Proceedings of the Royal Society of Edinburgh, **124A**, 1119–1134, 1994

Homogenisation of two-phase emulsions

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(MS received 27 April 1993. Revised MS received 1 August 1993)

We consider an emulsion of two Stokes fluids, one of which is periodically distributed in the form of small spherical bubbles. The effects of surface tension on the bubble boundaries are modelled mathematically, as in the work of G. I. Taylor, by a jump only in the normal component of the traction. For a given volume fraction of bubbles, we consider the two-scale convergence, and in the fine phase limit we find that the bulk flow is described by an anisotropic Stokes fluid. The effective viscosity tensor is consistent with the bulk stress formula obtained by Batchelor [2].

1. Introduction

In this paper we provide a rigorous framework for demonstrating the dependence of the bulk properties of an emulsion on its microstructure.

We consider flows of suspensions of n fluid drops in a second fluid. In such flows, the velocities of the drops must be determined simultaneously with the flow. For the case of fixed drops, the problem reduces to the one studied in [5]. The bubbles are assumed to be small with respect to macroscopic length scales. We model the effects of surface tension on the bubble boundary using the zeroth order approximation introduced by Taylor [10]. In this approximation the bubbles are assumed spherical and only the normal component of the traction is allowed to jump at the bubble interface.

We suppose that at the initial time the bubbles are periodically distributed in the emulsion. The scale of the period is assumed to be of the same order as the bubble diameter. Since the characteristic length scales of the body force and flow region are much greater than the local period, it follows that periodicity is preserved in the flow.

It is assumed that the flow is quasistatic and satisfies the time stationary Stokes equations at each instant. Under these hypotheses the flow equations of the emulsion are given by those derived by Keller, Rubenfeld and Molyneux [4].

We show that as the period of the suspension approaches zero, the associated family of flows approaches a *homogenised* flow with velocity field satisfying the stationary Stokes equation, where the constitutive relation is given by an anisotropic viscosity that depends upon the microscopic geometry.

The emulsion considered here is given by a simple cubic lattice of spherical drops. The associated effective viscosity is therefore cubically symmetric and of the form:

$$2\mu^H_{ijkl} = 2\mu^S \mathbf{P}^S_{ijkl} + 2\mu^D \mathbf{P}^D_{ijkl}, \tag{1.1}$$

where $\mathbf{P}^S$ is the projection onto off diagonal strain rates and $\mathbf{P}^D$ is the projection onto diagonal trace free strain rates (see [**12**]). We remark that our homogenisation result applies immediately to arbitrary lattice geometries.

The paper is outlined as follows. In Section 4 we normalise the pressure in order to get a uniform $L^2(\Omega)$ estimate for the sequence of pressures. Global and local conservation of mass equations are obtained for the homogenised velocity and its first corrector in Section 5. In Section 6 we obtain the local kinematic condition on the bubble interface by identifying the two scale limit of a suitably normalised velocity field with respect to the local bubble velocity. The two-scale convergence method is applied to the momentum balance equations to obtain the local balance laws for the homogenised stress (see Lemma 7.1). These results, together with those in Sections 5 and 6, give the local problem. In Section 8 we obtain the homogenised momentum equation and the formula for the effective viscosity. Unlike problems with continuity of the traction across phases, the effective property for this problem contains a term encoding the effects of the work done against the bubble boundary due to viscous forces. The effective viscosity obtained here is consistent with the bulk stress formula obtained by Batchelor [**2**] (see [**6**]). Variational formulation and bounds for the effective viscosity are also given in [**6**].

The work here constitutes the rigorous proof of the two-scale asymptotic expansions given in [**6**].

2. Formulation

We consider a bounded domain Ω in $\Re^3$, containing an emulsion of two fluids. The viscosities of the bubbles and of the surrounding fluid are μ_1 and μ_2, respectively, with $0<\mu_1<\mu_2$.

The local fluid velocity is denoted by $v(x)$. We consider the local strain rate tensor $e(v)=(\nabla v+\nabla v^T)/2$ and the local stress tensor $\sigma=2\mu e(v)-pI$, where p is the local pressure and:

$$\mu=\begin{cases}\mu_1 & \text{in the bubbles,}\\ \mu_2 & \text{in the continuous fluid phase.}\end{cases} \tag{2.1}$$

For a prescribed body force f, the equations of motion in each phase are:

$$\operatorname{div}\sigma+f=0 \tag{2.2}$$

and the incompressibility condition is:

$$\operatorname{div} v=0. \tag{2.3}$$

On the boundary of Ω a no-slip condition is imposed.

Following Taylor [**10**] and others [**3, 8**], we assume that the fluid velocity is continuous across the bubble surfaces. For a suspension of n bubbles, we denote the velocity of the centre of mass of the ith bubble, $1\leqq i\leqq n$, by V^i. The kinematic condition on the bubble surface Γ^i is given by:

$$v\cdot n=V^i\cdot n. \tag{2.4}$$

The associated dynamic condition is given by:

$$[\sigma n]=[\sigma n]\cdot nn, \tag{2.5}$$

where n is the exterior unit normal to the bubble surface Γ_i. Here the notation [ ] denotes the jump of the bracketed quantity across the bubble surface.

The balance of forces on each bubble B^i is given by:

$$\int_{B^i} f\,dx + \int_{\Gamma^i} \sigma n\,ds = 0. \tag{2.6}$$

Here the surface integral of the normal stress is evaluated on the exterior of the bubble. The above condition is easily seen from equations (2.2) and (2.5) to be equivalent to:

$$\int_{\Gamma^i} [\sigma n]\,ds = 0. \tag{2.7}$$

The balance of torque on each bubble B^i is given by:

$$\int_{B^i} x \times f\,dx + \int_{\Gamma^i} x + \sigma n\,ds = 0. \tag{2.8}$$

In view of (2.2), (2.5) and (2.7) and the fact that the bubbles are spherical, equation (2.8) is automatically satisfied.

The flow problem is to simultaneously find the flow v, pressure p and the bubble velocities V^i, $i = 1, \dots, n$ satisfying (2.1)–(2.6). We remark that the problem given by (2.1)–(2.6) is a specialisation of the suspension flow problem formulated in [4], to emulsions.

