BRUGGEMAN in the 1930's [4], has been used to show the optimality of many different bounds; see, for example, [8–11, 25–29, 38, 44, 50] and the review [36]. Our calculations are greatly simplified by—indeed, could not have been done without—an iterative approach developed by MURAT & TARTAR for scalar problems [50], and extended to the case of elasticity by FRANCFORT & MURAT [9].

2. A New Lower Bound on the Effective Energy Quadratic Form

We are interested in bounding the effective energy of a composite material made by mixing two incompressible, isotropic, linearly elastic solids in prescribed proportions. There are at least three different mathematical models that can be used to describe such a composite. Easiest to work with but most restrictive is the class of spatially periodic composites, *e.g.* [3, 43]. A more general notion is that of a random, statistically homogeneous composite, *e.g.* [12, 41]. The most general viewpoint is based on the theory of G-convergence (also called H-convergence) of elliptic operators, *e.g.* [9, 45, 46, 48, 54]. For proving bounds such as (1.1), however, it is sufficient to consider *spatially periodic composites with a cube as the period cell*. Indeed, the effective moduli (and volume fractions) of any composite can be approximated arbitrarily well by those of a spatially periodic one, so bounds proved in the periodic context extend to the other models by continuity. (This is proved in [12] for the random theory, and in [20] using G-convergence.)

We shall work in $\mathbb{R}^n$ for any $n \geqq 2$. Though the case of primary physical interest is $n = 3$, considering all $n \geqq 2$ at once serves to clarify the structure of the arguments. Since the upper bound (1.4) is well known, our attention will be focused entirely on the lower bound.

We are considering periodic mixtures of two incompressible, isotropic, linearly elastic materials, with shear moduli $\mu_1 < \mu_2$ $(0 < \mu_1, \mu_2 < \infty)$. This means that we are studying the elliptic system

$$e_{ij}(x) = \tfrac{1}{2}\left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\right), \quad \operatorname{tr} e = 0, \tag{2.1 a}$$

$$\sigma_{ij}(x) = 2\mu_\varepsilon(x)\, e_{ij}(x) + p(x)\, \delta_{ij}, \tag{2.1 b}$$

$$\sum_{j=1}^{n} \partial\sigma_{ij}/\partial x_j = f_i \tag{2.1 c}$$

with a rapidly varying coefficient $\mu_\varepsilon(x)$ of the form

$$\mu_\varepsilon(x) = \mu_1 \chi_1(x/\varepsilon) + \mu_2 \chi_2(x/\varepsilon). \tag{2.2}$$

Here $\chi_j(x/\varepsilon)$ is the characteristic function of the set occupied by material j (it equals one on that set and zero elsewhere). Since there are only two components, $\chi_2 = 1 - \chi_1$; and since the structure is presumed periodic, $\chi_j(y)$ is a periodic function defined on all $\mathbb{R}^n$ with period cell $Q = (0, 1]^n$. The volume fraction of material j is evidently

$$\theta_j = \int_Q \chi_j \, dy, \quad j = 1, 2. \tag{2.3}$$

The system (2.1 a–c) is to be solved in some domain $\Omega \subset \mathbb{R}^n$, with a specified boundary condition (*e.g.* $u = 0$ at $\partial\Omega$). The body load f is part of the data, but the pressure p is not—it can be viewed as a Lagrange multiplier for the constraint of incompressibility. The fundamental convergence theorem of homogenization says that for any (reasonable) Ω and f, the solution of (2.1 a–c) tends as $\varepsilon \to 0$ to that of the constant coefficient system obtained when (2.1 b) is replaced by

$$\sigma_{ij}(x) = \sum_{k,l=1}^{n} C_{ijkl}\, e_{kl}(x) + p(x)\, \delta_{ij}. \tag{2.1 b)*}$$

The tensor C_{ijkl} is the effective *Hooke's law* of the mixture. It is independent of Ω and f, and it defines a symmetric, positive definite linear operator on the space of $n \times n$, trace-free, symmetric matrices, equipped with the Hilbert-Schmidt inner product. It can be characterized through the solutions of various "cell problems", or, for our purposes better, variationally by the formula

$$(C\xi, \xi) = \inf_{\nabla \cdot \varphi = 0} \int_Q 2c(y) \, |\xi + e(\varphi)|^2 \, dy, \tag{2.4}$$

where φ ranges over *periodic* divergence-free vector fields on $\mathbb{R}^n$, $e(\varphi) = \frac{1}{2}(\nabla\varphi + \nabla\varphi^T)$, and $c(y)$ is the periodically varying shear modulus

$$c(y) = \mu_1 \chi_1(y) + \mu_2 \chi_2(y). \tag{2.5}$$

For justification of these assertions and further discussion we refer to [3, 7, 39, 43, 47]. (Some of these references consider unconstrained linear elasticity, but the incompressible case can be treated similarly).

Our task, then, is to establish a lower bound for the quadratic form (2.4) which depends on μ_1, μ_2, θ_1, θ_2, and ξ, but not on the particular form of χ_1 and χ_2. It is easy to see that the bound can depend on ξ only through its eigenvalues $\xi_1 \leqq \ldots \leqq \xi_n$, the "principal strains," since the class of all possible composites is invariant under rotations.

