Optimal design of gradient fields with applications to electrostatics *

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1 Introduction.

Consider a two dimensional design domain Ω, containing two isotropic dielectric materials. The dielectric permittivity is specified by $\varepsilon(x)$ and is piece-wise constant taking the values α and β where $\beta > \alpha > 0$. For a prescribed charge density f the associated electric potential φ satisfies the Poisson equation given by

$$-\text{div}\,(\varepsilon(x)\nabla\varphi) = f, \tag{1}$$

and $\varphi = 0$ on the boundary of Ω. In order to include the broadest class of charge densities we suppose that f lies in $W^{-1,2}(\Omega)$ and that φ is a $W_0^{1,2}(\Omega)$ solution of the Poisson equation. The associated electric field in the domain is $-\nabla\varphi$. We introduce a "target" electric field $\hat{\mathbf{E}}$. For a given charge density, our objective is to design a two phase dielectric that supports an electric field $-\nabla\varphi$ that is as close as possible to $\hat{\mathbf{E}}$. Here $\hat{\mathbf{E}} = -\nabla\hat{\varphi}$, where $\hat{\varphi}$ is a potential in $W^{1,2}(\Omega)$. Placing a constraint on the amount of the better dielectric β, the design problem is to minimize the difference

$$\int_\Omega |\nabla\varphi - \nabla\hat{\varphi}|^2\,dx, \tag{2}$$

over all configurations of the two dielectrics.

In general, material layout problems of this type fail to have an optimal design given by a configuration of the two materials. Instead one must study the behavior of minimizing sequences of configurations. The purpose of the analysis given here is to provide the methodology for the recovery of optimal configurations when they exist and to identify minimizing sequences of configurations for (2) otherwise. We introduce a tractable method for the numerical computation of minimizing sequences of configurations. These minimizing sequences are associated with materials with graded dielectric properties that may exhibit a fine scale structure composed of layers of the two dielectrics. Moreover, for a dense class of target fields we are able to characterize all fine scale structure that can appear in minimizing sequences of configurations, see Theorem 1 of this Section. We illustrate our approach through numerical examples provided in Section 6. The examples illustrate how the electric field can be controlled using functionally graded materials.

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2 Background and main theoretical results.

The nonexistence of an optimal configuration for the design problem coincides with the appearance of minimizing sequences containing regions of finite measure where the dielectric permittivity becomes highly oscillatory. As one follows these minimizing sequences the dielectric permittivity oscillates between the values α and β on progressively finer scales. To describe this mathematically we denote the subset of the design domain Ω containing the β dielectric by ω. The characteristic function of this set is written as χ where $\chi = 1$ for x in ω and $\chi = 0$ otherwise. The piece-wise constant dielectric permittivity is given by

$$\varepsilon(x) = \varepsilon(\chi) \stackrel{\Delta}{=} \beta\chi + \alpha(1-\chi). \tag{3}$$

Oscillation of a sequence of designs $\{\omega^\nu\}_{\nu=1}^\infty$ is described by the weak $L^\infty(\Omega)$ **star** convergence of the associated sequence of characteristic functions $\{\chi^\nu\}_{\nu=1}^\infty$ to a density θ in $L^\infty(\Omega)$ where $0 \leq \theta \leq 1$. The issue of nonexistence of optimal configurations for problems of material layout has been the object of much interest. The classic example is illustrated in the problem of minimizing the energy dissipation associated with configurations of two materials. In the context of two phase dielectric materials the energy dissipation for a configuration is given by

$$\int_\Omega \varepsilon(x)\nabla\varphi \cdot \nabla\varphi\, dx.$$

The problem of nonexistence was resolved in an elegant fashion by extending the design space to include all effective dielectric permittivities that could be obtained through oscillation, see [6], [10], and [11]. The crucial connection between minimizing sequences of configurations and optimal designs in the extended design space is established through a continuity property of the energy dissipation given in [3]. This continuity property is an example of the theory of compensated compactness developed in [7] and [12]. Here continuity is given in the context of *weakly* convergent sequences in $L^2(\Omega)^2$. Indeed, consider sequences $\{\varepsilon^\nu(x)\nabla\varphi^\nu\}_{\nu=1}^\infty$ and $\{\nabla\varphi^\nu\}_{\nu=1}^\infty$, such that $-\text{div}\,(\varepsilon(x)^\nu\nabla\varphi^\nu) = f$. If these sequences weakly converge to the limits $\varepsilon^\infty\nabla\varphi^\infty$ and $\nabla\varphi^\infty$ where ε^∞ is an effective tensor in the extended space of designs then one has the continuity expressed by

$$\lim_{\nu\to\infty} \int_\Omega \varepsilon^\nu(x)\nabla\varphi^\nu \cdot \nabla\varphi^\nu\, dx = \int_\Omega \varepsilon^\infty(x)\nabla\varphi^\infty \cdot \nabla\varphi^\infty\, dx.$$

For the design problem treated here we can attempt to resolve the nonexistence problem by extending the design space to include effective properties. However unlike the energy dissipation and other continuous functionals treated earlier, the objective functional given by (2) is not continuous with respect to weak convergence. Thus additional theoretical work is required to provide the connection between an extended space of designs and minimizing sequences of configurations. In this presentation we outline a methodology for the identification of minimizing sequences of configurations. The method is based on a careful extension of the design space and by replacing (2) with a suitable "relaxed" functional that is associated with the extended design space.

It is evident that any attempt to identify minimizing sequences of configurations must account for the possibility of oscillations in the sequence of gradients associated with minimizing sequences of designs. As above an oscillatory sequence of gradients is characterized by the weak convergence of the sequence in $L^2(\Omega)^2$. To fix ideas, let $\{\nabla\varphi_\nu\}_{\nu=1}^\infty$ denote a weakly converging sequence of gradients associated with a minimizing sequence of designs. The weak limit of the sequence is denoted by $\nabla\tilde{\varphi}$, and one has

$$\begin{aligned}
&\lim_{\nu\to\infty} \int_\Omega |\nabla\varphi_\nu - \nabla\hat{\varphi}|^2\, dx \\
&= \lim_{\nu\to\infty} \int_\Omega |\nabla\varphi_\nu - \nabla\tilde{\varphi}|^2\, dx + \int_\Omega |\nabla\tilde{\varphi} - \nabla\hat{\varphi}|^2\, dx.
\end{aligned} \tag{4}$$

The oscillatory behavior of minimizing sequences is naturally linked to the dependence of the limit

$$\lim_{\nu\to\infty}\int_\Omega |\nabla\varphi_\nu - \nabla\tilde{\varphi}|^2\,dx, \tag{5}$$

on the weak limits $\nabla\tilde{\varphi}$, θ, together with other moments of measures associated with weak limits of geometric quantities, (e.g., the H measure introduced by L. Tartar [14]). Our methodology for identifying minimizing sequences is based upon on writing (5) as an explicit function of the relevant weak limits. Although at this time we are unable to produce a formula for every type of oscillation we note that an explicit closed form expression is available when the oscillations consist of layers of the two materials. The formula follows directly from the corrector theory of homogenization given in [1] and [9]. To be precise we introduce the characteristic function $\chi(x,t)$. Here $\chi(x,t)$ is piece-wise constant in the first variable and unit periodic in the scalar t variable. As is done in homogenization theory [1], we define a locally layered material by $\chi(x) = \chi(x, x\cdot n)$. Here $n = n(x)$ represents the normal to the layers given by $n = (\cos(\gamma(x)), \sin(\gamma(x)))$. An oscillatory sequence is given by $\chi^\nu(x) = \chi(x, \nu x\cdot n)$, $\nu = 1,2\ldots,\infty$. For this case we have the closed form expression given by

$$\lim_{\nu\to\infty}\int_\Omega |\nabla\varphi_\nu - \nabla\tilde{\varphi}|^2\,dx = \int_\Omega R(\gamma(x))H(\theta(x))R^T(\gamma(x))\nabla\tilde{\varphi}(x)\cdot\nabla\tilde{\varphi}(x)\,dx. \tag{6}$$

Here $R(\gamma)$ is the orthogonal matrix associated with a rotation of γ radians and the matrix $H(\theta)$ is a function of the density θ given by

$$H(\theta) = \begin{pmatrix} (\frac{1}{\alpha}-\frac{1}{\beta})^2\theta(1-\theta)h_\theta^2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{7}$$

where $h_\theta = (\frac{1-\theta}{\alpha}+\frac{\theta}{\beta})^{-1}$ is the harmonic mean of the two dielectric permittivities. Here, the sequence of gradients $\{\nabla\varphi_\nu\}_{\nu=1}^\infty$ is related to the sequence of configurations through the equilibrium condition

$$-\text{div}\,(\varepsilon(\chi^\nu)\nabla\varphi_\nu) = f. \tag{8}$$

The "homogenized " equilibrium equation satisfied by the weak limit $\tilde{\varphi}$ is given by

$$-\text{div}\,(\varepsilon(\theta(x),\gamma(x))\nabla\tilde{\varphi}) = f, \tag{9}$$

where

$$\varepsilon(\theta(x),\gamma(x)) = R(\gamma(x))\;\Lambda(\theta(x))\;R^T(\gamma(x)), \tag{10}$$

and the diagonal tensor $\Lambda(\theta)$ is given by

$$\Lambda(\theta) = \begin{pmatrix} h_\theta & 0 \\ 0 & m_\theta \end{pmatrix},$$

with $m_\theta = \alpha\,(1-\theta) + \beta\,\theta$. Here the tensor $\varepsilon(\theta,\gamma)$ is the G-limit associated with the sequence of dielectric tensors $\{\varepsilon(\chi^\nu)\}_{n=1}^\infty$, see [9]. (Since the dielectric tensors are symmetric, the G-convergence [18] and the H-convergence [9] of any sequence of dielectric tensors is the same.)