We observe that the bubble velocities are related to the flow by the following:

$$V^i = \frac{1}{|B^i|} \int_{B^i} v\,dx. \tag{2.9}$$

This follows immediately from (2.4) and the identity:

$$\int_{B^i} v_j\,dx = \int_{\Gamma^i} v \cdot n x_j\,ds, \tag{2.10}$$

which holds for divergence-free flows.

3. Homogenisation result

We suppose that at some instant in time the emulsion is periodic, with the ratio between the period and the characteristic length of the domain given by ε. We consider a unit periodic reference emulsion of bubbles B^i with centres specified by the vectors r^i, such that r^0 coincides with the origin. The bubbles of the ε-periodic emulsion are denoted by $B^{i\varepsilon}$ and their centres are given by εr^i. Thus the coordinates of a point in the emulsion will be given by:

$$x = \varepsilon r^i + \varepsilon y, \tag{3.1}$$

with $y \in (-\frac{1}{2}, \frac{1}{2})^3$.

The emulsion is equivalently characterised by an ε-periodic viscosity μ^ε given by:

$$\mu^\varepsilon = \mu\left(\frac{x}{\varepsilon}\right), \quad \text{where } \mu(y) = \mu_1 \chi_1(y) + \mu_2 \chi_2(y), \tag{3.2}$$

where χ_1 and χ_2 are the characteristic functions of the bubble and of the surrounding fluid in the unit period cell $Q=(-\frac{1}{2},\frac{1}{2})^3$.

We consider the associated family of emulsion flow problems with solutions v^ε, p^ε, $V^{i\varepsilon}$ satisfying:

$$\begin{aligned}
&\operatorname{div}\sigma^\varepsilon+f=0 && \text{in } \Omega-\cup_i\partial B^{i\varepsilon},\\
&\sigma^\varepsilon=2\mu^\varepsilon e(v^\varepsilon)-p^\varepsilon I && \text{in } \Omega,\\
&\operatorname{div}v^\varepsilon=0 && \text{in } \Omega,\\
&[v^\varepsilon]=0 && \text{on } \Gamma^{i\varepsilon}=\partial B^{i\varepsilon},\\
&v^\varepsilon\cdot n=V^{i\varepsilon}\cdot n && \text{on } \Gamma^{i\varepsilon},\\
&[\sigma^\varepsilon n]=[\sigma^\varepsilon n]\cdot nn && \text{on } \Gamma^{i\varepsilon},\\
&\int_{\Gamma^{i\varepsilon}}[\sigma^\varepsilon n]\,ds=0 &&\\
&v^\varepsilon=0 && \text{on } \partial\Omega.
\end{aligned} \tag{3.3}$$

We observe that the pressure can be adjusted by a constant in each bubble and still satisfy (3.3).

To obtain the asymptotic behaviour of the flow, we define the following local problem:

$$\begin{gathered}
\operatorname{div}_y\tau^{ij}=0,\\
\operatorname{div}_y v^{ij}=0,\\
[v^{ij}]=0,\\
(\tfrac{1}{2}(\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl}-\tfrac{2}{3}\delta_{ij}\delta_{lm})y_l+v^{ij}_m)n_m=\frac{1}{|B|}\left(\int_B v^{ij}_m\,dy\right)\cdot n_m,\\
[\tau^{ij}n]=[\tau^{ij}n]\cdot nn,\\
\int_{\partial B}[\tau^{ij}n]\,ds_y=0,
\end{gathered} \tag{3.4}$$

where:

$$\tau^{ij}_{lm}=2\mu(\tfrac{1}{2}(\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl})-\tfrac{1}{3}\delta_{ij}\delta_{lm}+e_{lm}(v^{ij}))-p^{ij}\delta_{lm}$$

and

$$\int_Q p^{ij}\,dy=0.$$

Here v^{ij} is a Q-periodic vector field, and p^{ij} is normalised such that its average is zero.

We introduce a security region S inside the unit period containing the bubble B, and denote by B_+ the set:

$$B_+=S-\bar{B}. \tag{3.5}$$

The region homothetic to B_+ about the bubble $B^{i\varepsilon}$ is denoted by $B^{i\varepsilon}_+$.

We introduce a normalised pressure field $\bar{p}^\varepsilon$ by subtracting off a constant pressure $C^{i\varepsilon}$ inside each bubble given by:

$$C^{i\varepsilon} = \frac{1}{|B^{i\varepsilon}|}\int_{B^{i\varepsilon}} p^\varepsilon\, dx - \frac{1}{|B^{i\varepsilon}_+|}\int_{B^{i\varepsilon}_+} p^\varepsilon\, dx. \tag{3.6}$$

From earlier remarks, the normalised pressure also satisfies the emulsion flow problem (3.3). For each ε, the constant $C^{i\varepsilon}$ is a measure of the difference between the average pressures of the fluid inside and outside the bubble. We show in Lemma 4.5 that the normalised pressure is uniformly bounded in $L^2(\Omega)$.

The asymptotic behaviour of the flow is described by the following homogenisation result:

THEOREM 3.1. *As ε tends to zero, we have: for any body force f in $H^{-1}(\Omega)$, the sequence of flow fields and pressures $(v^\varepsilon, \bar{p}^\varepsilon)$ converges weakly in $H^1_0(\Omega)^3 \times L^2(\Omega)$ to (v^0, q) satisfying the* ***homogenised*** *flow equation given by:*

$$\begin{aligned} &\operatorname{div} \sigma^H + f = 0 && \text{in } \Omega,\\ &\sigma^H_{ij} = 2\mu^H_{ijkl} e_{kl}(v^0) - q\delta_{ij} && \text{in } \Omega,\\ &\operatorname{div} v^0 = 0 && \text{in } \Omega,\\ &v^0 = 0 && \text{on } \partial\Gamma, \end{aligned} \tag{3.7}$$

where the effective viscosity μ^H_{ijkl} is defined by:

$$2\mu^H_{lkij} = \int_Q \left(\tau^{ij}_{lk} - \tfrac{1}{3}\delta_{lk}\tau^{ij}_{pp}\right) dy - \int_\Gamma \left([\tau^{ij}_{lm} n_m] y_k - \tfrac{1}{3}[\tau^{ij}_{pm} n_m] y_p \delta_{lk}\right) ds, \tag{3.8}$$

with $\mu^H_{ijkl} = \mu^H_{jikl} = \mu^H_{klij}$.