2A. The Hashin-Shtrikman variational principle

This approach to bounding the effective moduli of a composite was first introduced by HASHIN & SHTRIKMAN in [16]. It has since been clarified and applied by many authors, including HILL [17] and WILLIS [52]. In the present context the principle says this: *for any periodic, square-integrable field of symmetric, trace-free tensors* σ *which vanishes on material* 1 (*i.e.* $\sigma\chi_1 = 0$),

$$(C\xi, \xi) \geqq \int_Q [2(\sigma, \xi) - \tfrac{1}{2}(\mu_2 - \mu_1)^{-1} |\sigma|^2 + 2\mu_1 |\xi|^2]\, dy + \int_Q (\sigma, e(\psi^*))]\, dy, \tag{2.6}$$

where ψ^* *minimizes*

$$\int_Q [2(\sigma, e(\psi)) + 2\mu_1 |e(\psi)|^2]\, dy \tag{2.7}$$

among all periodic, divergence-free vector fields ψ on Q. Notice that ψ^* is characterized by the Euler equation

$$\begin{aligned} \mu_1 \Delta\psi^* + \nabla p^* &= -\operatorname{div} \sigma \\ \nabla \cdot \psi^* &= 0. \end{aligned} \tag{2.8}$$

The proof of (2.6) is easy. Let σ be as above, and let φ be the minimizer of (2.4). Then

$$(C\xi, \xi) = \int_Q 2c(y)\, |\xi + e(\varphi)|^2\, dy, \tag{2.9}$$

and

$$|(2c(y) - 2\mu_1)^{1/2} (\xi + e(\varphi)) - (2c(y) - 2\mu_1)^{-1/2} \sigma|^2 \geqq 0 \tag{2.10}$$

a.e. in Q; here we interpret the second term of (2.10) as being zero on material 1, where $c(y) = \mu_1$ but $\sigma = 0$. Integrating (2.10) and combining the result with (2.9) gives, after an integration by parts,

$$\begin{aligned} (C\xi, \xi) - 2\mu_1 |\xi|^2 + \int_Q [\tfrac{1}{2}(\mu_2 - \mu_1)^{-1} |\sigma|^2 - 2(\xi, \sigma)]\, dy \\ \geqq \int_Q [2(\sigma, e(\varphi)) + 2\mu_1 |e(\varphi)|^2]\, dy. \end{aligned} \tag{2.11}$$

The expression on the right of (2.11) is the same as (2.7), so it is minimized by $\varphi = \psi^*$. An integration by parts using (2.8) gives

$$\int_Q [2(\sigma, e(\psi^*)) + 2\mu_1 |e(\psi^*)|^2]\, dy = \int_Q (\sigma, e(\psi^*))\, dy. \tag{2.12}$$

Therefore the left side of (2.11) is bounded by the right side of (2.12). After rearrangement, this gives (2.6).

We now make the customary choice of σ:

$$\sigma = \lambda\chi_2 \tag{2.13}$$

for some *constant* trace-free, symmetric tensor λ. The desired lower bound will be obtained by first estimating the nonlocal term $\int_Q (\sigma, e(\psi^*))\, dy$ when σ has the form (2.13), then optimizing over choices of λ.

2B. Estimation of the nonlocal term

Since ψ^* is periodic in each variable with period 1, it has a Fourier series representation

$$\psi^*(y) = \sum_{k \in Z^n} \hat{\psi}^*(k)\, e^{2\pi i k \cdot y}.$$

It is determined only up to an additive constant, so we may take

$$\hat{\psi}^*(0) = 0. \tag{2.14}$$

Representing the pressure p^* in (2.8) by

$$p^*(y) = \sum_{k \in Z^n} \hat{p}^*(k)\, e^{2\pi i k \cdot y}$$

and the characteristic function χ_2 by

$$\chi_2(y) = \sum_{k \in Z^n} \hat{\chi}_2(k)\, e^{2\pi i k \cdot y}, \tag{2.15}$$

one easily sees that (2.8) holds if and only if for each $k \neq 0$,

$$\hat{p}^*(k) = -\frac{(\lambda k, k)}{|k|^2} \hat{\chi}_2(k),$$

$$\hat{\psi}^*(k) = \frac{i}{2\pi |k|^2 \mu_1} (\hat{\chi}_2(k)\, \lambda k + \hat{p}^*(k)\, k).$$

It follows that $e^* = e(\psi^*)$ has Fourier coefficients

$$\hat{e}^*(k) = \frac{-1}{\mu_1} \hat{\chi}_2(k) \left(\frac{(\lambda k) \cdot k}{|k|^2} - \frac{(\lambda k, k)}{|k|^2} \frac{k \cdot k}{|k|^2} \right), \tag{2.16}$$

where $a \cdot b = \frac{1}{2}(a \otimes b + b \otimes a)$ represents the symmetric tensor product of a pair of vectors in $\mathbb{R}^n$. Let us introduce the notation

$$P_k(\lambda) = \frac{(\lambda k) \cdot k}{|k|^2} - \frac{(\lambda k, k)}{|k|^2} \frac{k \cdot k}{|k|^2} \tag{2.17}$$

for the expression on the right in (2.16); it is simply the projection of $(\lambda k) \cdot k/|k|^2$ orthogonal to $k \cdot k$ with respect to the Hilbert-Schmidt inner product on symmetric tensors. By the Plancherel formula and (2.15)–(2.17),

$$\int_Q (\sigma, e(\psi^*))\, dy = \frac{-1}{\mu_1} \sum_{k \neq 0} |\hat{\chi}_2(k)|^2 (P_k(\lambda), \lambda).$$

A second application of Plancherel's theorem gives

$$\sum_{k \neq 0} |\hat{\chi}_2(k)|^2 = \int_Q (\chi_2 - \theta_2)^2 = \theta_1 \theta_2,$$

in view of the definitions (2.3) of θ_1 and θ_2. Therefore

$$\int_Q (\sigma, e(\psi^*))\, dy \geqq \frac{-1}{\mu_1} \theta_1 \theta_2 \cdot \sup_{k \in Z^n} (P_k(\lambda), \lambda). \tag{2.18}$$

Since $P_k(\lambda)$ depends on k only through $k/|k|$, and the unit vectors $v = k/|k|$ are dense in the sphere S^{n-1}, (2.18) can be rewritten as

$$\int_Q (\sigma, e(\Psi^*))\, dy \geqq \frac{-1}{\mu_1} \theta_1 \theta_2 \cdot \max_{v \in S^{n-1}} (P_v(\lambda), \lambda). \tag{2.19}$$

The maximum on the right of (2.19) can be computed explicitly by using the following

Lemma 2.1. *For any $n \times n$ symmetric matrix λ with eigenvalues $\lambda_1 \leqq \ldots \leqq \lambda_n$,*

$$\max_{v \in S^{n-1}} |\lambda v|^2 - (\lambda v, v)^2 = \tfrac{1}{4} (\lambda_n - \lambda_1)^2, \tag{2.20}$$

achieved precisely when $v = \frac{1}{\sqrt{2}} (e_1 + e_n)$, where e_1 is an eigenvector of λ with eigenvalue λ_1 and e_n is an eigenvector of λ with eigenvalue λ_n.