The methodology presented here uses the explicit formula given by (6). Our approach is to replace χ and $\varepsilon(\chi)$ with the new design variables θ, γ, and $\varepsilon(\theta,\gamma)$ given by (10). In addition we introduce the new objective functional given by

$$RF(\theta,\gamma,\varepsilon(\theta,\gamma),\nabla\hat{\varphi}) = \int_\Omega |\nabla\varphi - \nabla\hat{\varphi}|^2\,dx + \int_\Omega R(\gamma(x))H(\theta(x))R^T(\gamma(x))\nabla\varphi\cdot\nabla\varphi\,dx, \tag{11}$$

where the state variable is the $W_0^{1,2}$ solution of

$$-\text{div}\ (\varepsilon(\theta(x),\gamma(x))\nabla\varphi) = f. \tag{12}$$

In order to state our results we formulate the original design problem in a precise way. We introduce the constant Θ, such that $0 < \Theta < 1$. The space of admissible configurations and associated dielectric permittivities is denoted by ad_Θ, and

$$ad_\Theta = \{\chi : \int_\Omega \chi\, dx \leq \Theta\ meas(\Omega)\}. \tag{13}$$

The objective functional is denoted by $F(\chi, \varepsilon(\chi), \nabla\hat{\varphi})$ and is given by

$$F(\chi, \varepsilon(\chi), \nabla\hat{\varphi}) = \int_\Omega |\nabla\varphi - \nabla\hat{\varphi}|^2\, dx, \tag{14}$$

where the state variable φ is a solution of (1). The original design problem is formulated as

$$P = \inf_{\chi\in ad_\Theta} F(\chi, \varepsilon(\chi), \nabla\hat{\varphi}). \tag{15}$$

The admissible space of designs for the new design problem is given by

$$D_\Theta = \{\ (\theta,\gamma,\varepsilon(\theta,\gamma))\,|\,\theta\in L^\infty(\Omega;[0,1]);\,\gamma\in L^\infty(\Omega;[0,2\pi]) : \\ \int_\Omega \theta\ dx \leq \Theta\ meas\Omega\ ;\ \varepsilon(\theta(x),\gamma(x)) = R(\gamma(x))\ \Lambda(\theta(x))\ R^T(\gamma(x))\ \}, \tag{16}$$

and the new design problem is formulated as

$$RP = \inf_{(\theta,\gamma,\varepsilon(\theta,\gamma))\in D_\Theta} RF(\theta,\gamma,\varepsilon(\theta,\gamma),\nabla\hat{\varphi}). \tag{17}$$

We point out that the extended space of designs D_Θ contains the original space of designs ad_Θ. Indeed, choosing $\theta = \chi$ we have $\varepsilon(\theta,\gamma) = \varepsilon(\chi)$, $H(\theta) = 0$ and

$$F(\chi,\varepsilon(\chi),\nabla\hat{\varphi}) = RF(\theta,\gamma,\varepsilon(\theta,\gamma),\nabla\hat{\varphi}). \tag{18}$$

The first result that we describe is given in Theorem 8. It states that for every $\hat{\varphi}$ in $W^{1,2}(\Omega)$ and for every f in $W^{-1,2}(\Omega)$ that

$$P = RP. \tag{19}$$

In deriving (19) we identify a class of minimizing sequences for P that is tractable for numerical computation. Our approach is based upon a discrete approximation of the design space D_Θ. We consider any partition T_κ of Ω consisting of a finite number of pair-wise disjoint subdomains $\Omega_i \subset \Omega$, $i = 1,\ldots,N(\kappa)$ such that:

$$\Omega = \bigcup_{i=1}^{N(\kappa)} \Omega_i \ \text{ and } \max_{i=1,\ldots N(\kappa)} (diam(\Omega_i)) \leq \kappa. \tag{20}$$

We fix the partition T_κ and the discrete approximation D_Θ^κ is given by the piece-wise constant functions $\theta^\kappa(x)$, $\gamma^\kappa(x)$ taking constant values in each subdomain. We denote the restriction of $\theta^\kappa(x)$ and $\gamma^\kappa(x)$ to Ω_i by θ_i^κ and γ_i^κ respectively. Here $0 \leq \theta_i^\kappa \leq 1$, $0 \leq \gamma_i^\kappa \leq 2\pi$, and

$$\Sigma_{i=1}^{N(\kappa)} (\theta_i^\kappa meas(\Omega_i)) = \Theta meas(\Omega). \tag{21}$$

The piece-wise constant dielectric permittivity tensor is given by

$$\varepsilon(\gamma^\kappa(x), \theta^\kappa(x)) = R(\gamma^\kappa(x))\Lambda(\theta^\kappa(x))R^T(\gamma^\kappa(x)), \tag{22}$$

and the associated state variable φ^κ solves the Poisson equation

$$-\text{div}\ (\varepsilon(\gamma^\kappa(x), \theta^\kappa(x))\nabla\varphi^\kappa) = f. \tag{23}$$

It is clear that D_Θ^κ is contained in the larger set D_Θ and the design problem posed on this smaller set of designs is written

$$RP^\kappa = \inf_{(\theta^\kappa,\gamma^\kappa,\varepsilon(\theta^\kappa,\gamma^\kappa))\in D_\Theta^\kappa} RF(\theta^\kappa, \gamma^\kappa, \varepsilon(\theta^\kappa, \gamma^\kappa), \nabla\hat{\varphi}). \tag{24}$$

It is shown that a minimizing vector of design variables $(\overline{\theta}^\kappa, \overline{\gamma}^\kappa)$ exists for this problem, see Theorem 4. Most importantly it follows from (6) that there exists a recovery sequence of configurations χ^ν in ad_Θ for which

$$\lim_{\nu\to\infty} F(\chi^\nu, \varepsilon(\chi^\nu), \nabla\hat{\varphi}) = RP^\kappa, \tag{25}$$

see Theorem 5. For any given partition T_κ we consider its refinements, i.e., the nested family of partitions $\{T_\epsilon\}_{\epsilon>0}$ that includes T_κ. We show that the optimal design vectors associated with the refinements represent a minimizing sequence of designs for the problem RP, see Theorem 7. Moreover, using (25) we are able to recover the explicit form of minimizing sequences for the original problem P, see Theorem 5 and equation (57). We point out that the choice of the initial partition T_κ is arbitrary so this method generates minimizing sequences of designs for any initial choice of partition.

It is instructive to write $RF(\theta, \gamma, \varepsilon(\theta, \gamma), \nabla\hat{\varphi})$ in a form where $\varepsilon(\theta, \gamma)$ appears explicitly. Manipulation gives

$$R(\gamma)H(\theta)R^T(\gamma) = \frac{(m_\theta I - \varepsilon(\theta, \gamma))^2}{(1-\theta)\beta(\beta-\alpha)} + \frac{(m_\theta I - \varepsilon(\theta, \gamma))}{\beta}. \tag{26}$$

It is clear from (11) and (26) that if $\varepsilon(\theta, \gamma)\nabla\varphi = m_\theta\nabla\varphi$ then

$$\int_\Omega R(\gamma)H(\theta)R^T(\gamma)\nabla\varphi\cdot\nabla\varphi\, dx = 0.$$

It now follows from (6) that the gradients $\nabla\varphi^\nu$ associated with a recovery sequence of configurations $\{\varepsilon(\chi_\omega^\nu)\}_{\nu=0}^\infty$ G-converging to $\varepsilon(\theta, \gamma)$ converge strongly to $\nabla\varphi$. Conversely if the gradients $\nabla\varphi^\nu$ associated with a recovery sequence of configurations converge strongly to $\nabla\varphi$ then the term

$$\int_\Omega R(\gamma)H(\theta)R^T(\gamma)\nabla\varphi\cdot\nabla\varphi\, dx$$

vanishes and $\varepsilon(\theta, \gamma)\nabla\varphi = m_\theta\nabla\varphi$ follows from (26). In this context we mention that the earlier work of [5] focuses on the energy dissipation to show that the condition $\varepsilon^e\nabla\varphi = \varepsilon^*\nabla\varphi$ is necessary and sufficient for the strong convergence of gradients associated with sequences $\{\varepsilon_\nu^e\}_{\nu=0}^\infty$ G-converging to ε^e and weak L^∞ **star** converging to ε^*.