It is shown in [6] that the effective viscosity delivered by homogenisation is equivalent to the bulk stress formula obtained using formal averaging procedures of the type given in [2].

4. Pressure extension and estimates

The emulsion flow problem (3.3) has the equivalent variational formulation: for $f \in L^2(\Omega)$, find $v^\varepsilon \in \mathbf{V}^\varepsilon$ such that:

$$\int_\Omega 2\mu^\varepsilon e(v^\varepsilon) : e(w)\, dx = \int_\Omega f w\, dx, \quad \text{for any } w \in \mathbf{V}^\varepsilon, \tag{4.1}$$

where $\mathbf{V}^\varepsilon$ is the closed subspace of $(H^1_0(\Omega))^3$ given by:

$$\mathbf{V}^\varepsilon = \left\{ w \in (H^1_0(\Omega))^3 \mid \operatorname{div} w = 0 \text{ in } \Omega,\ w \cdot n = W^{i\varepsilon} \cdot n \text{ on } \Gamma^{i\varepsilon},\ W^{i\varepsilon} = \frac{1}{|B^{i\varepsilon}|}\int_{B^{i\varepsilon}} w\, dx \right\}. \tag{4.2}$$

We remark that the last equation in (4.2) is a compatibility condition for a divergence-free vector field with normal component constant on $\Gamma^{i\varepsilon}$.

The existence and uniqueness of the solution of the emulsion flow problem follows from the Lax–Milgram lemma.

It also follows immediately from (4.1) that the velocity field is bounded uniformly in ε:

$$\| v^{\varepsilon} \|_{H_0^1(\Omega)} < C. \tag{4.3}$$

The pressure gradients delivered by (4.1) are linear functionals on subspaces of $H_0^1(\Omega)$. Following the ideas of Tartar [9], we construct an extension for the pressure gradient as a linear functional on $H_0^1(\Omega)$. We show as in [5] that the extension is equivalent to a suitable normalisation of pressures.

LEMMA 4.1. *There exists a restriction operator*:

$$R^{\varepsilon} \in \mathscr{L}[(H_0^1(\Omega))^3; H^{\varepsilon}] \tag{4.4}$$

satisfying: (4.5)

(1) $R^{\varepsilon}u = u$, *for any* $u \in H^{\varepsilon}$;

(2) *if* $\operatorname{div} u = 0$ *then* $\operatorname{div} R^{\varepsilon}u = 0$;

(3) $\varepsilon \| \nabla R^{\varepsilon} u \|_{L^2(\Omega)} + \| R^{\varepsilon} u \|_{L^2(\Omega)} \leqq C(\varepsilon \| \nabla u \|_{L^2(\Omega)} + \| u \|_{L^2(\Omega)})$.

Here:

$$H^{\varepsilon} = \{ w \in (H_0^1(\Omega))^3 \mid w \cdot n = W^{i\varepsilon} \cdot n \text{ on } \Gamma^{i\varepsilon} \}.$$

Proof. Let us introduce for any $u \in H^1(\Omega)$, $v^+ \in H^1(B_+)$, $q^+ \in L^2(B_+)$ and $v \in H^1(B)$, $q \in L^2(B)$ solutions of the following nonhomogeneous Stokes problems:

$$\begin{aligned}
&\Delta v^+ = \Delta u - \nabla q^+ && \text{in } B_+, \\
&\operatorname{div} v^+ = \operatorname{div} u + C^+ && \text{in } B_+, \\
&v^+ = u && \text{on } \Gamma^+ = \partial S, \\
&v^+ \cdot n = V \cdot n && \text{on } \Gamma, \\
&v^+ \cdot \tau = u \cdot r && \text{on } \Gamma;
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
&\Delta v = \Delta u - \nabla q && \text{in } B, \\
&\operatorname{div} v = \operatorname{div} u + C && \text{in } B, \\
&v \cdot n = V \cdot n && \text{on } \Gamma, \\
&v \cdot \tau = u \cdot \tau && \text{on } \Gamma,
\end{aligned} \tag{4.7}$$

where τ is any tangent vector to Γ and

$$V = \frac{1}{|B|} \int_B (u + (\operatorname{div} u)x)\, dx, \tag{4.8}$$

$$C^+ = \frac{1}{|B_+|} \int_\Gamma u \cdot n \, ds, \tag{4.9}$$

$$C = -\frac{1}{|B|} \int_\Gamma u \cdot n \, ds. \tag{4.10}$$

The existence and uniqueness of the solutions for (4.5) and (4.6) follow from general results for nonhomogeneous Stokes problems (see [11]).

We define the restriction operator on $(H^1(Q))^3$:

$$R(u)=\begin{cases} u & \text{in } Q-S,\\ v^+ & \text{in } B_+,\\ v & \text{in } B,\end{cases}$$

and applying R to every εQ period we define R^ε.

The first property of the restriction follows from the uniqueness of the solutions of (4.5) and (4.6) and from the fact that, for any $u\in H^1(Q)$ with $u\cdot n=U\cdot n$ on Γ, we have that $V=U$.

Since the normal component of Ru is continuous across Γ_+ and Γ, and $\operatorname{div} u=0$, the second property follows immediately from (4.8) and (4.9).