Proof. By the method of Lagrange multipliers, at any critical point $v \in S^{n-1}$ of the function $|\lambda v|^2 - (\lambda v, v)^2$ there exists $c \in \mathbb{R}$ such that

$$\lambda^2 v - 2(\lambda v, v)\, \lambda v = cv.$$

If we choose the eigenvectors of λ as a basis for $\mathbb{R}^n$, then the components $v_j = (v, e_j)$ of v satisfy

$$\lambda_j^2 v_j - 2(\lambda v, v)\, \lambda_j v_j = c v_j, \qquad 1 \leqq j \leqq n.$$

If $v_j \neq 0$ for only one (distinct) λ_j then v is an eigenvector of λ, and one easily varifies that $|\lambda v|^2 - (\lambda v, v)^2 = 0$; these are the minima. It cannot occur that $v_j \neq 0$ for three (distinct) λ_j, since then the quadratic polynomial $x^2 - 2(\lambda v, v)\, x - c = 0$ would have three distinct roots. Hence at the remaining critical points there are two distinct eigenvalues, say λ_i and λ_j, such that

$$\lambda_i^2 - 2(\lambda v, v)\, \lambda_i = c = \lambda_j^2 - 2(\lambda v, v)\, \lambda_j. \tag{2.21}$$

Adjusting the basis if necessary in the case of multiple eigenvalues, we may suppose that the only nonzero components of v are v_i and v_j. From (2.21),

$$\lambda_i^2 - \lambda_j^2 = 2(\lambda v, v)\,(\lambda_i - \lambda_j);$$

since $\lambda_i \neq \lambda_j$ it follows that

$$2(\lambda v, v) = \lambda_i + \lambda_j.$$

With $|v|^2 = 1$ this implies that $v_i^2 = v_j^2 = 1/2$. Thus $v = (e_i + e_j)/\sqrt{2}$ for some pair of eigenvectors e_i, e_j (the case $v_i = -1/\sqrt{2}$ is included, by replacing e_i with $-e_i$). For such v one computes that $|\lambda v|^2 - (\lambda v, v)^2 = \frac{1}{4} (\lambda_i - \lambda_j)^2$. The maximum is clearly achieved precisely when λ_i and λ_j are the smallest and largest eigenvalues of λ.

We return to the task for evaluating the right side of (2.19). Since

$$(P_v(\lambda), \lambda) = |\lambda v|^2 - (\lambda v, v)^2 \tag{2.22}$$

for any $v \in S^{n-1}$, (2.19), and (2.20) give

$$\int_Q (\sigma, e(\psi^*))\, dy \geqq \frac{-1}{4\mu_1} \theta_1\theta_2 (\lambda_n - \lambda_1)^2. \tag{2.23}$$

Assembling (2.6), (2.19), and (2.23), we have established that

$$\begin{aligned}(C\xi, \xi) &\geqq 2\theta_2(\lambda, \xi) - \tfrac{1}{2}\theta_2(\mu_2 - \mu_1)^{-1} |\lambda|^2 + 2\mu_1 |\xi|^2 - \frac{\theta_1\theta_2}{\mu_1} \max_{v \in S^{n-1}} (P_v(\lambda), \lambda) \\ &= 2\theta_2(\lambda, \xi) - \tfrac{1}{2}\theta_2(\mu_2 - \mu_1)^{-1} |\lambda|^2 + 2\mu_1 |\xi|^2 - \frac{\theta_1\theta_2}{4\mu_1}(\lambda_n - \lambda_1)^2 \end{aligned} \tag{2.24}$$

for any pair of symmetric, trace-free matrices ξ and λ, where λ_1 and λ_n are the smallest and largest eigenvalues of λ.

2C. Optimization over λ

Our new lower bound

$$(C\xi, \xi) \geqq f_- \tag{2.25}$$

is obtained by maximizing the right side of (2.24) over λ:

$$f_- = \max_\lambda 2\theta_2(\lambda, \xi) - \tfrac{1}{2}\theta_2(\mu_2 - \mu_1)^{-1} |\lambda|^2 + 2\mu_1 |\xi|^2 - \frac{\theta_1\theta_2}{4\mu_1}(\lambda_n - \lambda_1)^2. \tag{2.26}$$

It suffices in (2.26) to consider choices of λ which are simultaneously diagonal with ξ, since the maximum of (λ, ξ) over all λ with specified spectrum $\lambda_1 \leqq \ldots \leqq \lambda_n$ is

$$\max_{R \text{ orthogonal}} (R^T\lambda R, \xi) = \sum_{j=1}^n \lambda_j \xi_j, \tag{2.27}$$

see *e.g.* [37]. (Our convention here, as always, is that $\xi_1 \leqq \ldots \leqq \xi_n$ are the eigenvalues of ξ.) Thus we have proved:

Theorem 2.2. *Define the set $K \subset \mathbb{R}^n$ by*

$$K = \left\{(\lambda_1, \ldots, \lambda_n) : \lambda_1 \leqq \ldots \leqq \lambda_n, \ \sum_{i=1}^n \lambda_i = 0\right\}, \tag{2.28}$$

and let

$$M = \max_{(\lambda_1, \ldots, \lambda_n) \in K} \left\{2\Sigma\lambda_j\xi_j - \tfrac{1}{2}(\mu_2 - \mu_1)^{-1} \Sigma\lambda_j^2 - \frac{\theta_1}{4\mu_1}(\lambda_n - \lambda_1)^2\right\}. \tag{2.29}$$