For a dense set of target fields we show that our method accounts for all oscillations appearing in minimizing sequences of configurations. We consider target fields of the form

$$-\nabla\hat{\varphi},\ \hat{\varphi}\in W_0^{1,2}(\Omega) \tag{27}$$

and the relaxed version of the original problem is given by

Theorem 1. *There exists a dense G_δ subset K of $W_0^{1,2}(\Omega)$ such that for $\hat{\varphi}$ in K:*

(1) There exists a minimizer of RP in D_Θ,

(2) $P = RP$,

(3) Any cluster point of any minimizing sequence in ad_Θ of P is a minimizer of RP and any minimizer of RP in D_Θ is a limit of a minimizing sequence for P.

(4) Let $\bar{\varphi}$ be the potential associated with the minimizer $(\bar{\theta}, \bar{\gamma}, \varepsilon(\bar{\theta}, \bar{\gamma}))$ of RP, then

$$RF(\bar{\theta}, \bar{\gamma}, \varepsilon(\bar{\theta}, \bar{\gamma}), \nabla\hat{\varphi}) = \int_\Omega |\nabla\bar{\varphi} - \nabla\hat{\varphi}|^2 \, dx \quad \text{and} \quad \varepsilon(\bar{\theta}, \bar{\gamma})\nabla\bar{\varphi} = m_{\bar{\theta}}\nabla\bar{\varphi}.$$

Here the convergence of sequences of designs are with respect to the G-convergence [18].

The class of targets appearing in Theorem 1 is motivated by the following theorem of L. Tartar [2], [13], which is an improvement of a result of M. Edelstein [4].

Theorem 2. *:* Let S be a non-empty strongly closed subset of a Hilbert space H. Then there exists a dense G_δ subset K of H such that for any $x \in$ K, the minimizing sequences $\{c_n\}_{n=1}^{\infty} \in$ S of the function $c \to ||x - c||$ are Cauchy sequences. In particular the subset of points of H with a unique projection on S contains a dense G_δ subset, as it contains K.

With Theorem 2 in mind we can take advantage of the geometry of the set of effective tensors for two dimensional problems and establish Theorem 1. This topic is taken up in Section 5 where Theorem 1 is proved.

The recent work of P. Pedregal [15], [16] approaches similar design problems from a different perspective. In that work the equilibrium equation for the potential, together with the resource constraint is incorporated into the cost functional and a new type of envelope for the augmented functional is introduced. The envelope is shown to be weakly lower semicontinuous in $W_0^{1,2}(\Omega) \times L^\infty(\Omega)$ see [15], [16] and can be thought of as a constrained quasiconvexification of the original augmented functional. The constrained quasiconvex envelope can be expressed in terms of a class $\mathcal{A}$ of gradient Young measures, see [15]. To proceed further, the envelope needs to be given in terms of explicit formulas. This requires knowledge of the set $\mathcal{A}$. However at this stage the characterization of $\mathcal{A}$ is not known. In principle the methods of this paper can be used to deduce the part of $\mathcal{A}$ containing Young measures associated with simple one scale laminates, see [15]. Theorem 1 shows that the knowledge of gradient Young measures associated with layered microstructures (a.k.a. laminates) is sufficient for the computation of the constrained quasi convex hull when the target fields are in the class $K \subset W_0^{1,2}$.

We point out that the discrete problem given by (24) is of interest on its own right. From a practical perspective there is a prohibitive manufacturing cost incurred when attempting to make a graded material with possibly different anisotropic dielectric properties at every point. Instead there is a smallest scale κ over which the dielectric properties change. The scale is set by the manufacturing cost. Practically speaking one partitions the design domain into subdomains of diameter κ and inside these subdomains one optimizes the dielectric properties. This approach to the design of graded materials is naturally incorporated in the formulation of the discrete problem given here and is discussed in the context of the numerical examples given in Section 6.

3 The Discrete Problem.

In this Section we analyze the design problem on the discretized space of designs. The existence of an optimal design is established in this space. Next we apply the corrector theory to exhibit a recovery sequence of configurations of the two dielectrics.

We consider any partition T_κ of Ω consisting of a finite number of pair-wise disjoint subdomains $\Omega_i \subset \Omega$, $i = 1, \ldots, N(\kappa)$ such that:

$$\Omega = \bigcup_{i=1}^{N(\kappa)} \Omega_i \quad \text{and} \quad \max_{i=1,\ldots N(\kappa)} (diam(\Omega_i)) \leq \kappa.$$

We fix the partition T_κ and the discrete approximation D_Θ^κ is described by equations (20–23) given in the introduction. The design problem over the discrete space is given by (24). Existence of the optimal design is established using the direct method of the calculus of variations. We start by introducing the type of convergence relevant to the discrete problem. A design $(\theta^\kappa, \gamma^\kappa, \varepsilon(\theta^\kappa, \gamma^\kappa))$ in D_Θ^κ can be identified with the vector $(\theta_i^\kappa, \gamma_i^\kappa)$ for $i = 1, \ldots, N(\kappa)$ in $R^{2N(\kappa)}$. Thus D_Θ^κ is identified with a compact subset of $R^{2N(\kappa)}$ and convergence of designs in D_Θ^κ is given by sequential convergence in $R^{2N(\kappa)}$. Existence of an optimal design will follow once we show that the functional $RF(\theta^\kappa, \gamma^\kappa, \varepsilon(\theta^\kappa, \gamma^\kappa), \nabla\hat{\varphi})$ is continuous with respect to sequential convergence in $R^{2N(\kappa)}$.

Theorem 3. Given a sequence of designs $\{(\theta^{\kappa,n}, \gamma^{\kappa,n})\}_{n=1}^\infty$ and a design $(\bar{\theta}^\kappa, \bar{\gamma}^\kappa)$ such that

$$\lim_{n\to\infty} (\theta^{\kappa,n}, \gamma^{\kappa,n}) = (\bar{\theta}^\kappa, \bar{\gamma}^\kappa) \tag{28}$$

as elements of $R^{2N(\kappa)}$, then

$$\lim_{n\to\infty} RF(\theta^{\kappa,n}, \gamma^{\kappa,n}, \varepsilon(\theta^{\kappa,n}, \gamma^{\kappa,n}), \nabla\hat{\varphi}) = RF(\bar{\theta}^\kappa, \bar{\gamma}^\kappa, \varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa), \nabla\hat{\varphi}). \tag{29}$$

Proof. The state variable associated with the limit design $(\bar{\theta}^\kappa, \bar{\gamma}^\kappa)$ is denoted by $\bar{\varphi}^\kappa$ and is the $W_0^{1,2}(\Omega)$ solution of

$$-\text{div}\ \left(\varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa)\nabla\bar{\varphi}^\kappa\right) = f. \tag{30}$$

The convergence of the sequence of designs given by (28) implies that the associated conductivities $\{\varepsilon(\theta^{\kappa,n}, \gamma^{\kappa,n})\}_{n=1}^\infty$ converge to $\varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa)$ almost everywhere. From the theory of G-convergence [18] we also know that the sequence G-converges to $\varepsilon(\bar{\theta}^\kappa, \bar{\gamma}^\kappa)$. The definition of G-convergence implies that the state variables φ_n^κ associated with the sequence $\{\varepsilon(\theta^{\kappa,n}, \gamma^{\kappa,n})\}_{n=1}^\infty$ converge weakly in $W_0^{1,2}(\Omega)$ to $\bar{\varphi}^\kappa$. In order to establish the continuity given by (29) we first show that the sequence φ_n^κ converges strongly to $\bar{\varphi}^\kappa$ in $W_0^{1,2}(\Omega)$. To do this we recall the formula (10) for $\varepsilon(\theta, \gamma)$ to easily see that

$$0 < \alpha \leq \varepsilon(\theta^{\kappa,n}, \gamma^{\kappa,n}) \leq \beta$$

We apply this estimate to obtain,

$$\begin{aligned} \int_\Omega \alpha|\nabla\varphi_n^\kappa - \nabla\bar{\varphi}^\kappa|^2 dx \quad &\leq \quad \int_\Omega \varepsilon(\theta^{\kappa,n}, \gamma^{\kappa,n})(\nabla\varphi_n^\kappa - \nabla\bar{\varphi}^\kappa) \cdot (\nabla\varphi_n^\kappa - \nabla\bar{\varphi}^\kappa) dx & (31) \\ &= \quad \int_\Omega \varepsilon(\theta^{\kappa,n}, \gamma^{\kappa,n})\nabla\varphi_n^\kappa \cdot \nabla\varphi_n^\kappa dx - 2\int_\Omega \varepsilon(\theta^{\kappa,n}, \gamma^{\kappa,n})\nabla\varphi_n^\kappa \cdot \nabla\bar{\varphi}^\kappa dx \\ &+ \quad \int_\Omega \varepsilon(\theta^{\kappa,n}, \gamma^{\kappa,n})\nabla\bar{\varphi}^\kappa \cdot \nabla\bar{\varphi}^\kappa dx. & (32) \end{aligned}$$

Passing to the limit as $n \to \infty$ in (31,32) , we apply the well known properties of G-convergence together with the almost everywhere convergence of $\{\varepsilon(\theta^{\kappa,n}, \gamma^{\kappa,n})\}_{n=1}^\infty$ and the Lebesgue convergence theorem to find that

$$\lim_{n\to\infty} \| \nabla\varphi_n^\kappa - \nabla\bar{\varphi}^\kappa \|_{L^2}^2 = 0, \tag{33}$$

and strong convergence of φ_n^κ to $\bar{\varphi}^\kappa$ in $W_0^{1,2}$ follows.