From standard trace and lift estimates, we obtain:

$$\|R(u)\|_{H^1(Q)}\leqq C\|u\|_{H^1(Q)}$$

and the third property is obtained by rescaling. □

Substitution of smooth test fields in $\mathbf{V}^\varepsilon$ with support in each phase in (4.1) delivers pressure fields p^ε in each phase such that:

$$\operatorname{div}(2\mu_2 e(v^\varepsilon)-p^\varepsilon I)=f \text{ in } H^{-1}(\Omega-\cup_i\bar{B}^{i\varepsilon}), \tag{4.11}$$

$$\operatorname{div}(2\mu_1 e(v^\varepsilon)-p^\varepsilon I)=f \text{ in } H^{-1}(B^{i\varepsilon}). \tag{4.12}$$

DEFINITION 4.2. The normalised pressure $\bar{p}^\varepsilon$ is defined by:

$$\bar{p}^\varepsilon=p^\varepsilon-\sum_i C^{i\varepsilon}\chi_{B^{i\varepsilon}}, \tag{4.13}$$

where the constants $C^{i\varepsilon}$ are given by:

$$C^{i\varepsilon}=-\frac{1}{|B_+^{i\varepsilon}|}\int_{B_+^{i\varepsilon}}p^\varepsilon\,dx+\frac{1}{|B^{i\varepsilon}|}\int_{B^{i\varepsilon}}p^\varepsilon\,dx. \tag{4.14}$$

LEMMA 4.3. *The normalised pressure gradient* $\nabla\bar{p}^\varepsilon$ *is in* $H^{-1}(\Omega)$ *and satisfies*

$$\int_\Omega \bar{p}^\varepsilon \operatorname{div} u\,dx=\int_\Omega \bar{p}^\varepsilon \operatorname{div} R^\varepsilon(u)\,dx=\int_\Omega p^\varepsilon \operatorname{div} R^\varepsilon(u)\,dx \tag{4.15}$$

for any $u\in H_0^1(\Omega)$.

Proof. Indeed:

$$\begin{aligned}\int_\Omega(\bar{p}^\varepsilon \operatorname{div} u-\bar{p}^\varepsilon \operatorname{div} R^\varepsilon(u))\,dx&=\sum_i\left(\int_{B_+^{i\varepsilon}}p^\varepsilon(\operatorname{div} u-\operatorname{div} v^+)\,dx\right.\\&\quad\left.+\int_{B^{i\varepsilon}}(p^\varepsilon+C^{i\varepsilon})(\operatorname{div} u-\operatorname{div} v)\,dx\right)\\&=\sum_i\left(-\int_{B_+^{i\varepsilon}}p^\varepsilon C^+\,dx-\int_{B^{i\varepsilon}}(p^\varepsilon+C^{i\varepsilon})C\,dx\right)=0.\end{aligned}$$

Here the last equality follows from (4.8), (4.9) and (4.14).

The second equality in (4.15) is a direct consequence of the definition of the normalised pressure and the restriction operator. □

We denote by $\bar{\sigma}^\varepsilon$ the stress associated with the normalised pressure (see Definition 4.2) i.e.:

$$\bar{\sigma}^\varepsilon = -\bar{p}^\varepsilon I + 2\mu^\varepsilon e(v^\varepsilon).$$

REMARK 4.4. Since the solution of the emulsion flow problem is unique up to a constant pressure in each phase, we see that we may replace σ^ε by $\bar{\sigma}^\varepsilon$ in (3.3).

We are now in a position to estimate the normalised pressure.

LEMMA 4.5. *The normalised pressure is uniformly bounded in* $L^2(\Omega)$:

$$\|\bar{p}^\varepsilon\|_{L^2(\Omega)|\mathfrak{R}} \leqq C. \tag{4.16}$$

Proof. We first estimate $\varepsilon\bar{p}^\varepsilon$ and then use this to prove the uniform bound on the sequence $\bar{p}^\varepsilon$.

Indeed, if follows from the emulsion flow problem (3.3) and its variational formulation (4.1) that

$$-\text{div}\, \bar{\sigma}^\varepsilon = f \tag{4.17}$$

as regular distributions in each phase, thus for u in $(H_0^1(\Omega))^3$ we have

$$(\text{div}\, \bar{\sigma}^\varepsilon, R^\varepsilon(u)) = -(f, R^\varepsilon(u)). \tag{4.18}$$

Integrating by parts using an adequate Stokes formula (cf. [11]) yields:

$$\int_\Omega p^\varepsilon \,\text{div}\, R^\varepsilon(u)\, dx = \int_\Omega 2\mu^\varepsilon e(v^\varepsilon) : e(R(u)) - \int_\Omega f \cdot R(u). \tag{4.19}$$

Combining Lemma 4.3 with (4.19) and the estimate $(4.5)_3$, we find that the normalised pressure gradient satisfies

$$\|\nabla \bar{p}^\varepsilon\|_{H^{-1}(\Omega)} \leqq \varepsilon^{-1} C. \tag{4.20}$$

Thus from [11] it follows that

$$\|\varepsilon\bar{p}^\varepsilon\|_{L^2(\Omega)|\mathfrak{R}} \leqq C. \tag{4.21}$$

Multiplying (3.3) by $w^\varepsilon\varphi$ where $\varphi \in \mathscr{D}(\Omega)$, and $w^\varepsilon(x) = w(x/\varepsilon)$ with $w \in (H^1_{\text{per}}(Q))^3$ and integrating by parts, we obtain the identity:

$$\int_\Omega \bar{\sigma}^\varepsilon : e(w^\varepsilon\varphi)\, dx - \sum_i \int_{\Gamma^{i\varepsilon}} [\bar{\sigma}^\varepsilon n] \cdot w^\varepsilon\varphi\, ds = \int_\Omega f w^\varepsilon \varphi\, dx. \tag{4.22}$$