Then (2.25) *holds with*

$$f_- = \theta_2 M + 2\mu_1 |\xi|^2. \tag{2.30}$$

To make f_- totally explicit, one must solve the constrained, quadratic maximization problem (2.29). In principle this is possible for any n; in practice the answer becomes increasingly unwieldy with n. However, we have these two results:

Proposition 2.3. *If* ξ *has rank two, and in particular if* $n = 2$, *then*

$$f_- = 2(\theta_1\mu_1^{-1} + \theta_2\mu_2^{-1})^{-1} |\xi|^2. \tag{2.31}$$

If, however, $\operatorname{rank}(\xi) \geqq 3$ *then*

$$f_- > 2(\theta_1\mu_1^{-1} + \theta_2\mu_2^{-1})^{-1} |\xi|^2. \tag{2.32}$$

Proposition 2.4. *For* $n = 3$, *define*

$$A = \tfrac{3}{2}\,\xi_2\theta_1\left(\frac{\mu_2 - \mu_1}{\mu_1}\right) - (\xi_1 - \xi_2) \tag{2.33 a}$$

$$B = -\tfrac{3}{2}\,\xi_2\theta_1\left(\frac{\mu_2 - \mu_1}{\mu_1}\right) + (\xi_3 - \xi_2). \tag{2.33 b}$$

Then

$$f_- = \begin{cases} 2(\theta_1\mu_1^{-1} + \theta_2\mu_2^{-1})^{-1} |\xi|^2 + \dfrac{3\theta_1\theta_2(\mu_2 - \mu_1)^2\,\xi_2^2}{[\mu_2\theta_1 + \mu_1\theta_2]} & \text{if } A \geqq 0, B \geqq 0 \\ 2\mu_1 |\xi|^2 + \dfrac{12\theta_2(\mu_2 - \mu_1)\,\mu_1\xi_1^2}{4\mu_1 + 3\theta_1(\mu_2 - \mu_1)} & \text{if } A \geqq 0, B < 0 \\ 2\mu_1 |\xi|^2 + \dfrac{12\theta_2(\mu_2 - \mu_1)\,\mu_1\xi_3^2}{4\mu_1 + 3\theta_1(\mu_2 - \mu_1)} & \text{if } A < 0, B \geqq 0. \end{cases} \tag{2.34}$$

(*The case* $A < 0$, $B < 0$ *never occurs.*)

Proof of Proposition 2.3. If ξ has rank two, *i.e.* $\xi_1 = -\xi_n$ and $\xi_2 = \ldots = \xi_{n-1} = 0$ then the best choice for (2.29) is easily seen to be

$$\lambda_j = t\xi_j \quad \text{with } t = \frac{2\mu_1(\mu_2 - \mu_1)}{\mu_1\theta_2 + \mu_2\theta_1}. \tag{2.35}$$

It gives (2.31). If, on the other hand, $\operatorname{rank}\xi \geqq 3$, then (2.35) is still admissible for (2.29); therefore

$$M \geqq 2t\,|\xi|^2 - t^2\left(\tfrac{1}{2}(\mu_2 - \mu_1)^{-1} |\xi|^2 + \frac{\theta_1}{4\mu_1}(\xi_1 - \xi_n)^2\right).$$

But

$$(\xi_1 - \xi_n)^2 \leqq 2(\xi_1^2 + \xi_n^2) < 2\,|\xi|^2,$$

so

$$M > [2t - (\tfrac{1}{2}(\mu_2 - \mu_1)^{-1} + \tfrac{1}{2}\mu_1^{-1}\theta_1)\,t^2]\,|\xi|^2 = t\,|\xi|^2.$$

This gives (2.32).

Proof of Proposition 2.4. Consider the quadratic form on the right of (2.29):

$$2\sum_{j=1}^{3} \lambda_j \xi_j + \tfrac{1}{2}(\mu_2 - \mu_1)^{-1} \sum_{j=1}^{3} \lambda_j^2 - \frac{\theta_1}{4\mu_1}(\lambda_3 - \lambda_1)^2. \tag{2.36}$$

Its maximum on all $\mathbb{R}^3$ is achieved at

$$\bar{\lambda}_1 = \frac{(\xi_1 + \xi_3)(\mu_2 - \mu_1)\,\theta_1 + 2\xi_1\mu_1}{(\theta_1\mu_2 + \theta_2\mu_1)}(\mu_2 - \mu_1),$$

$$\bar{\lambda}_2 = 2(\mu_2 - \mu_1)\,\xi_2,$$

$$\bar{\lambda}_3 = \frac{(\xi_1 + \xi_3)(\mu_2 - \mu_1)\,\theta_1 + 2\xi_3\mu_1}{(\theta_1\mu_2 + \theta_2\mu_1)}(\mu_2 - \mu_1).$$

One verifies that $\bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_3 = 0$, and that

$$\bar{\lambda}_2 \leqq \bar{\lambda}_3 \Leftrightarrow B \geqq 0, \quad \bar{\lambda}_1 \leqq \bar{\lambda}_2 \Leftrightarrow A \geqq 0,$$

where A and B are defined by (2.33a, b). If $\bar{\lambda}_1 \leqq \bar{\lambda}_2 \leqq \bar{\lambda}_3$ then they are admissible for (2.29), and M is the value of (2.36) at this point. This is the case $A \geqq 0$, $B \geqq 0$, and substitution gives the first formula in (2.34). If $\bar{\lambda}_2 > \bar{\lambda}_3$ and $\bar{\lambda}_1 \leqq \bar{\lambda}_2$, then the extremum of (2.36) on K is easily seen to be achieved at a point where $\lambda_2 = \lambda_3 = -\frac{1}{2}\lambda_1$. This is the case $A \geqq 0$, $B < 0$; a one-dimensional maximization yields the value of M and, after some calculation, the second formula in (2.34). The case $\bar{\lambda}_2 \leqq \bar{\lambda}_3$, $\bar{\lambda}_1 > \bar{\lambda}_2$, in other words $A < 0$, $B \geqq 0$, is symmetrical, the roles of ξ_1 and ξ_3 being interchanged. Finally, it is impossible for A and B both to be negative, so the case $\bar{\lambda}_3 < \bar{\lambda}_2 < \bar{\lambda}_1$ does not occur.