From (7) we easily obtain the following estimate for the sequence

$$\{R(\gamma^{\kappa,n}(x))H(\theta^{\kappa,n}(x))R^T(\gamma^{\kappa,n}(x))\}_{n=1}^{\infty}$$

given by

$$R(\gamma^{\kappa,n}(x))H(\theta^{\kappa,n}(x))R^T(\gamma^{\kappa,n}(x))\eta\cdot\eta \leq \frac{\beta^2}{4}(\frac{1}{\alpha}-\frac{1}{\beta})\ |\eta|^2, \tag{34}$$

for every vector η in R^2. Moreover, from the convergence given by (28) the sequence converges almost everywhere to

$$R(\bar{\gamma}^{\kappa}(x))H(\bar{\theta}^{\kappa}(x))R^T(\bar{\gamma}^{\kappa}(x)).$$

Finally since $\nabla\varphi_n^{\kappa} \to \nabla\bar{\varphi}^{\kappa}$ strongly in L^2 we apply (34) together with the Lebesgue convergence theorem to conclude that:

$$\lim_{n\to\infty} RF(\theta^{\kappa,n},\gamma^{\kappa,n},\varepsilon(\theta^{\kappa,n},\gamma^{\kappa,n}),\nabla\hat{\varphi}) =$$

$$= \lim_{n\to\infty}\ \left(\int_{\Omega} |\nabla\varphi_n^{\kappa} - \nabla\hat{\varphi}|^2\,dx + \int_{\Omega} R(\gamma^{\kappa,n})H(\theta^{\kappa,n})R^T(\gamma^{\kappa,n})\ \nabla\varphi_n^{\kappa}\cdot\nabla\varphi_n^{\kappa}\,dx \right)$$

$$\begin{aligned} &= \int_{\Omega} |\nabla\bar{\varphi}^{\kappa} - \nabla\hat{\varphi}|^2\,dx + \int_{\Omega} R(\bar{\gamma}^{\kappa})H(\bar{\theta}^{\kappa})R^T(\bar{\gamma}^{\kappa})\ \nabla\bar{\varphi}^{\kappa}\cdot\nabla\bar{\varphi}^{\kappa}\,dx \\ &= RF(\bar{\theta}^{\kappa},\bar{\gamma}^{\kappa},\varepsilon(\bar{\theta}^{\kappa},\bar{\gamma}^{\kappa}),\nabla\hat{\varphi}). \end{aligned} \tag{35}$$

which proves the theorem.

Collecting our results we have shown that

Theorem 4. There exists an optimal design $(\bar{\theta}^{\kappa},\bar{\gamma}^{\kappa},\varepsilon(\bar{\theta}^{\kappa},\bar{\gamma}^{\kappa}))$ in D_{Θ}^{κ} for the discrete problem, i.e.,

$$\begin{aligned} RP^{\kappa} &= RF(\bar{\theta}^{\kappa},\bar{\gamma}^{\kappa},\varepsilon(\bar{\theta}^{\kappa},\bar{\gamma}^{\kappa}),\nabla\hat{\varphi}) \\ &= \min_{(\theta^{\kappa},\gamma^{\kappa},\varepsilon(\theta^{\kappa}\gamma^{\kappa}))\in D_{\Theta}^{\kappa}} RF(\theta^{\kappa},\gamma^{\kappa},\varepsilon(\theta^{\kappa}\gamma^{\kappa}),\nabla\hat{\varphi}). \end{aligned} \tag{36}$$

Now we show how to construct a sequence of configurations described by with the sequence of characteristic functions $\{\chi^n\}_{n=1}^{\infty}$ for which

$$\lim_{n\to\infty} F(\chi^n,\varepsilon(\chi^n),\nabla\hat{\varphi}) = RP^{\kappa}. \tag{37}$$

In view of Theorem 4 it is sufficient to consider any design given by $(\theta^{\kappa},\gamma^{\kappa})$ in D_{θ}^{κ} and show how to construct a sequence $\{\chi^{\kappa,n}\}_{n=1}^{\infty}$ for which

$$\lim_{n\to\infty} F(\chi^{\kappa,n},\varepsilon(\chi^{\kappa,n}),\nabla\hat{\varphi}) = RF(\theta^{\kappa},\gamma^{\kappa},\varepsilon(\theta^{\kappa},\gamma^{\kappa}),\nabla\hat{\varphi}). \tag{38}$$

We start by observing that for $\theta = 0$ or $\theta = 1$ that $\varepsilon(\theta,\gamma) = \alpha I$ or βI respectively, where I is the 2×2 identity. Thus for a design specified by $(\theta^{\kappa},\gamma^{\kappa})$ we proceed to construct the sequence $\{\chi^{\kappa,n}\}_{n=1}^{\infty}$ in the following way. In the subdomains Ω_i for which $\theta_i^{\kappa} = 0$ we set $\chi^{\kappa,n} = 0$, $n = 1,2\ldots\infty$ and in the subdomains Ω_i for which $\theta_i^{\kappa} = 1$ we set $\chi^{\kappa,n} = 1$, $n = 1,2\ldots\infty$. Next we consider the subdomains Ω_i where $0 < \theta_i^{\kappa} < 1$. In these subdomains we have $0 \leq \gamma_i^{k} \leq 2\pi$ and we set $\chi^{\kappa,n} = \mu(nx\cdot\mathbf{n}(\gamma_i^{\kappa}))$, where $\mu(t)$ is a periodic function on the real line of period unity taking the values 1 for $0 \leq t \leq \theta_i^{\kappa}$ and 0 for $\theta_i^{\kappa} < t < 1$ and $\mathbf{n}(\gamma_i^{\kappa}) = (\cos\gamma_i^{k},\sin\gamma_i^{k})$. We summarize our construction in the following equation,

$$\chi^{\kappa,n} = \begin{cases} 0, & \text{in } \Omega_i \text{ for which } \theta_i^{\kappa} = 0, \\ 1, & \text{in } \Omega_i \text{ for which } \theta_i^{\kappa} = 1, \\ \mu(nx\cdot\mathbf{n}(\gamma_i^{\kappa})), & \text{in } \Omega_i \text{ for which } 0 < \theta_i^{\kappa} < 1. \end{cases} \tag{39}$$

The associated dielectric permittivity $\varepsilon(\chi^{\kappa,n})$ corresponds to pure α dielectric in the subdomains Ω_i where $\theta^\kappa = 0$, pure β dielectric in the subdomains Ω_i where $\theta^\kappa = 1$, and layers of α and β dielectric with layer normal in the direction $(\cos\gamma_i^k, \sin\gamma_i^k)$ in the subdomains where $0 < \theta_i^\kappa < 1$. The associated state variables $\varphi^{\kappa,n}$ are the $W_0^{1,2}(\Omega)$ solutions to the equilibrium equation

$$-\text{div}\ (\varepsilon(\chi^{\kappa,n})\nabla\varphi^{\kappa,n}) = f. \tag{40}$$

The sequence $\varepsilon(\chi^{\kappa,n})$ G-converges to $\varepsilon(\theta^\kappa,\gamma^\kappa)$ [9], hence the sequence $\varphi^{\kappa,n}$ converges weakly in $W_0^{1,2}(\Omega)$ to the state variable φ^κ associated with the design $(\theta^\kappa,\gamma^\kappa)$. With this construction in mind we state the following Theorem that guarantees the existence of recovery sequences of configurations.