Here the integral over $\Gamma^{i\varepsilon}$ is understood in the sense of traces: $\bar{\sigma}^\varepsilon n \in (H^{-\frac{1}{2}}(\Gamma^{i\varepsilon}))^3$ and $w^\varepsilon\varphi \in (H^{\frac{1}{2}}(\Gamma^{i\varepsilon}))^3$. Choosing $w \cdot n = 0$ on Γ, it follows from $(3.3)_6$ that the second term in (4.22) vanishes and multiplication of the result by ε gives:

$$\int_\Omega \bar{p}^\varepsilon(\text{div}_y\, w)^\varepsilon \varphi\, dx = -\varepsilon \int_\Omega f w^\varepsilon \varphi\, dx + \int_\Omega 2\mu^\varepsilon e(v^\varepsilon) : ((e_y w)^\varepsilon \varphi - \varepsilon w^\varepsilon \cdot \nabla\varphi)\, dx$$
$$- \int_\Omega \varepsilon\bar{p}^\varepsilon w^\varepsilon \cdot \nabla\varphi\, dx, \tag{4.23}$$

where $(e_y w)^\varepsilon = (e_y w)(x/\varepsilon)$ and $(\mathrm{div}_y\, w)^\varepsilon = (\mathrm{div}_y\, w)(x/\varepsilon)$. It now follows from the uniform $L^2(\Omega)$ bound on $\varepsilon\bar{p}^\varepsilon$ and (4.3) that the right-hand side of (4.23) is uniformly bounded. Therefore:

$$\left| \int_\Omega \bar{p}^\varepsilon (\mathrm{div}_y\, w)^\varepsilon \varphi \, dx \right| < C, \tag{4.24}$$

for all $\varphi \in \mathscr{D}(\Omega)$, $w \in (H^1_{\mathrm{per}}(Q))^3$, $w \cdot n = 0$ on Γ. This is equivalent to:

$$\left| \int_\Omega \bar{p}^\varepsilon s^\varepsilon \varphi \, dx \right| < C, \tag{4.25}$$

for all $\varphi \in \mathscr{D}(\Omega)$, $s \in L^2_{\mathrm{per}}(Q)$, $\int_B s\, dy = \int_{Q-B} s\, dy = 0$. By a density argument, for ε fixed, we may choose $s^\varepsilon\varphi$ to approximate $\bar{p}^\varepsilon$ and obtain the estimate:

$$\| \bar{p}^\varepsilon \|_{L^2(\Omega)|\mathbb{R}} \leqq C. \quad \square \tag{4.26}$$

5. Convergence of the conservation of mass

The sequence v^ε of flow fields is uniformly bounded in $H^1_0(\Omega)$, therefore it follows from the two-scale convergence introduced by Nguetseng [7] (see also [1]) that there exists $v^0 \in H^1(\Omega)$ and $v^1 \in L^2(\Omega, H^1_{\mathrm{per}}(Q))$ such that

$$v^\varepsilon \rightharpoonup v^0, \quad \text{weakly in } H^1(\Omega) \quad \text{and} \tag{5.1}$$

$$\frac{\partial v^\varepsilon}{\partial x_i} \rightsquigarrow \frac{\partial v^0}{\partial x_i} + \frac{\partial v^1}{\partial y_i}, \tag{5.2}$$

where the convergence indicated in (5.2) is in the two-scale sense, i.e. for $u^\varepsilon \in L^2(\Omega)$, $u^0 \in L^2(\Omega, L^2_{\mathrm{per}}(Q))$, $u^\varepsilon \rightsquigarrow u^0$ if and only if:

$$\int_\Omega u^\varepsilon w^\varepsilon \varphi \, dx \to \int_{\Omega \times Q} u^0(x, y) w(y) \varphi(x) \, dy \, dx \tag{5.3}$$

for any $w \in L^2_{\mathrm{per}}(Q)$, $w^\varepsilon(x) = w(x/\varepsilon)$ and $\varphi \in \mathscr{D}(\Omega)$. To expedite the presentation, the symbol "$\rightsquigarrow$" will be used to indicate two-scale convergence.

Applying the two-scale convergence to the conservation of mass law $(3.2)_3$ gives

$$\mathrm{div}_x\, v^0 + \mathrm{div}_y\, v^1 = 0. \tag{5.4}$$

Integration in the y variable of (5.4) over the unit cell Q yields from periodicity:

$$\mathrm{div}_x\, v^0 = 0, \quad \mathrm{div}_y\, v^1 = 0. \tag{5.5}$$

6. Convergence of the kinematic condition on the bubble interface

In this section we rigorously prove the asymptotic behaviour of the kinematic condition $(3.3)_5$ as ε tends to zero. In view of (5.1) and (5.2), the asymptotic behaviour of the kinematic condition is given by the following theorem:

THEOREM 6.1. *The limits v^0 and v^1 delivered by the two-scale convergence of the* *emulsion flow fields v^ε satisfy*

$$\left(\frac{\partial v_j^0}{\partial x_i} y_i + v_j^1(x, y)\right) n_j = \frac{1}{|B|}\left(\int_B v_j^1 \, dy\right) n_j \tag{6.1}$$

for y on Γ and x in Ω.

The proof of Theorem 6.1 follows immediately from the following two lemmas:

LEMMA 6.2. *Let $\{v^\varepsilon\}$ be a bounded sequence in $H^1(\Omega)$ and let $\{V^\varepsilon\}$ be an $L^2(\Omega)$ approximation to $\{v^\varepsilon\}$, i.e.:*

$$V^\varepsilon = \sum_i \left(\frac{1}{|B^{i\varepsilon}|}\int_{B^{i\varepsilon}} v^\varepsilon \, dx\right) \chi(Q^{i\varepsilon}); \tag{6.2}$$

then there exists $c_i \in L^2(\Omega)$ such that:

$$\frac{1}{\varepsilon}(v_i^\varepsilon - V_i^\varepsilon) \rightsquigarrow \frac{\partial v_j^0}{\partial x_i} y_j + v_i^1(x, y) + c_i(x). \tag{6.3}$$

Proof. From (6.2) and Poincare's inequality we have:

$$\| v^\varepsilon - V^\varepsilon \|_{L^2(\Omega)} \leqq C\varepsilon \tag{6.4}$$

and thus, from the two-scale convergence theorem, there exists $s \in L^2(\Omega, L^2_{\text{per}}(Q))$ such that

$$\frac{1}{\varepsilon}(v^\varepsilon - V^\varepsilon) \rightsquigarrow s. \tag{6.5}$$