3. Attainability

The bounds (1.1), with f_+ and f_- given by (1.4) and (2.30), are optimal in the sense that no better bounds for $(C\xi, \xi)$ are possible when ξ, μ_1, μ_2, θ_1, and θ_2 are fixed. To show this, we shall display a pair of microstructures which use the specified materials in the prescribed proportions, and whose effective moduli C_+ and C_- satisfy $(C_+\xi, \xi) = f_+$ and $(C_-\xi, \xi) = f_-$.

Our method is that of *sequential lamination*. This is an iterative construction, producing microstructures having several different length scales. A *laminar composite of rank* 1 is obtained by layering the two initial materials μ_1 and μ_2, specifying the proportion of each and the layer direction, and using a small parameter ε_1 as the layer thickness. As $\varepsilon_1 \to 0$, the elastic behavior is described by an effective Hooke's law C_1. A *laminar composite of rank* 2 is obtained by layering two laminar composites of rank one, again specifying the proportion of each and the layer direction, and using another small parameter ε_2 for the layer thickness. As $\varepsilon_1, \varepsilon_2 \to 0$ with $\varepsilon_1 \ll \varepsilon_2$, the elastic behavior is described by an effective Hooke's law C_2. This process can clearly be continued any finite number of times (and even countably many times, using a suitable limiting procedure). Such composites have been discussed by many authors, *e.g.* [1, 2, 8–11, 25–32, 36, 44, 50], and they have been used to prove the attainability of many different bounds. Our application is very close to those in [1, 11, 25, 28, 29, 49]. The "strongest" composite C_+ is achieved, it turns out, by a rank 1 laminate; the "weakest" one C_- requires rank $n-1$ in $\mathbb{R}^n$.

It may seem like cheating that we use *periodic* composites to establish the bounds, but *sequentially laminated* ones to attain them. This is, however, perfectly legitimate: the bounds, once established for the periodic case, extend by continuity to all composites—understood in the sense of G-convergence—including sequentially laminated ones. From another perspective: though sequentially laminated composites are not themselves spatially periodic, their effective tensors can be approximated arbitrarily well by ones associated with periodic microstructures. Actually, it is quite natural to use the most restrictive possible setting for establishing bounds, and the most general one for showing that they are achieved.

3A. *Effective moduli of a sequentially laminated composite*

As discussed above, the general sequentially laminated composite of rank r is obtained by mixing two arbitrarily chosen laminates of rank $r-1$. We shall consider here only a special case, in which *one of these two materials is the isotropic one with shear modulus μ_1 at each successive stage*. The resulting microstructure (for $r > 1$) has many small inclusions of material 2 embedded in a matrix of material 1. An elegant, iterative formula for representing the effective moduli of such a composite was given in [50] for scalar equations, and generalized in [9] to (compressible) elasticity. The analogous formula for the incompressible case can be obtained from that in [9] by passage to the limit $\varkappa_i \to \infty$. For the sake of completeness, however, we shall repeat the derivation.

The basic building block is a formula for the effective tensor C corresponding to a layered mixture of material 1 with a general incompressible material B. Recall that B and C are really symmetric linear maps on the space of trace-free, symmetric tensors; so is $2\mu_1 = 2\mu_1 I$, where I represents the identity on this space. We shall assume (for convenience only) that $B > 2\mu_1 I$; this implies that $C > 2\mu_1 I$ by the analogue of PAUL's lower bound (1.3), so that $B - 2\mu_1 I$ and $C - 2\mu_1 I$ are both invertible.

Lemma 3.1. *With B and μ_1 as above, consider a layered composite in which the isotropic material with shear modulus μ_1 occurs with volume fraction ϱ_1, and B with volume fraction $\varrho_B = 1 - \varrho_1$. Let $v \in \mathbb{R}^n$ be the unit vector orthogonal to the layers. Then the effective Hooke's law C is determined by the formula*

$$\varrho_B (C - 2\mu_1 I)^{-1} \lambda = (B - 2\mu_1 I)^{-1} \lambda + \frac{\varrho_1}{\mu_1} P_v(\lambda) \qquad (3.1)$$

for any trace-free, symmetric tensor λ, with the notation

$$P_v(\lambda) = (\lambda v) \cdot v - (\lambda v, v)\, v \cdot v. \tag{3.2}$$

Proof. The layered structure under consideration is in fact periodic (in a suitable coordinate system), so it can be treated using the theory sketched at the beginning of Section 2. But the solutions of the "cell-problems"—or equivalently, of the analogoues of (2.4)—have piecewise constant stress, strain, and pressure. Therefore, through arguing as in [9, 50, 51], the calculation of $C\xi$ given ξ is easily reduced to this algebraic problem: find a pair of trace-free, symmetric matrices ξ_1 and ξ_B (representing the strain in the layers occupied by materials 1 and B respectively) and a pair of real numbers p_1 and p_B (the pressures in the respective layers) such that

$$\varrho_1 \xi_1 + \varrho_B \xi_B = \xi, \tag{3.3a}$$

$$\xi_B - \xi_1 = v \cdot w \quad \text{for some } w \in \mathbb{R}^n, \tag{3.3b}$$