Theorem 5. Given a design $(\theta^\kappa,\gamma^\kappa)$ in D_θ^κ the sequence of configurations $\{\chi^{\kappa,n}\}_{n=1}^\infty$ given by (39) is a recovery sequence, i.e.,

$$\lim_{n\to\infty} F(\chi^{\kappa,n}, \varepsilon(\chi^{\kappa,n}), \nabla\hat{\varphi}) = RF(\theta^\kappa,\gamma^\kappa,\varepsilon(\theta^\kappa,\gamma^\kappa),\nabla\hat{\varphi}). \tag{41}$$

Proof. We have

$$\lim_{n\to\infty} F(\chi^{\kappa,n}, \varepsilon(\chi^{\kappa,n})) = \int_\Omega |\nabla\varphi^\kappa - \nabla\hat{\varphi}|^2\,dx + \lim_{n\to\infty}\int_\Omega |\nabla\varphi^{\kappa,n} - \nabla\varphi^\kappa|^2\,dx \tag{42}$$

Applying the corrector theory of homogenization given in [9] we can write

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega |\nabla\varphi^{\kappa,n} - \nabla\varphi^\kappa|^2\,dx\\ &= \int_\Omega |(P^{\kappa,n} - I)\nabla\varphi^\kappa + z^{\kappa,n}|^2\,dx.\end{aligned} \tag{43}$$

Where $P^{\kappa,n}$ is the corrector matrix associated with $\chi^{\kappa,n}$ and on each subdomain Ω_i it is given by:

$$P^{\kappa,n} = R(\gamma_i^\kappa)\begin{pmatrix}\frac{h_{\theta_i^\kappa}}{[\ \alpha(1-\chi^{\kappa,n})+\beta\chi^{\kappa,n}\ ]} & 0\\ 0 & 1\end{pmatrix} R^T(\gamma_i^\kappa),\ \text{and}\ P^{\kappa,n} \rightharpoonup I\ \text{ in } L^2 \text{ as } n\to\infty.$$

Since $P^{\kappa,n} \in L^\infty(\Omega)^{2\times 2}$ it follows from the corrector theorem of F. Murat and L. Tartar [9] that $z^{\kappa,n} \to 0$ strongly in L^2. As a consequence we have:

$$\lim_{n\to\infty}\int_\Omega |(P^{\kappa,n} - I)\nabla\varphi^\kappa + z^{\kappa,n}|^2\,dx =$$

$$\lim_{n\to\infty}\sum_{i=1}^{N(\kappa)}\left(\int_{\Omega_i} R(\gamma_i^\kappa)\begin{pmatrix}(\frac{h_{\theta_i^\kappa}}{[\ \alpha(1-\chi^{\kappa,n})+\beta\chi^{\kappa,n}\ ]} - 1)^2 & 0\\ 0 & 0\end{pmatrix} R^T(\gamma_i^\kappa)\nabla\varphi^\kappa\cdot\nabla\varphi^\kappa\ dx\right) =$$

$$= \int_\Omega R(\gamma^\kappa)H(\theta^\kappa)R^T(\gamma^\kappa)\nabla\varphi^\kappa\cdot\nabla\varphi^\kappa\ dx, \tag{44}$$

and the Theorem follows.

4 Minimizing sequences of configurations.

In this Section we identify minimizing sequences of design vectors for the problem RP. These sequences are associated with the refinements of a given partition T_κ. Next we employ Theorem 5 to deduce that $RP = P$ and identify a special class of minimizing sequences of configurations for P.

We recall that a nested family of partitions $\{T_\kappa\}_{\kappa\leq\epsilon}$ of Ω is a family that satisfies:

$$\kappa_1 < \kappa_2 \;\Rightarrow\; \forall\Omega_i^1 \in T_{\kappa_1} \;\; \exists\Omega_j^2 \in T_{\kappa_2} : \; \Omega_i^1 \subset \Omega_j^2. \tag{45}$$

For any given partition $T_{\tilde\kappa}$ the sequence of refinements of this partition is denoted by $\{T_\kappa\}_{\kappa\leq\tilde\kappa}$ and is a nested family of partitions as described by (45). We show that the space of discrete designs D_Θ^κ associated with the refinements of $T_{\tilde\kappa}$ is dense in D_Θ.

Theorem 6. The system of designs $\{D_\Theta^\kappa\}_{\kappa\to 0}$ is dense in D_Θ. Indeed, for every $(\theta, \gamma, \varepsilon(\theta,\gamma)) \in D_\Theta$, there exists a sequence $(\theta^\kappa, \gamma^\kappa, \varepsilon(\theta^\kappa,\gamma^\kappa)) \in D_\Theta^\kappa$ for which

$$\theta^\kappa \to \theta,\; \gamma^\kappa \to \gamma \text{ a.e. in } \Omega \text{ and } \varepsilon(\theta^\kappa,\gamma^\kappa) \text{ G-converges to } \varepsilon(\theta,\gamma) \text{ as } \kappa \to 0, \tag{46}$$

furthermore*:*

$$\lim_{\kappa\to 0} RF(\theta^\kappa, \gamma^\kappa, \varepsilon(\theta^\kappa,\gamma^\kappa), \nabla\hat\varphi) = RF(\theta,\gamma,\varepsilon(\theta,\gamma),\nabla\hat\varphi)\;. \tag{47}$$

Proof. For a given design $(\theta, \gamma, \varepsilon(\theta,\gamma)) \in D_\Theta$ we choose any partition $T_{\tilde\kappa}$ of Ω and consider its refinements $\{T_\kappa\}_{\kappa\leq\tilde\kappa}$. For any refinement T_κ , we construct $(\theta^\kappa, \gamma^\kappa, \varepsilon(\theta^\kappa,\gamma^\kappa)) \in D_\Theta^\kappa$ as described below:

$$\theta_i^\kappa = \frac{1}{meas\ \Omega_i}\int_{\Omega_i}\theta(x)\,dx$$

$$\gamma_i^\kappa = \frac{1}{meas\ \Omega_i}\int_{\Omega_i}\gamma(x)\,dx$$

$$\varepsilon(\theta^\kappa,\gamma^\kappa) = R(\gamma_i^\kappa)\Lambda(\theta_i^\kappa)R^T(\gamma_i^\kappa) \;\text{ on } \Omega_i.$$

We consider the intersection of Lebesgue points for the functions $\theta(x)$ and $\gamma(x)$. On this set we have:

$$\theta^\kappa(x) \to \theta(x),\;\; \gamma^\kappa(x) \to \gamma(x) \quad as\ \kappa \to 0.$$

This delivers the convergence

$$\varepsilon(\theta^\kappa(x),\gamma^\kappa(x)) \to \varepsilon(\theta(x),\gamma(x)) = R(\gamma)\Lambda(\theta)R^T(\gamma) \text{ a.e. in } \Omega \text{ as } \kappa \to 0$$

and

$$R(\gamma^\kappa)H(\theta^\kappa)R^T(\gamma^\kappa) \to R(\gamma)H(\theta)R^T(\gamma) \text{ a.e. in } \Omega \text{ as } \kappa \to 0.$$

From the properties of G-convergence [18] we deduce as in Theorem 3 that $\varepsilon(\theta^\kappa(x),\gamma^\kappa(x))$ G-converges to $\varepsilon(\theta(x),\gamma(x))$ and this establishes (46). This implies that the sequence of state variables φ^κ, satisfying $\varphi^\kappa \in W_0^{1,2}(\Omega)$ and

$$-\text{div}\;(\varepsilon(\theta^\kappa,\gamma^\kappa)\nabla\varphi^\kappa) = f, \tag{48}$$

converges weakly in $W_0^{1,2}(\Omega)$ to the $W_0^{1,2}(\Omega)$ solution φ of

$$-\text{div } (\varepsilon(\theta,\gamma)\nabla\varphi) = f. \tag{49}$$

Following the same arguments given in the proof of Theorem 3, we see that the sequence $\{\varphi^\kappa\}_{\kappa>0}$ converges strongly in $W_0^{1,2}(\Omega)$ to φ. Moreover, the same estimate as given in (34) holds for the sequence

$$\{R(\gamma^\kappa)H(\theta^\kappa)R^T(\gamma^\kappa)\}_{\kappa>0}$$

and we can proceed along the same lines as in the proof of Theorem 3 to show that

$$\lim_{\kappa\to 0} RF(\theta^\kappa,\gamma^\kappa,\varepsilon(\theta^\kappa,\gamma^\kappa),\nabla\hat{\varphi}) = RF(\theta,\gamma,\varepsilon(\theta,\gamma),\nabla\hat{\varphi}). \tag{50}$$

We now identify minimizing sequences of designs for the RP problem. We consider any nested family of partitions denoted by $\{T_\kappa\}_{\kappa>0}$. For each value of κ we consider the optimal design for the discrete problem RP^κ denoted by $(\bar{\theta}^\kappa,\bar{\gamma}^\kappa,\varepsilon(\bar{\theta}^\kappa,\bar{\gamma}^\kappa))$.