In order to identify s, we introduce $w \in L^2(Q)$, such that $\operatorname{div} w \in L^2(Q)$, $w \cdot n = 0$ on ∂Q and write the following identity:

$$\frac{1}{\varepsilon}\int_\Omega (v^\varepsilon - V^\varepsilon)(\operatorname{div}_y w)^\varepsilon \varphi \, dx = \int_\Omega (v^\varepsilon - V^\varepsilon) \operatorname{div} w^\varepsilon \varphi \, dx \tag{6.6}$$

for any $\varphi \in \mathscr{D}(\Omega)$. The left-hand side converges from (6.5) to:

$$\int_{\Omega \times Q} s(x, y) \operatorname{div}_y w \varphi(x) \, dy \, dx \tag{6.7}$$

and integration by parts on the right-hand side yields:

$$\int_\Omega (v^\varepsilon - V^\varepsilon) \operatorname{div} w^\varepsilon \varphi \, dx = -\int_\Omega \frac{\partial v^\varepsilon}{\partial x_i} w_i^\varepsilon \varphi \, dx - \int_\Omega (v^\varepsilon - V^\varepsilon) w^\varepsilon \cdot \nabla \varphi, \tag{6.8}$$

which by (6.4) and two-scale convergence has the limit:

$$-\int_{\Omega \times Q} \left(\frac{\partial v^0}{\partial x_i} + \frac{\partial v^1}{\partial y_i}\right) w_i(y) \varphi(x) \, dy \, dx. \tag{6.9}$$

Equating (6.7) and (6.9), we see that:

$$\frac{\partial s}{\partial y_i} = \frac{\partial v^0}{\partial x_i} + \frac{\partial v^1}{\partial y_i} \tag{6.10}$$

and the lemma is proved. □

LEMMA 6.3. *Let $\{v^\varepsilon\}$ and $\{V^\varepsilon\}$ satisfy the conditions of the previous lemma and moreover*

$$\operatorname{div} v^\varepsilon = 0, \quad v^\varepsilon \cdot n = V^\varepsilon \cdot n \text{ on } \partial B^{i\varepsilon}; \tag{6.11}$$

then on ∂B:

$$\left(\frac{\partial v_j^0}{\partial x_i} y_i + v_j^1(x, y)\right) n_j = \frac{1}{|B|}\left(\int_B v_j^1 \, dy\right) n_j. \tag{6.12}$$

Proof. We let χ_B denote the characteristic function of the bubble B and consider the identity given by:

$$\begin{aligned}\frac{1}{\varepsilon}\int_\Omega (v_j^\varepsilon - V_j^\varepsilon)\left(\frac{\partial w}{\partial y_j}\chi_B\right)^\varepsilon \varphi = \sum_i \Bigg(&\int_{\partial B^{i\varepsilon}} (v_j^\varepsilon - V_j^\varepsilon) n_j w^\varepsilon \varphi \, dx \\ &- \int_{B^{i\varepsilon}} \operatorname{div}(v^\varepsilon - V^\varepsilon) w^\varepsilon \varphi \, dx \\ &- \int_{B^{i\varepsilon}} (v_j^\varepsilon - V_j^\varepsilon) w^\varepsilon \frac{\partial \varphi}{\partial x_j} dx\Bigg).\end{aligned} \tag{6.13}$$

Applying the hypotheses to the right-hand side of (6.13), and using the estimate (6.4), we see that the limit of the right-hand side is zero.

Passing to the limit on the left-hand side using Lemma 6.2, we find:

$$\int_{\Omega\times B}\left(\frac{\partial v_i^0}{\partial x_j} y_j + v_i^1 + c_i\right)\frac{\partial w}{\partial y_i}\varphi(x)\, dy\, dx = 0. \tag{6.14}$$

Thus

$$\left(\frac{\partial v_i^0}{\partial x_j} y_j + v_i^1\right) n_i = C \cdot n \text{ on } \partial B. \tag{6.15}$$

Since C is constant in y and the vector:

$$\frac{\partial v_i^0}{\partial x_j} y_j + v^1 \tag{6.16}$$

is from (5.4) divergence-free in y, it follows from (2.10) that:

$$C = \frac{1}{|B|}\left(\int_B v^1 \, dy\right) \tag{6.17}$$

and the lemma follows. □

7. Derivation of the cell problem

In this section we obtain the local balance laws of the flow. From the estimates (4.3) and (4.25) on the velocity field and extended pressure we have $2\mu^\varepsilon e(v^\varepsilon)$ and $\bar{p}^\varepsilon$ bounded in $L^2(\Omega)$. Thus from two-scale convergence:

$$\bar{p}^\varepsilon \rightsquigarrow p^0, \quad \bar{\sigma}^\varepsilon \rightsquigarrow \sigma^0 = -p^0 I + 2\mu(y)(e_x(v^0) + e_y(v^1)). \tag{7.1}$$

LEMMA 7.1. *The two-scale limit of the stress satisfies:*

$$\operatorname{div}_y \sigma^0 = 0, \quad \int_\Gamma [\sigma^0 n]\, ds = 0, \quad [\sigma^0 n] = ([\sigma^0 n]\cdot n)n, \tag{7.2}$$

where n is the unit outer normal to Γ

Proof. We first observe that for any function $\varphi \in \mathscr{D}(\Omega)$, the step function approximation

$$\varphi^\varepsilon = \sum_i \varphi(\varepsilon r^i)\chi(Q^{i\varepsilon}) \tag{7.3}$$

converges to φ in $L^\infty(\Omega)$. Thus from the Hölder inequality we have that:

$$\int_\Omega w^\varepsilon \varphi^\varepsilon \operatorname{div} \bar{\sigma}^\varepsilon\, dx - \int_\Omega w^\varepsilon \varphi \operatorname{div} \bar{\sigma}^\varepsilon\, dx \to 0, \tag{7.4}$$

as $\varepsilon \to 0$, since w^ε is bounded in $L^2(\Omega)$ and $\operatorname{div} \bar{\sigma}^\varepsilon = f$ in $(L^2(\Omega - \Gamma^{i\varepsilon}))^3$.