$$(2\mu_1 \xi_1 - B\xi_B)\, v + (p_1 - p_B)\, v = 0. \tag{3.3c}$$

The first relation says that ξ is the mean strain; the second is the consistency condition for the existence of a deformation with the specified piecewise constant strain; and the third represents the continuity of the normal stress at the layer interface. In terms of these quantities, $C\xi$ is determined by

$$C\xi = 2\varrho_1 \mu_1 \xi_1 + \varrho_B B \xi_B, \tag{3.3d}$$

which identifies as it the average deviatoric stress. The solution of (3.3a–d) is easiest to represent in terms of the variable

$$\lambda = (C - 2\mu_1 I)\, \xi.$$

A straightforward if lengthy calculation shows that

$$\xi_B = \varrho_B^{-1}(B - 2\mu_1 I)^{-1} \lambda;$$

$$\xi_1 = \xi_B - v \cdot w \quad \text{with } w = \frac{1}{\varrho_B \mu_1} [(\lambda v, v)\, v - \lambda v];$$

$$p_1 - p_B = \varrho_B^{-1}(\lambda v, v).$$

Substitution yields

$$\begin{aligned}(C - 2\mu_1 I)^{-1} \lambda &= \xi \\ &= \xi_B - \varrho_1\, v \cdot w \\ &= \varrho_B^{-1}(B - 2\mu_1 I)^{-1} \lambda + \frac{\varrho_1}{\mu_1 \varrho_B} \{(\lambda v) \cdot v - (\lambda v, v)\, v \cdot v\},\end{aligned}$$

which is precisely the desired formula (3.1).

Remark 3.2. We note that the operator P_v defined by (3.2) is the *same* as the operator P_k defined by (2.17), with the identification $v = k/|k|$. The fact that same operator occurs in both the bound—see *e.g.* (2.24)—and the lamination formula (3.1) is crucial to the success of sequential lamination in achieving the bounds.

Now consider a sequence $C_0, C_1, C_2, \ldots$ of effective tensors such that

$$C_0 = 2\mu_1 I$$

and, for $r \geqq 1$,

C_r is obtained by layering material 1 with C_{r-1} in volume fractions α_r and $1-\alpha_r$ respectively, using the unit vector v_r as the layer normal. (3.4)

Evidently, C_r represents the effective behavior of a certain sequentially laminated composite of rank r. The volume fraction of material 1 in C_r is

$$\varkappa_r = 1 - \prod_{i=1}^{r} (1-\alpha_i), \quad r \geqq 1; \quad \varkappa_0 = 0. \tag{3.5}$$

A formula for C_r is easily obtained by iterating (3.1):

$$(1-\varkappa_r)(C_r - 2\mu_1 I)^{-1}\lambda = \tfrac{1}{2}(\mu_2-\mu_1)^{-1}\lambda + \frac{1}{\mu_1}\sum_{i=0}^{r-1}(\varkappa_{i+1}-\varkappa_i)\,P_{v_i}(\lambda) \tag{3.6}$$

for any trace-free, symmetric tensor λ. As a consequence, we have

Lemma 3.3. *Fix an integer* $N \geqq 1$; *unit vectors* $\{v_i\}_{i=1}^N$ *in* $\mathbb{R}^n$; *real numbers* $\{m_i\}_{i=1}^N$ *with* $0 \leqq m_i \leqq 1$ *and* $\Sigma m_i = 1$; *and a real number* θ_1, $0 < \theta_1 < 1$. *Then there is a sequentially laminated composite made by mixing materials* 1 *and* 2 *as in* (3.4), *with overall volume fractions* θ_1 *and* $\theta_2 = 1-\theta_1$ *of materials* 1 *and* 2 *respectively, whose effective Hooke's law tensor C is characterized by*

$$\theta_2(C - 2\mu_1 I)^{-1}\lambda = \tfrac{1}{2}(\mu_2-\mu_1)^{-1}\lambda + \frac{\theta_1}{\mu_1}\sum_{i=1}^{N} m_i P_{v_i}(\lambda) \tag{3.7}$$

for every trace-free, symmetric tensor λ.

Proof. Clearly (3.7) coincides with (3.6) when

$$r = N, \quad \varkappa_N = \theta_1, \quad \text{and } \varkappa_i - \varkappa_{i-1} = \theta_1 m_i, \quad 1 \leqq i \leqq N.$$

This is easily achieved by choosing

$$\varkappa_0 = 0, \quad \varkappa_j = \theta_1 \sum_{i=1}^{j} m_i, \quad 1 \leqq j \leqq N,$$

which corresponds through (3.5) to

$$\alpha_j = \frac{\varkappa_j - \varkappa_{j-1}}{1-\varkappa_{j-1}}, \quad 1 \leqq j \leqq N.$$

Notice that $0 \leqq \alpha_j \leqq 1$ for each r, since $0 = \varkappa_0 \leqq \varkappa_1 \leqq \ldots \leqq \varkappa_N = \theta_1 < 1$.

3B. Optimality of Paul's upper bound

Let ξ be symmetric and trace-free, and let $\theta_1, \theta_2 = 1 - \theta_1$ be fixed. Consider the rank-one laminate C_+ obtained by choosing $B = 2\mu_2 I$ in (3.1), with $\varrho_1 = \theta_1$, $\varrho_B = \theta_2$, and using any normalized eigenvector of ξ as the layer normal v. Then $P_v(\xi) = 0$, so (3.1) with $\lambda = \xi$ gives

$$\theta_2(C_+ - 2\mu_1 I)^{-1}\,\xi = \tfrac{1}{2}(\mu_1 - \mu_2)^{-1}\,\xi.$$

Algebraic manipulation leads to

$$C_+\xi = 2(\theta_1\mu_1 + \theta_2\mu_2)\,\xi,$$

hence

$$(C_+\xi, \xi) = 2(\theta_1\mu_1 + \mu_2\theta_2)\,|\xi|^2.$$

We have proved:

Proposition 3.4. *In any space dimension, and for any symmetric, trace-free ξ, Paul's upper bound* (1.4) *is achieved by a layered composite of rank one.*

3C. Optimality of the lower bound when ξ has rank 2

According to Proposition 2.3, our lower bound agrees with PAUL'S (1.3) in two space dimensions, and also in higher dimensions when ξ happens to have rank 2. The proof of optimality is much easier in this case than in general, so we present it separately. Given a rank-two, trace-free, symmetric tensor ξ and volume fractions $\theta_1, \theta_2 = 1 - \theta_1$, consider the rank-one laminate C_- obtained by choosing $B = 2\mu_2 I$ in (3.1), with $\varrho_1 = \theta_1$, $\varrho_B = \theta_2$, and using $v = (e_1 + e_2)/\sqrt{2}$ as the layer normal, where e_1 and e_2 are normalized eigenvectors of ξ with nonzero eigenvalues. An elementary calculation shows that

$$P_v\xi = \tfrac{1}{2}\,\xi,$$

so (3.1), with $\lambda = \xi$ gives

$$\theta_2(C_- - 2\mu_1 I)^{-1}\,\xi = \tfrac{1}{2}(\mu_2 - \mu_1)^{-1}\,\xi + \frac{\theta_1}{2\mu_1}\,\xi.$$

Algebraic manipulation leads to

$$C_-\xi = 2(\mu_1^{-1}\theta_1 + \mu_2^{-1}\theta_2)^{-1}\,\xi,$$

hence

$$(C_-\xi, \xi) = 2(\theta_1\mu_1^{-1} + \theta_2\mu_2^{-1})^{-1}\,|\xi|^2.$$

We have proved:

Proposition 3.5. *In two space dimensions, and in higher space dimensions when ξ has rank* 2, *the lower bound* (1.3) *is achieved by a layered composite of rank one.*

3D. Optimality of the new lower bound

Let ξ be symmetric and trace-free, with eigenvalues $\xi_1 \leqq \ldots \leqq \xi_n$, and fix $\theta_1, \theta_2 = 1 - \theta_1$, $0 < \theta_i < 1$. Our goal is to construct a (sequentially laminated) composite whose effective Hooke's law C_- satisfies

$$(C_- \xi, \xi) = f_-(\mu_1, \mu_2; \theta_1, \theta_2; \xi),$$

with f_- given by (2.30) or, equivalently, by (2.26). We shall of course use Lemma 3.3. The proper choice of the parameters m_i and v_i will emerge from the optimality conditions for (2.26).

The first step, then, is derive those optimality conditions. We shall use the subdifferential calculus; since this technique may be unknown to some readers, specific references will be given for the key steps. (A more traditional argument, using the Kuhn-Tucker conditions for (2.29), will be found in [24].)

Lemma 3.6. *If λ^* achieves the maximum of* (2.26) *then λ^* is simultaneously diagonal with ξ and*

$$\xi - \tfrac{1}{2}(\mu_2 - \mu_1)^{-1} \lambda^* = \frac{\theta_1}{\mu_1} \int_{S^{n-1}} P_v(\lambda^*)\, dm(v) \tag{3.8}$$

for some probability measure m on S^{n-1}. Moreover, if $E_{\min}$ and $E_{\max}$ are respectively the eigenspaces of λ^* with minimum and maximum eigenvalues, then the support of m lies in

$$N(\lambda^*) = \left\{ v : v = \frac{e + e'}{\sqrt{2}},\ e \in E_{\min},\ e' \in E_{\max} \right\}. \tag{3.9}$$

Proof. We begin by rewriting (2.26) using the first version of (2.24) instead of the second:

$$f_- = \max_\lambda 2\theta_2(\lambda, \xi) - \tfrac{1}{2}\theta_2(\mu_2 - \mu_1)^{-1} |\lambda|^2 + 2\mu_1 |\xi|^2 - \frac{\theta_1 \theta_2}{\mu_1} g(\lambda) \tag{3.10}$$

where λ ranges over trace-free, symmetric matrices and

$$g(\lambda) = \max_{v \in S^{n-1}} (P_v(\lambda), \lambda). \tag{3.11}$$

For fixed $v \in S^{n-1}$, $(P_v(\lambda), \lambda)$ is a nonnegative, quadratic, and hence convex function of λ. Therefore $g(\lambda)$ is convex, so that (3.10) gives f_- as the maximum value of a strictly concave function of λ. That the unique extremal λ^* is simultaneously diagonal with ξ has already been shown in the course of proving Theorem 2.2.

By the subdifferential calculus ([6, 2.3.1–2.3.3 and Corollary 1, § 2.3]), the condition for λ^* to be extremal is

$$0 \in 2\theta_2 \xi - \theta_2(\mu_2 - \mu_1)^{-1} \lambda^* - \frac{\theta_1 \theta_2}{\mu_1} \partial g(\lambda^*), \tag{3.12}$$

where $\partial g(\lambda^*)$ is the subdifferential of g at λ^*. Moreover, g is given by (3.11) as the maximum of a continuously parametrized family of convex, quadratic functions. For such g, the subdifferential is the convex hull of the gradients $\nabla(P_v(\lambda), \lambda)) = 2P_v(\lambda)$, as v ranges over the subset of S^{n-1} where the maximum (3.11) is achieved ([6, § 2.8, Corollary 1]). By Lemma 2.1, that subset of S^{n-1} is precisely $N(\lambda^*)$, so

$$\partial g(\lambda^*) = \left\{ \int_{S^{n-1}} 2P_v(\lambda^*)\, dm(v) \colon \begin{array}{l} m \text{ is a probability measure} \\ \text{on } S^{n-1} \text{ supported on } N(\lambda^*) \end{array} \right\}. \tag{3.13}$$

The desired (3.8) is an immediate consequence of (3.12) and (3.13).