Theorem 7. The sequence $\{(\overline{\theta}^\kappa,\overline{\gamma}^\kappa,\varepsilon(\bar{\theta}^\kappa,\bar{\gamma}^\kappa))\}_{\kappa>0}$, is a minimizing sequence for the RP problem and satisfies the monotonicity condition:

$$\text{for } \kappa<\kappa',\ RP^\kappa = RF(\overline{\theta}^\kappa,\overline{\gamma}^\kappa,\varepsilon(\bar{\theta}^\kappa,\bar{\gamma}^\kappa),\nabla\hat{\varphi}) \le RP^{\kappa'} = RF(\overline{\theta}^{\kappa'},\overline{\gamma}^{\kappa'},\varepsilon(\bar{\theta}^{\kappa'},\bar{\gamma}^{\kappa'}),\nabla\hat{\varphi}),$$

and

$$\lim_{\kappa\to 0} RF(\overline{\theta}^\kappa,\overline{\gamma}^\kappa,\varepsilon(\bar{\theta}^\kappa,\bar{\gamma}^\kappa),\nabla\hat{\varphi}) = RP.$$

Proof. The monotonicity follows immediately from the fact that $\kappa<\kappa'$ implies that $D_\Theta^{\kappa'}\subset D_\Theta^\kappa$. We note that the monotonicity property implies the existence of the limit

$$\lim_{\kappa\to 0} RF(\overline{\theta}^\kappa,\overline{\gamma}^\kappa,\varepsilon(\bar{\theta}^\kappa,\bar{\gamma}^\kappa),\nabla\hat{\varphi}).$$

Since $D_\Theta^\kappa\subset D_\Theta$ we have:

$$RP \le RF(\bar{\theta}^\kappa,\bar{\gamma}^\kappa,\varepsilon(\bar{\theta}^\kappa,\bar{\gamma}^\kappa),\nabla\hat{\varphi}), \tag{51}$$

for every $\kappa>0$. On the other hand, for a nested family of partitions $\{T_\kappa\}_{\kappa>0}$ and for any given $(\theta,\gamma,\varepsilon(\theta,\gamma))$ in D_Θ, it follows from Theorem 6 that there exists a sequence $\{(\theta^\kappa,\gamma^\kappa,\varepsilon(\theta^\kappa,\gamma^\kappa))\}_{\kappa>0}$ for which:

$$RF(\overline{\theta}^\kappa,\overline{\gamma}^\kappa,\varepsilon(\bar{\theta}^\kappa,\bar{\gamma}^\kappa),\nabla\hat{\varphi}) \le RF(\theta^\kappa,\gamma^\kappa,\varepsilon(\theta^\kappa,\gamma^\kappa),\nabla\hat{\varphi}), \tag{52}$$

and

$$\begin{aligned}\lim_{\kappa\to 0+} RF(\overline{\theta}^\kappa,\overline{\gamma}^\kappa,\varepsilon(\bar{\theta}^\kappa,\bar{\gamma}^\kappa),\nabla\hat{\varphi}) &\le \lim_{\kappa\to 0+} RF(\theta^\kappa,\gamma^\kappa,\varepsilon(\theta^\kappa,\gamma^\kappa),\nabla\hat{\varphi})\\ &= RF(\theta,\gamma,\varepsilon(\theta,\gamma),\nabla\hat{\varphi}).\end{aligned} \tag{53}$$

It is now evident that:

$$\lim_{\kappa\to 0+} RF(\overline{\theta}^\kappa,\overline{\gamma}^\kappa,\varepsilon(\bar{\theta}^\kappa,\bar{\gamma}^\kappa),\nabla\hat{\varphi}) \le \inf_{(\theta,\gamma,\varepsilon(\theta,\gamma))\in D_\Theta} RF(\theta,\gamma,\varepsilon(\theta,\gamma),\nabla\hat{\varphi}) = RP \tag{54}$$

and the theorem follows from (51) and (54).

With Theorems 5 and 7 in hand it is possible to identify a sequence of configurations specified by χ^j for which

$$RP = \lim_{j\to\infty} F(\chi^j,\varepsilon(\chi^j),\nabla\hat{\varphi}). \tag{55}$$

Indeed we consider a minimizing sequence for RP as given by Theorem 7. To each element $(\overline{\theta}^{\kappa}, \overline{\gamma}^{\kappa}, \varepsilon(\bar{\theta}^{\kappa}, \bar{\gamma}^{\kappa}))$ of the sequence we can apply Theorem 5 to find a recovery sequence of configurations $\{\chi^{\kappa,n}\}_{n=1}^{\infty}$. In this way we see that

$$RP = \lim_{\kappa \to \infty} \lim_{n \to \infty} F(\chi^{\kappa,n}, \varepsilon(\chi^{\kappa,n}), \nabla \hat{\varphi}), \tag{56}$$

and it follows that we can extract a sequence of configurations $\{\chi^{\kappa_j,n_j}\}_{j=1}^{\infty}$ for which

$$RP = \lim_{j \to \infty} F(\chi^{\kappa_j,n_j}, \varepsilon(\chi^{\kappa_j,n_j}), \nabla \hat{\varphi}). \tag{57}$$

We now establish the following result.

Theorem 8.

$$P = RP$$

i.e.,

$$\inf_{(\chi,\varepsilon(\chi)) \in ad_\Theta} F(\chi, \varepsilon(\chi), \nabla \hat{\varphi}) = \inf_{(\theta,\gamma,\varepsilon(\theta,\gamma)) \in D_\Theta} RF(\theta, \gamma, \varepsilon(\theta, \gamma), \nabla \hat{\varphi}). \tag{58}$$

Proof. Since $ad_\Theta \subset D_\Theta$ and from (18) it follows that $P \geq RP$. Moreover from Theorem 7 and (57), there exist $\{(\overline{\theta}^{\kappa}, \overline{\gamma}^{\kappa}, \varepsilon(\bar{\theta}^{\kappa}, \bar{\gamma}^{\kappa})\}_{\kappa>0} \in \{D_\Theta^{\kappa}\}_{\kappa>0}$, such that

$$RP = \lim_{\kappa \to 0} RF(\overline{\theta}^{\kappa}, \overline{\gamma}^{\kappa}, \varepsilon(\bar{\theta}^{\kappa}, \bar{\gamma}^{\kappa}), \nabla \hat{\varphi}) = \lim_{j \to \infty} F(\chi^{\kappa_j,n_j}, \varepsilon(\chi^{\kappa_j,n_j}), \nabla \hat{\varphi}). \tag{59}$$

On the other hand

$$F(\chi^{\kappa_j,n_j}, \varepsilon(\chi^{\kappa_j,n_j}), \nabla \hat{\varphi}) \geq P, \text{ for all } j. \tag{60}$$

Thus (59) and (60) imply that $RP \geq P$, and we conclude that $RP = P$.

The results presented in this Section provide the way for the identification of a class of minimizing sequences of configurations of the two conductors. Theorem 7 shows how to generate a minimizing sequence of generalized designs coming from discrete problems. Theorem 5 and (39) provide the methodology for constructing an optimizing sequence of configurations based upon the information given in the solution of the generalized design problem. These results give rigorous rules of thumb for the design of two phase conductors. The numerical implementation is given in Section 6.

5 A complete characterization of minimizing sequences.

In this Section we provide the proof of Theorem 1. We consider a dense class of target fields for which we can account for all oscillations in minimizing sequences of designs. For $\chi \in ad_\theta$ we introduce the set of gradients given by

$$\mathcal{S}_\Theta = \left\{ \begin{array}{l} \nabla u \mid u \text{ is a } W_0^{1,2}(\Omega) \text{ solution of } -\text{div } (\varepsilon(\chi)\nabla u) = f, \\ \chi \in ad_\theta. \end{array} \right. \tag{61}$$

The strong $L^2(\Omega)$ closure of the set $\mathcal{S}_\Theta$ is denoted by the set $\bar{\mathcal{S}}_\Theta$. It is evident that

$$\begin{aligned} P = \inf_{\chi \in ad_\Theta} F(\chi, \varepsilon(\chi), \nabla \hat{\varphi}) &= \inf_{\nabla u \in \mathcal{S}_\Theta} \int_\Omega |\nabla u - \nabla \hat{\varphi}|^2 \, dx \\ &= \inf_{\nabla u \in \bar{\mathcal{S}}_\Theta} \int_\Omega |\nabla u - \nabla \hat{\varphi}|^2 \, dx. \end{aligned} \tag{62}$$

In light of (62) and the definition of $\bar{\mathcal{S}}_\Theta$ we apply Theorem 2 of the introduction to conclude the existence of a G_δ subset K of $W_0^{1,2}(\Omega)$ such that

Theorem 9. Given a target field $\hat{\varphi} \in \mathrm{K}$ and a minimizing sequence $\{(\chi^n, \varepsilon(\chi^n))\}_{n=1}^{\infty} \in ad_{\Theta}$ for P then the associated sequence of state variables $\{\varphi^n\}_{n=1}^{\infty}$ solving the equilibrium equation

$$-\text{div } (\varepsilon(\chi^n)\nabla\varphi^n) = f \tag{63}$$

is Cauchy in the $W_0^{1,2}(\Omega)$ norm given by $||u||^2 = \int_\Omega |\nabla u|^2\, dx$.