We now multiply (3.3) by $w^\varepsilon \varphi^\varepsilon$. Here we shall take $w \in (H^1_{\text{per}}(Q))^3$, such that $w = 0$ on ∂Q, $w \cdot n = W \cdot n$ on Γ with W constant and φ^ε as defined above. Integrating by parts, we obtain the identity:

$$\int_\Omega \bar{\sigma}^\varepsilon : e(w^\varepsilon)\varphi^\varepsilon\, dx = \int_\Omega f w^\varepsilon \varphi^\varepsilon. \tag{7.5}$$

Multiplying (7.5) by ε and taking the limit as ε goes to zero gives:

$$\int_{\Omega\times Q} \sigma^0(x, y) : e_y(w)\phi(x)\, dy\, dx = 0. \tag{7.6}$$

Thus we see that

$$\operatorname{div}_y \sigma^0 = 0 \tag{7.7}$$

for y in B and $Q - \bar{B}$. Equation $(7.2)_2$ follows by integrating (7.6) over the period cell. Finally, integrating by parts and choosing w with support on Γ and taking arbitrary tangential variations gives:

$$[\sigma^0 n] = ([\sigma^0 n]\cdot n)n \tag{7.8}$$

for y on Γ. □

Collecting the results of Lemma 7.1, equations (5.5) and (6.12), we find that for each point x, v^1 solves the following flow problem in the unit cell:

$$\operatorname{div}_y (2\mu(y)e_x(v^0) + 2\mu(y)e_y v^1) = \nabla_y p^0, \tag{7.9}$$

$$\operatorname{div}_y v^1 = 0, \tag{7.10}$$

$$[v^1] = 0, \tag{7.11}$$

$$\left(\frac{\partial v^0}{\partial x_i} y_i + v^1\right)\cdot n = \frac{1}{|B|}\int_B v^1(x, y)\, dy \cdot n, \tag{7.12}$$

$$[\sigma^0 n] = [\sigma^0 n]\cdot nn, \tag{7.13}$$

$$\int_B [\sigma^0 n] = 0. \tag{7.14}$$

We observe that on the surface of the spherical bubble, the normal vector $n = y$ and only the symmetric part of $\partial v_j^0/\partial x_i$ appears in the contraction $(\partial v_j^0/\partial x_i) y_i n_j$; i.e.:

$$\frac{\partial v_j^0}{\partial x_i} y_i n_j = e_{ij}(v^0) y_i n_j. \tag{7.15}$$

Therefore we see that, for each x, the solution v^1, p^0 of (7.9)–(7.14) depends linearly on $e(v^0)$ and we write:

$$v^1 = v^{ij}(y) e_{ij}(v^0) + w(x), \tag{7.16}$$

$$p^0 = p^{ij}(y) e_{ij}(v^0) + q(x), \tag{7.17}$$

where v^{ij}, p^{ij} are solutions of the local problem:

$$\operatorname{div}_y \tau^{ij} = 0, \tag{7.18}$$

$$\operatorname{div}_y v^{ij} = 0, \tag{7.19}$$

$$[v^{ij}] = 0, \tag{7.20}$$

$$(\tfrac{1}{2}(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl} - \tfrac{2}{3}\delta_{ij}\delta_{lm}) y_l + v_m^{ij}) n_m = \frac{1}{|B|}\left(\int_B v_m^{ij}\, dy\right) \cdot n_m, \tag{7.21}$$

$$[\tau^{ij} n] = [\tau^{ij} n] \cdot nn, \tag{7.22}$$

$$\int_{\partial B} [\tau^{ij} n]\, ds_y = 0, \tag{7.23}$$

where:

$$\tau_{lm}^{ij} = 2\mu(\tfrac{1}{2}(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}) - \tfrac{1}{3}\delta_{ij}\delta_{lm} + e_{lm}(v^{ij})) - p^{ij}\delta_{lm} \tag{7.24}$$

and

$$\int_Q p^{ij}\, dy = 0. \tag{7.25}$$

In view of (7.24) we may write σ^0 as

$$\sigma_{lm}^0 = e_{xij}(v^0)\tau_{lm}^{ij} - q\delta_{lm}. \tag{7.26}$$

8. Homogenised momentum equation and formula for the effective viscosity

THEOREM 8.1. *The two-scale limit of the stress satisfies the macroscopic momentum balance law given by:*

$$\int_\Omega \left(\int_Q \sigma_{il}^0\, dy - \int_\Gamma [\sigma_{ij}^0 n_j] y_l\, ds_y\right) e_{il} v\, dx = \int_\Omega f \cdot v\, dx \tag{8.1}$$

for all $v \in (H_0^1(\Omega))^3$.

Proof. Multiplying (3.3) by $v \in \mathscr{D}(\Omega)$, we obtain:

$$-\sum_i \int_{\Gamma^{i\varepsilon}} [\bar{\sigma}^\varepsilon n] \cdot v\, ds + \int_\Omega \bar{\sigma}^\varepsilon : e(v)\, dx = \int_\Omega f \cdot v. \tag{8.2}$$

It is evident from (7.1) that the second term has the limit:

$$\int_{\Omega\times Q} \sigma^0 : e(v)\, dx\, dy. \tag{8.3}$$

To compute the limit of the first term, we introduce $w \in L^2(Q)$ such that for a given constant W, we have $w = W$ on Γ and $w = 0$ on ∂Q. For a given $v \in \mathscr{D}(\Omega)$, we introduce the piecewise constant approximation:

$$V^\varepsilon = \sum_i v(\varepsilon r^i)\chi(Q^{i\varepsilon}). \tag{8.4}$$

In the sequel we use the fact that $v - V^\varepsilon$ converges to zero $L^\infty(\Omega)$ as ε tends to zero. Multiplying the first term in (8.2) by W, writing $v = V^\varepsilon + (v - V^\varepsilon)$ and application of $(3.3)_7$ gives:

$$-\sum_i W \int_{\Gamma^{i\varepsilon}} [\bar{\sigma}^\varepsilon n] \cdot v\, ds = -\sum_i \int_{\Gamma^{i\varepsilon}} W[\bar{\sigma}^\varepsilon_{im} n_m] \cdot (v_i - V^\varepsilon_i)\, ds. \tag{8.5}$$