The space of trace-free, symmetric tensors has dimension $\frac{1}{2}n(n+1) - 1$. By Carathéodory's theorem, any point in the compact, convex set $\partial g(\lambda^*)$ is a convex combination of at most $\frac{1}{2}n(n+1)$ extreme points. Therefore the measures m in (3.13) may be assumed to be supported at just $\frac{1}{2}n(n+1)$ points of $N(\lambda^*)$. However, by using all of the available information we can reduce the number of points further, to just $n-1$:

Lemma 3.7. *In the context of Lemma 3.6,*

$$\xi - \tfrac{1}{2}(\mu_2 - \mu_1)^{-1}\lambda^* = \frac{\theta_1}{\mu_1} \sum_{i=1}^{n-1} m_i P_{v_i}(\lambda^*) \tag{3.14}$$

for some $v_i \in N(\lambda^*)$ *and* m_i, $0 \leqq m_i \leqq 1$, *with* $\Sigma m_i = 1$.

Proof. Let $\lambda_{\min}$ and $\lambda_{\max}$ be the smallest and largest eigenvalues of λ^*; as in Lemma 3.6, $E_{\min}$ and $E_{\max}$ are the associated eigenspaces. If $v = \frac{1}{\sqrt{2}}(e + e')$ with $e \in E_{\min}$, $e' \in E_{\max}$, then calculation gives

$$P_v(\lambda^*) = \frac{\lambda_{\min} - \lambda_{\max}}{4}(e \cdot e - e' \cdot e'). \tag{3.15}$$

Therefore for any probability measure m supported on $N(\lambda^*)$, the matrix of $\int P_v(\lambda^*)\, dm(v)$ relative to the basis of eigenvectors of ξ is block diagonal:

$$\int P_v(\lambda^*)\, dm(v) = \frac{\lambda_{\min} - \lambda_{\max}}{4} \left[\begin{array}{c|c|c} A & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & -B \end{array} \right]. \tag{3.16}$$

The blocks A and B correspond to the basis vectors in $E_{\min}$ and $E_{\max}$ respectively; it is clear from (3.15) that $A \geqq 0$, $B \geqq 0$, and $\operatorname{tr} A = \operatorname{tr} B = 1$.

Now let us use the additional information that λ^* is simultaneously diagonal with ξ and satisfies (3.8). Evidently the matrices A and B in (3.16) are *diagonal*:

$$A = \sum_{i=1}^{c} a_i e_i \cdot e_i, \quad B = \sum_{j=n-d+1}^{n} b_j e_j \cdot e_j,$$

where $\{e_j\}_{j=1}^{c}$ are the eigenvectors of ξ that lie in $E_{\min}$, $\{e_j\}_{j=n-d+1}^{n}$ are those that lie in $E_{\max}$, and

$$a_i \geq 0, \quad b_j \geq 0, \quad \sum_{i=1}^{c} a_i = \sum_{j=n-d+1}^{n} b_j = 1. \tag{3.17}$$

To prove (3.14) we must show that

$$A - B = \sum_{k=1}^{n-1} m_k (e_{i_k} \cdot e_{i_k} - e'_{j_k} \cdot e'_{j_k}) \tag{3.18}$$

with $0 \leqq m_k \leqq 1$, $\Sigma m_k = 1$, $1 \leqq i_k \leqq c$, $n - d + 1 \leqq j_k \leqq n$. A constructive, inductive argument is not difficult, but the easiest method is to apply Carathéodory's theorem again: the set of all tensors

$$\sum_{i=1}^{c} a_i e_i \cdot e_i - \sum_{j=n-d+1}^{n} b_j e_j \cdot e_j$$

with a_i, b_j restricted by (3.17) is a compact, convex subset of a $c + d - 2 \leqq n - 2$ dimensional affine space, so each element is convex combination of at most $c + d - 1 \leqq n - 1$ extreme points. It is easy to see that the extreme points are precisely the tensors $e_i \cdot e_i - e_j \cdot e_j$ with $1 \leqq i \leqq c$, $n - d + 1 \leqq j \leqq n$, so this yields the desired representation (3.18).

We are ready to prove that our lower bound is optimal.

Theorem 3.8. *In n spatial dimensions the lower bound f_-, given by* (3.10), *is achieved by a sequentially laminated composite of rank at most* $n - 1$.

Proof. Consider the sequentially laminated composite C_- obtained by using the parameters $\{m_i, v_i\}$ of (3.14) in (3.7): taking $\lambda = \lambda^*$ in (3.7) gives

$$\theta_2 (C_- - 2\mu_1 I)^{-1} \lambda^* = \tfrac{1}{2} (\mu_2 - \mu_1)^{-1} \lambda^* + \frac{\theta_1}{\mu_1} \sum_{i=1}^{n} m_i P_{v_i}(\lambda^*) = \xi,$$

using (3.14) in the latter step. Algebraic manipulation yields

$$(C_- \xi, \xi) = 2\mu_1 |\xi|^2 + \theta_2 (\lambda^*, \xi). \tag{3.19}$$

We claim that the right hand side of (3.19) equals f_-. Indeed, substitution of λ^* in (3.10) gives

$$f_- = 2\theta_2 (\lambda^*, \xi) - \tfrac{1}{2} \theta_2 (\mu_2 - \mu_1)^{-1} |\lambda^*|^2 + 2\mu_1 |\xi|^2 - \frac{\theta_1 \theta_2}{\mu_1} g(\lambda^*),$$

while taking the inner product of (3.14) with λ^* yields

$$(\xi, \lambda^*) - \tfrac{1}{2} (\mu_2 - \mu_1)^{-1} |\lambda^*|^2 - \frac{\theta_1}{\mu_1} g(\lambda^*) = 0.$$

It follows that

$$f_- = 2\mu_1 |\xi|^2 + \theta_2 (\lambda^*, \xi),$$

as desired.

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