From the completeness of $W_0^{1,2}(\Omega)$ there exists a potential $\bar{\varphi} \in W_0^{1,2}(\Omega)$ such that $\lim_{n\to\infty} \varphi^n = \bar{\varphi}$ strongly in $W_0^{1,2}(\Omega)$. Passing to subsequences if necessary, the sequence $\{\chi^n\}_{n=1}^{\infty}$ weak $L^\infty(\Omega)$ **star** converges to a density $\bar{\theta}$ and the compactness property of G-convergence implies that the sequence $\{\varepsilon(\chi^n)\}_{n=1}^{\infty}$ G-converges to an effective tensor ε^e where

$$-\text{div } (\varepsilon^e\nabla\bar{\varphi}) = f. \tag{64}$$

From the results given in [6] and [11] we have that the set of effective tensors associated with the density $\bar{\theta}(x)$ are all the symmetric 2×2 matrices with eigenvalues λ_1, λ_2 lying in the set $\mathrm{K}_{\bar{\theta}}$ for almost all x in Ω. The set $\mathrm{K}_{\bar{\theta}}$ is given by the inequalities

$$\begin{aligned} \sum_{k=1}^{2} \frac{1}{\lambda_j - \alpha} &\leq \frac{1}{h_{\bar{\theta}} - \alpha} + \frac{1}{m_{\bar{\theta}} - \alpha} \\ \sum_{k=1}^{2} \frac{1}{\beta - \lambda_j} &\leq \frac{1}{\beta - h_{\bar{\theta}}} + \frac{1}{\beta - m_{\bar{\theta}}}. \end{aligned} \tag{65}$$

On the other hand the work of J. Dvorak, J. Haslinger, and M. Miettinen [5] shows that the strong convergence of the sequence $\{\varphi^n\}_{n=1}^{\infty}$ delivers the local relation

$$\varepsilon^e\nabla\bar{\varphi} = m_{\bar{\theta}}\nabla\bar{\varphi}, \text{ a.e.} \tag{66}$$

This implies that $m_{\bar{\theta}}$ is an eigenvalue of ε^e. The constraints on the eigenvalues of ε^e given by (65) together with (66) allows us to uniquely identify ε^e as the effective tensor given by

$$\varepsilon^e = R(\bar{\gamma})\Lambda(\bar{\theta})R^T(\bar{\gamma}), \tag{67}$$

where the angle $\bar{\gamma}$ is chosen according to the requirement given by (66). For this choice of angle we also have the local relation

$$R(\bar{\gamma})H(\bar{\theta})R^T(\bar{\gamma})\nabla\bar{\varphi} = 0, \text{ a.e.}, \tag{68}$$

and we conclude that

$$P = \int_\Omega |\nabla\bar{\varphi} - \nabla\hat{\varphi}|^2\, dx = RF(\bar{\theta}, \bar{\gamma}, \varepsilon(\bar{\theta}, \bar{\gamma}), \nabla\hat{\varphi}). \tag{69}$$

In view of Theorem 8 we deduce that the design $(\bar{\theta}, \bar{\gamma}, \varepsilon(\bar{\theta}, \bar{\gamma}))$ is the optimal design for the problem RP. This establishes parts (1), (2), and (4) of Theorem 1.

To proceed we recall the notion of a cluster point (θ, ε^e) for a sequence of designs associated with a sequence configurations $\{\chi^n\}_{n=1}^{\infty}$. The definition of a cluster point (θ, ε^e) implies the existence of a subsequence $\{\chi^{n_j}, \varepsilon(\chi^{n_j})\}_{j=1}^{\infty}$ such that $\{\chi^{n_j}\}_{j=1}^{\infty}$ weak $L^\infty(\Omega)$ **star** converges to θ and $\{\varepsilon(\chi^{n_j})\}_{j=1}^{\infty}$ G-converges to ε^e. Arguments identical to those given above show that any cluster point of any minimizing sequence for the problem P is a minimizing design for the problem RP. This establishes the first part of (3) of Theorem 1. The second part of (3) of Theorem 1 follows immediately from the construction of a recovery sequence of configurations based upon Theorems 5 and 7, see equations (57).

Part (3) of Theorem 1 together with (66) and (68) point out what kinds of oscillations can occur in minimizing sequences of configurations. In the subregion of Ω where the minimizing sequence oscillates, i.e., the region where $0 < \bar{\theta} < 1$, we see that the oscillations are in the form of layers of the two conductors. The layers are asymptotically parallel to the optimal gradient $\nabla\bar{\varphi}$. This configuration allows for the best effective conductivity properties to be aligned with the direction of the gradient. This is consistent with physical intuition.

6 Numerical solution and a practical approach to design of graded materials.

We provide an outline of the method used for the numerical solution of the discrete design problem. For convenience the objective functional is denoted by $E(\theta,\gamma)$ and

$$\begin{aligned} E(\theta,\gamma) &= RF(\theta,\gamma,\varepsilon(\theta,\gamma),\nabla\hat{\varphi}) \\ &= \int_\Omega |\nabla\varphi-\nabla\hat{\varphi}|^2\,dx + \int_\Omega R(\gamma(x))H(\theta(x))R^T(\gamma(x))\nabla\varphi\cdot\nabla\varphi\,dx, \end{aligned} \tag{70}$$

where the state variable φ solves the equilibrium equation (12). For a given partition, the number of subdomains is $N(\kappa)$ and the design variable (θ,γ) is a vector of length $2N(\kappa)$. The components of (θ,γ) are the constant values (θ_i,γ_i) taken in each subdomain Ω_i. The components of the design vector are subject to the box constraints:

$$\begin{aligned} &0\le\theta_i\le 1,\ i=1\ldots,N(\kappa), \\ &0\le\gamma_i\le 2\pi,\ i=1\ldots,N(\kappa). \end{aligned} \tag{71}$$

We include the resource constraint $\int_\Omega \theta\,dx \le \Theta\, meas(\Omega)$ by adding a penalty term

$$\ell\times\left(\int_\Omega \theta\,dx - \Theta\, meas(\Omega)\right),$$

for $\ell\ge 0$. The discrete design problem is written

$$\min_{(\theta,\gamma)} E(\theta,\gamma)+\ell\times\left(\int_\Omega \theta\,dx - \Theta\, meas(\Omega)\right), \tag{72}$$

where (θ,γ) are subject to the constraints given by (71). The numerical procedure is a straight forward application of the steepest decent method, see [17]. Gradients of the objective $E(\theta,\gamma)$ are computed and increments of the design variables $(\delta\theta,\delta\gamma)$ are chosen to insure $E(\theta,\gamma)\ge E(\theta+\delta\theta,\gamma+\delta\gamma)$. The advantage of this procedure is that it is monotone and convergence is assured.

We provide numerical examples that illustrate how electrostatic fields can be controlled using functionally graded materials. For all examples the design domain is chosen to be the square centered at the origin given by $\Omega=(-1,1)\times(-1,1)$ and we choose the target field to be zero, i.e., $\nabla\hat{\varphi}=(0,0)$. The discrete design is associated with a partition of Ω into 20,000 subdomains of diameter on the order of 10^{-2}.

For the first two examples the charge distribution is taken to be uniform in Ω and given by $f=1$. We choose $\alpha=1$ and $\beta=2$ and constrain the amount of good dielectric to be 40% of the design domain. The density distribution, $\theta(x)$, of the better dielectric material in the optimized discrete design is given in Figure 1a. Here the darkest regions consist of pure β dielectric, the white regions are occupied by pure α dielectric and the regions of graded conductivity properties are given by the intermediate shades. The layer normals in the graded parts of the design are given by the arrows in Figure 1a. The contours are the level lines of the electric potential. Note that the layer normals are tangential to the level lines, hence perpendicular to the electric field. We emphasize that Figure 1a gives the necessary geometric information for manufacturing graded materials. Indeed, given $\theta(x),\gamma(x)$ we can we apply (39) to construct a sequence of graded materials. Because of the continuity expressed by Theorem 5 we are guaranteed that we can construct a two phase configuration thats nearly optimal.