Integrating by parts on each pave $Q^{i\varepsilon}$, we obtain:

$$\sum_i \left(\int_{Q^{i\varepsilon}} \frac{\partial w^\varepsilon}{\partial x_m} \bar{\sigma}^\varepsilon_{im}(v_i - V^\varepsilon_i)\, dx - \int_{Q^{i\varepsilon}} w^\varepsilon \operatorname{div} \bar{\sigma}^\varepsilon \cdot (v - V^\varepsilon)\, dx - \int_{Q^{i\varepsilon}} w^\varepsilon \bar{\sigma}^\varepsilon_{im} \frac{\partial(v_i - V^\varepsilon_i)}{\partial x_m}\, dx \right). \tag{8.6}$$

Here the contribution on the cell boundary $\partial Q^{i\varepsilon}$ vanishes as $w^\varepsilon = 0$ there. Passing to the limit in (8.6), we see that the second term converges to zero and the third reduces to:

$$-\sum_i \int_{Q^{i\varepsilon}} w^\varepsilon \bar{\sigma}^\varepsilon_{im} \frac{\partial v_i}{\partial x_m}\, dx \to -\int_{\Omega\times Q} \sigma^0_{im}(x, y) w(y) \frac{\partial v_i}{\partial x_m}\, dx\, dy. \tag{8.7}$$

For a given $\varepsilon > 0$, we have for each pave $Q^{i\varepsilon}$ the estimate:

$$\max_{x \in Q^{i\varepsilon}} |(v_i - V^\varepsilon_i) - \partial_l v_i(x)(x_l - r^{i\varepsilon}_l)| < \varepsilon^2 C. \tag{8.8}$$

Here the constant C is independent of ε and the estimate follows from Taylor's formula and from the uniform Lipschitz continuity of $\partial_{x_l} v_i$. Therefore we can approximate the first term by:

$$-\sum_i \int_{Q^{i\varepsilon}} \frac{\partial w^\varepsilon}{\partial x_m} \bar{\sigma}^\varepsilon_{im}(x_l - r^{i\varepsilon}_l) \frac{\partial v_i}{\partial x_l}\, dx = -\sum_i \int_{Q^{i\varepsilon}} \left(\frac{\partial w}{\partial x_m}\right)^\varepsilon \bar{\sigma}^\varepsilon_{im} y_l \frac{\partial v_i}{\partial x_l}\, dx$$
$$= -\int_\Omega \bar{\sigma}^\varepsilon_{im} \left(y_l \frac{\partial w}{\partial y_m}\right)^\varepsilon \frac{\partial v_i}{\partial x_l}\, dx \tag{8.9}$$

and passage to the limit gives:

$$-\int_{\Omega\times Q} \sigma^0_{im}(x, y) \frac{\partial w}{\partial y_m} y_l \frac{\partial v_i}{\partial x_l}\, dx\, dy. \tag{8.10}$$

In this way we see that (8.5) converges to

$$-\int_{\Omega\times Q} \sigma^0_{im}(x, y) \frac{\partial w}{\partial y_m} y_l \frac{\partial v_i}{\partial x_l}\, dx\, dy - \int_{\Omega\times Q} \sigma^0_{im}(x, y) w(y) \frac{\partial v_i}{\partial x_m}\, dx\, dy. \tag{8.11}$$

Integration by parts in the first term of (8.11) and applying (7.2) gives:

$$\lim_{\varepsilon\to 0}\left(-\sum_i W\int_{\Gamma^{i\varepsilon}}[\bar{\sigma}^\varepsilon n]\cdot v\,ds\right)=-W\int_\Omega \frac{\partial v_i}{\partial x_l}\left(\int_\Gamma [\sigma^0_{ij}n_j]y_l\,ds_y\right)dx. \quad (8.12)$$

Setting $W=1$ in (8.12), it follows from (8.6) and (8.7) that the limit of (8.5) is:

$$\int_\Omega\left(\int_Q \sigma^0_{il}\,dy-\int_\Gamma[\sigma^0_{ij}n_jy_l]\,ds_y\right)e_{il}v\,dx=\int_\Omega f\cdot v\,dx \quad (8.13)$$

and the theorem follows from density of $(\mathcal{D}(\Omega))^3$ in $(H^1_0(\Omega))^3$. □

It follows immediately from Theorem 8.1 that we can identify the deviatoric part of the homogenised stress as:

$$\sigma^H_{lk}-\tfrac{1}{3}\sigma^H_{ii}\delta_{lk}=\int_Q(\sigma^0_{lk}-\tfrac{1}{3}\sigma^0_{ii}\delta_{lk})\,dy-\int_\Gamma([\sigma^0_{ij}n_j]y_k-\tfrac{1}{3}[\sigma^0_{ij}n_j]y_i\delta_{lk})\,ds \quad (8.14)$$

and from (7.26) we see that this is related to the homogenised strain rate $e(v^0)$ through the effective viscosity tensor μ^H given by:

$$\sigma^H_{lk}-\tfrac{1}{3}\sigma^H_{ii}\delta_{lk}=2\mu^H_{lkij}e_{ij}v^0, \quad (8.15)$$

where

$$2\mu^H_{lkij}=\int_Q(\tau^{ij}_{lk}-\tfrac{1}{3}\delta_{lk}\tau^{ij}_{pp})\,dy-\int_\Gamma([\tau^{ij}_{lm}n_m]y_k-\tfrac{1}{3}[\tau^{ij}_{pm}n_m]y_p\delta_{lk})\,ds. \quad (8.16)$$

From Theorem 8.1 and equation (7.26), the hydrostatic part of the homogenised stress σ^H is given by the pressure of $q(x)$, i.e.:

$$\tfrac{1}{3}\sigma^H_{ii}\delta_{lk}=q(x)\delta_{lk}. \quad (8.17)$$

Collecting our results, we observe that Theorem 3.1 follows from Theorem 8.1, equations (5.5), (8.16) and (8.17).

Acknowledgments

The work of R. Lipton was supported by NSF grant DMS 9205158. The research was initiated while B. Vernescu was visiting the Institute for Mathematics and its Applications of the University of Minnesota, whose support is gratefully acknowledged.

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(*Issued* 13 *December* 1994)