For the second example we consider a subdomain D of the design domain Ω. Here we take $D=\Omega\setminus\{(-1/2,1/2)\times(-1/2,1/2)\}$. We consider the problem

$$P=\inf_{\chi\in ad_\Theta}\int_D |\nabla\varphi|^2\,dx. \tag{73}$$

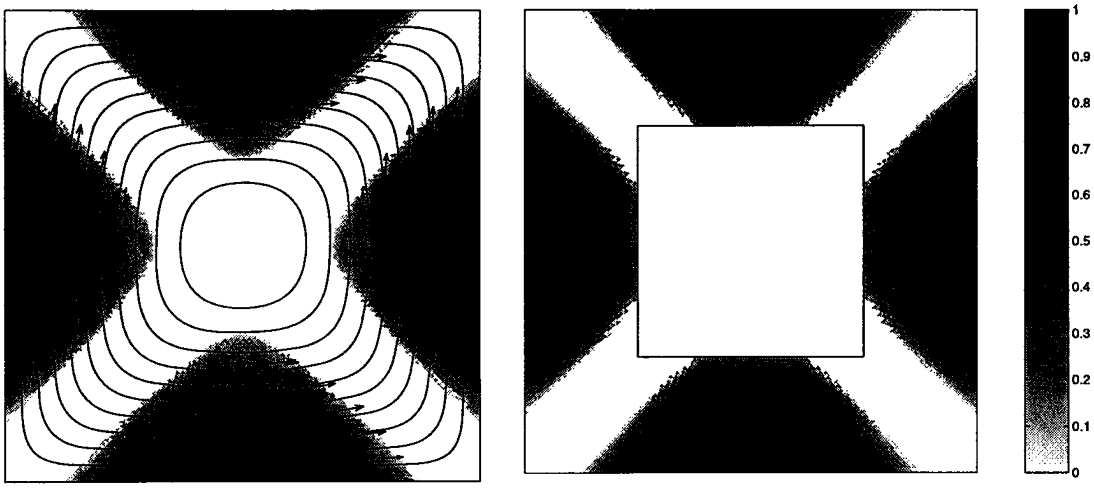

Figure 1: a.

Figure 1: b.

The theory presented in this paper easily generalizes to this case and the relaxed problem is

$$
\begin{aligned}
RP &= \inf_{(\theta,\gamma,e(\theta,\gamma))\in D_\Theta} \left\{ \int_D |\nabla\varphi|^2\,dx \right. \\
&+ \left. \int_D R(\gamma(x))H(\theta(x))R^T(\gamma(x))\nabla\varphi\cdot\nabla\varphi\,dx \right\}, \qquad (74)
\end{aligned}
$$

and $P = RP$.

Here the goal is to screen as much electric field away from the domain D as possible. The good dielectric is constrained to occupy 40% of Ω. The density distribution of the good dielectric in the optimal design is given in Figure 1b. We point out that we allow the two dielectrics to be placed anywhere in Ω, however the algorithm automatically uses the good dielectric only in D. This is consistent with intuition.

For the next example we take the charge distribution to be 1 everywhere outside of D and zero inside D. As before we take $\alpha = 1$ and $\beta = 2$.

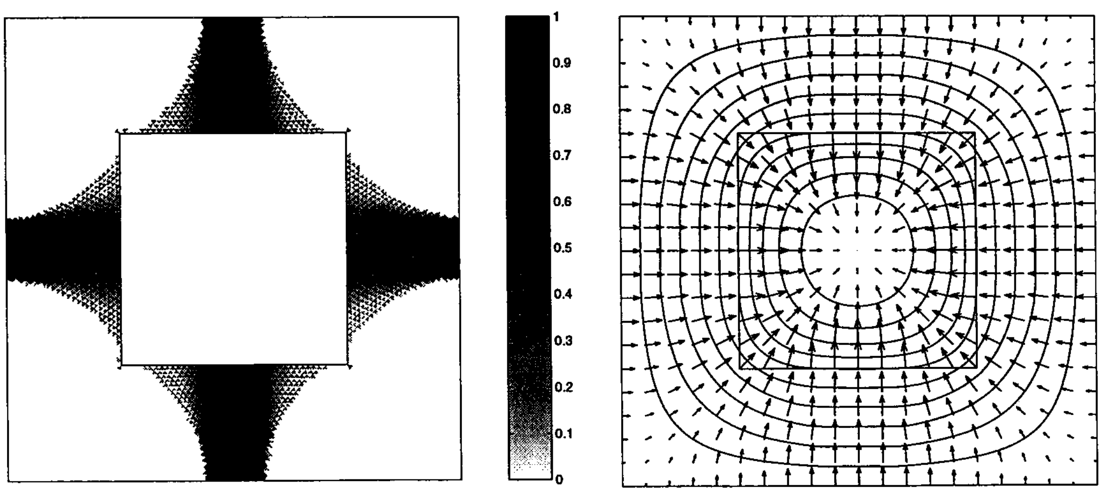

Figure 2: a.

Figure 2: b.

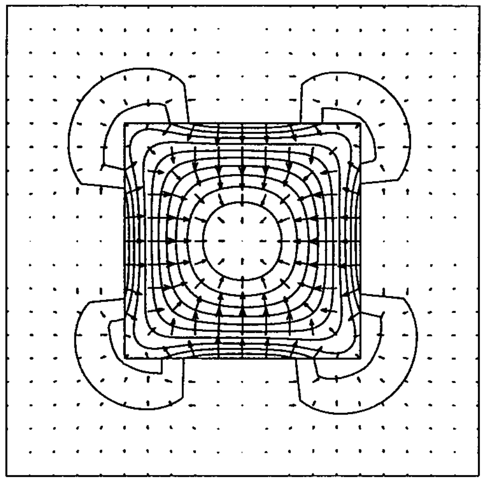

Figure 3: Electric field for $\beta = 1000$.

The good dielectric is constrained to occupy 15% of the design domain. The density distribution for the optimal design is given in Figure 2a. In Figure 2b we plot the level lines of the potential and the electric field associated with the design. Last we consider the same layout as in Figure 2a but with $\alpha = 1$ and $\beta = 1000$ and we plot the electric field for this case in Figure 3. For this layout and choice of β we see that the electric field has been screened away from D.

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References

[1] Bensoussan A., Lions J. L., and Papanicolaou G. *Asymptotic Analysis for Periodic Structures,* Studies in Mathematics and its Applications, **5**, (1978) North Holland, Amsterdam.

[2] Bidaut M. *Théorèms d'existence et d'existence en général d'un contrôle optimal pour des systemes régis par des équations aux dérivées partielles non linéaires,* PhD thesis, Université Paris VI, June 1973.

[3] De Giorgi E. and Spagnolo S. Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine, *Boll. U.M.I.*8 (1973), pp. 391–411.

[4] Edelstein M. On nearest points of sets in uniformly convex Banach spaces, *J. London Math. Soc.,* **43** (1968), pp. 375–377.

[5] Dvorak J., Haslinger J., and Miettinen M. On the Problem of Optimal Material Distribution, Preprint, (1996).

[6] Lurie K. A. and Cherkaev A. V. Exact estimates of conductivity of composites formed by two isotropically conducting media taken in prescribed proportion, *Proc. Roy. Soc. Edinburgh,* **99** A (1984), pp. 71–87.

[7] Murat F. Compacité par compensation, *Ann. Sc. Norm. Sup. Pisa.* **5** (1978), pp. 489–507.

[8] Murat F. and Tartar L. On the control of coefficients in partial differential equations. In Topics in the Mathematical Modeling of Composite Materials, Cherkaev A. and Kohn R. editors. Berkhauser, Boston, (1997), pp. 1–8.

[9] Murat F. and Tartar L. H-convergence. In Topics in the Mathematical Modeling of Composite Materials, Cherkaev A. and Kohn R. editors. Berkhauser, Boston, (1997) pp. 21–43.

[10] Murat F. and Tartar L. Calcul des variations et homogénéisation. In Les Méthodes de l'Homogénéisation: Théorie et Applications en Physique. Collection de la Direction des Etudes et Recherches d'Electricité de France, **57**, Eyrolles, Paris, (1985), pp. 319-369.

[11] Tartar L. Estimations fines de coefficients homogénéisés. In Ennio De Giorgi Colloquium, ed. by P. Kree, Research Notes in Mathematics **125**, Pitman, London 1985, pp. 168–187.

[12] Tartar L. Compensated compactness and applications to partial differential equations. In: *Nonlinear Analysis and Mechanics, Heriot-Watt Symposium,* Volume IV, R.J. Knops, ed., Research Notes in Mathematics, **39**. Pitman, Boston 1979, pp. 136–212.

[13] Tartar L. Remarks on optimal design problems. In Calculus of Variations, Homogenization and Continuum Mechanics, Buttazzo G., Bouchitte G., and Suquet P. editors., World Scientific, Singapore, (1994), pp. 279–296.

[14] Tartar L. H-measures, a new approach for studying homogenization, oscillations and concentration effects in partial differential equations, *Proc. Roy. Soc. Edinb.*, **115A**, (1990), pp. 193–230.

[15] Pedregal P. Optimal design and constrained quasiconvexity, Preprint, (1998).

[16] Pedregal P. Optimization, Relaxation and Young Measures, *Bull. Amer. Math. Soc.*, **36**, (1999), pp. 27–58.

[17] Pironneau O. *Optimal Shape Design for Elliptic Systems.* Springer Verlag, New York 1984.

[18] Spagnolo S. Convergence in energy for elliptic operators. In *Proceedings of the Third Symposium on Numerical Solutions of Partial Differential Equations*, (College Park,1975), ed. by Hubbard B., Academic Press, New York, pp. 496–498.